On the Stability of the Basic East-West Velocity Flow

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ABSTRACT

Instability due to the horizontal shear of the basic state velocity, $U$, is addressed. It is corroborated that instability happens whenever $\partial^2 U / \partial y^2$ covaries negatively with $U$.

Keywords: stabilizing effect, barotropic, instability, basic state velocity, perturbation

INTRODUCTION

Assume a westerly jet as shown in Fig. 1. $U$ may be viewed as a reservoir in such a way that a small loss of $U$ is enough to start the perturbation. It is an infinitesimal perturbation (Fig. 2).

Therefore, if there is a divergence of flux of momentum, energy is removed from the basic flow to the eddies, which will tend to grow. This may be done by displacing the parcel from the faster part of the flow to the slower part of the flow (Fig. 3).

The actual observed case is the opposite. Energy is removed from the eddies to the basic flow. This may be thought of as taking the faster particles of the slow part and interacting them with the slower particles of the quicker flow (Fig. 4).

If the eddies grow, there is an unstable situation. If they decay, it is stable. Otherwise, it is neutral.

THE PROBLEM

Let us consider a homogeneous, incompressible fluid with $U_o = U_o (y)$ only, where $U$ represents the basic state east-west (zonal) velocity flow. The two-dimensional perturbation equations are:

$$\begin{align}
\partial u / \partial t + U_o \partial u / \partial x + v \partial U_o / \partial y - f v + (1/\rho) \partial p / \partial x &= 0 \\
\partial v / \partial t + U_o \partial v / \partial x + f u + (1/\rho) \partial p / \partial y &= 0
\end{align}$$
Fig 1. Idealized westerly jet showing the basic east-west velocity

Fig 2. Infinitesimal perturbation of the basic state east-west velocity

Fig 3. Infinitesimal perturbation showing energy flowing from the basic state to the eddies
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Fig 4. Infinitesimal perturbation showing energy flowing from the eddies to the basic state

where:

$$U_0 = - (\rho_o f)^{-1} \frac{\partial P}{\partial y}$$  \hspace{1cm} (3)

and $u$ and $v$ represent the perturbation equations.

Upon multiplication of equations (1) and (2) by $u$ and $v$, respectively, the energy equation is obtained:

$$\partial \left( \frac{u^2}{2} + \frac{v^2}{2} \right) / \partial t + U_0 \partial \left( \frac{u^2}{2} + \frac{v^2}{2} \right) \partial x + u v \partial U_0 / \partial y + u \partial \pi / \partial x + v \partial \pi / \partial y = 0$$  \hspace{1cm} (4)

where $\pi = p/\rho$.

Upon integration over a closed volume, it yields:

$$\partial / \partial t \left[ \int \left( \frac{u^2}{2} + \frac{v^2}{2} / 2 \right) \right] d m = - \int (u v) \partial U_0 / \partial y d m = - \int (V \cdot \nabla \pi) d m$$  \hspace{1cm} (5)

where:

$$\int (V \cdot \nabla \pi) d m = \int V \cdot (\nabla \pi) d m - \int \pi \cdot (V) d m = 0$$  \hspace{1cm} (6)

It is convenient to define a space average of the form:

$$\overline{()} = \left[ \int ( ) \ dx \ dy \right] / \left[ \int \ dx \ dy \right]$$  \hspace{1cm} (7)

Therefore,

$$\partial K_x / \partial t = \int \overline{u' v'} \partial U_0 / \partial y d m = \int U_o \partial \overline{(u' v')} / \partial y d m$$  \hspace{1cm} (8)

Without specifying the profile, the vorticity equation is:

$$\partial \zeta / \partial t + U_0 \partial \zeta / \partial x - v \partial^2 U_0 / \partial y^2 + \beta v = 0$$  \hspace{1cm} (9)
where \( \psi = \nabla^2 \psi \) and \( c = \sigma/k \). It is obtained:

\[
[\partial / \partial t + U_o \partial / \partial x \nabla^2 \psi + (\partial \psi / \partial x)(\beta - U_o^m)] = 0
\]  

(10)

It is convenient to define \( \psi \) as:

\[
\psi = 0.5 [\phi(y) \exp[i(k x - \sigma t)] + \phi^*(y) [-i(k x - \sigma^* t)]]
\]  

(11)

where the asterisks represent the complex conjugate. Plugging equation (11) back into the vorticity equation, equation (9), it is obtained:

\[
(-i \sigma + i U_o k) (-k^2 \phi + d^2 \phi / dy^2) + i k (\beta - U_o^m) \phi = 0
\]  

(12)

\[
(-i \sigma^* - i U_o k) (-k^2 \phi^* + d^2 \phi^* / dy^2) + i k (\beta - U_o^m) \phi^* = 0
\]  

(13)

Upon division by \( k \) and multiplication by \( i \), it is obtained:

\[
(U_o - c) (d^2 /dy^2 - k^2) \phi + (\beta - U_o^m) \phi = 0
\]  

(14)

\[
(U_o - c^*) (d^2 /dy^2 - k^2) \phi^* + (\beta - U_o^m) \phi^* = 0
\]  

(15)

Assuming \( c \) to be complex, and dividing by \( (U_o - c) \neq 0 \), it is obtained:

\[
d^2 \phi /dy^2 - [k^2 - ([\beta - U_o^m] / (U_o - c))] \phi = 0
\]  

(16)

\[
d^2 \phi^* /dy^2 - [k^2 - ([\beta - U_o^m] / (U_o - c^*))] \phi^* = 0
\]  

(17)

It is convenient to define:

\[
u' = \partial \psi / \partial y,
\]  

(18a)

\[
v' = - \partial \psi / \partial x.
\]  

(18b)

Then:

\[
\bar{u'}v' = -(\partial \psi / \partial y)(\partial \psi / \partial x)
\]  

(19)

It is obtained:

\[
\partial \psi / \partial x = (i k/2) [\phi \exp[i(k x - \sigma t)] - \phi^* [-i(k x - \sigma^* t)]
\]  

(20)

\[
\partial \psi / \partial y = (1/2) [\phi' \exp[i(k x - \sigma t)] + (\phi^*)' \exp[-i (k x - \sigma^* t)]
\]  

(21)

Multiplication of equations (20) and (21) yields:

\[
\bar{u'}v' = (i k/4) [\phi^* \phi' - \phi (\phi^*)'] \exp(2 i \sigma t)
\]  

(22)

It follows naturally that:

\[
\partial \bar{u'}v'/\partial y = (i k/4) \exp(2 i \sigma t) [\phi^* \phi' - \phi (\phi^*)'/\partial y]
\]  

(23)

The value of \( \mathcal{K} = (1/2)[(u')^2 + (v')^2] \) is sought. After some algebraic manipulations, it is obtained:
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\[ K_\varepsilon = \left(1/2\right) \left(1/4\right) \left| \phi \right|^2 \left( 2 + \left(1/4\right) k^2 \left| \phi \right|^2 \right) \exp\left(2i\sigma t\right) \]  

(24)

On the other hand,

\[ \phi^* \frac{d^2 \phi}{d y^2} = \phi^* \frac{d}{d y} \left( \frac{d \phi}{d y} \right) = \frac{d}{d y} \left( \phi^* \frac{d}{d y} \phi \right) - \left( \frac{d \phi}{d y} \right)^2 \]  

(25)

\[ \phi \frac{d^2 \phi^*}{d y^2} = \phi \frac{d}{d y} \left( \phi^* \frac{d}{d y} \phi \right) = \frac{d}{d y} \left( \phi \frac{d}{d y} \phi^* \right) - \left( \frac{d \phi}{d y} \right)^2 \]  

(26)

Upon subtraction of the product of equation (25) by \( \phi^* \) minus the product of equation (26) by \( \phi \), it follows that:

\[ 0.5 \frac{d}{d y} \left( \phi^* \phi - \phi (\phi^*)' \right) + 0.5 \left( \beta - U_o^* \right) \left( U_o - c \right)^{-1} \left( U_o - c^* \right)^{-1} \left| \phi \right|^2 = 0 \]  

(27)

Upon addition of the product of equation (25) by \( \phi^* \) plus the product of equation (26) by \( \phi \), it yields:

\[ 0.5 \frac{d}{d y} \left( \phi^* \phi + \phi (\phi^*)' \right) - \left| \phi \right|^2 - k^2 \left| \phi \right|^2 + 0.5 \left( \beta - U_o^* \right) \left( U_o - c \right)^{-1} + \left( U_o - c^* \right)^{-1} \left| \phi \right|^2 = 0 \]  

(28)

Equation (27) may be written as:

\[ 0.5 \frac{d}{d y} \left( \phi^* \phi - \phi (\phi^*)' \right) + \left(1/2\right) \left( \beta - U_o^* \right) \left( U_o - c \right)^{-1} \left( U_o - c^* \right)^{-1} \left| \phi \right|^2 = 0 \]  

(29)

whereas equation (28) may be written as:

\[ 0.5 \frac{d}{d y} \left( \phi^* \phi + \phi (\phi^*)' \right) - \left| \phi \right|^2 - k^2 \left| \phi \right|^2 + \left(1/2\right) \left( \beta - U_o^* \right) \left( U_o - c \right)^{-1} \left( U_o - c^* \right)^{-1} \left| \phi \right|^2 = 0 \]  

(30)

After some algebraic manipulations, it yields:

\[ \left( U_o - c^* \right) - \left( U_o - c \right) = 2i c \]  

(31)

\[ \left( U_o - c^* \right) + \left( U_o - c \right) = 2 \left( U_o - c^* \right) \]  

(32)

\[ \left( U_o - c \right) \left( U_o - c^* \right) = \left| U_o - c \right|^2 \]  

(33)

where \( c = c_r + i c_i \)

Therefore, equation (29) may be written as:

\[ 0.5 \frac{d}{d y} \left( \phi^* \phi - \phi (\phi^*)' \right) + i \frac{c}{c} \left( \beta - U_o^* \right) \left| U_o - c \right|^2 \left| \phi \right|^2 = 0 \]  

(34)

whereas equation (30) may be rewritten as:

\[ 0.5 \frac{d}{d y} \left( \phi^* \phi + \phi (\phi^*)' \right) - \left| \phi \right|^2 - k^2 \left| \phi \right|^2 + 0.5 \left( \beta - U_o^* \right) \left| U_o - c \right|^2 \left| \phi \right|^2 = 0 \]  

(35)

The boundary conditions for this problem are:

\[ \phi = \phi = 0, \text{ at the boundaries.} \]  

(36)
DISCUSSION

Whenever the profiles of $\phi$ and $\phi^*$ look more or less than in Fig. 5, upon integration over the entire region, it is obtained:

$$\int d(\phi^* \phi' - \phi(\phi^*))'/dy = 0$$  \hspace{1cm} (37)

Therefore, it is obtained:

$$i \ c_i \int (\beta - U_o^*) \ |U_o - c|^2 \ |\phi|^2 = 0$$  \hspace{1cm} (38)

Because of the fact that both $|U_o - c|^2$ and $|\phi|^2$ are positive, from equation (38) it may be inferred that either $c_i$ is zero or the integral must vanish. If $c_i$ is zero, we expect neutral solutions. The existence of the integral becomes questionable. If $c_i \neq 0$, equation (38) cannot be satisfied unless $(\beta - U_o^*)$ has at least one zero (Rayleigh, 1888). This is the necessary condition for barotropic instability. Namely that $(\beta - U_o^*)$ changes sign somewhere in the region.

On the other hand, the same reasoning as before applies for the integration over the entire region of equation (35). It follows that:

$$\int d(\phi^* \phi' + \phi(\phi^*))'/dy = 0$$  \hspace{1cm} (39)

Therefore, the integration of equation (35) yields:

$$\int_2^3 (U_o - c_r)(\beta - U_o^*) \ |U_o - c|^2 \ |\phi|^2 = \int_2^3 \left\{ |\phi|^2 + k^2 |\phi|^2 \right\} dy$$  \hspace{1cm} (40)

As $c_r$ is constant, it yields:

$$c_r \int_2^3 (\beta - U_o^*) \ |U_o - c|^2 \ |\phi|^2 \ dy = 0$$  \hspace{1cm} (41)

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Fig 5. Profiles of $\phi$ and $\phi^*$
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due to equation (38). Therefore, equation (40) yields:

\[ \frac{\partial K_t}{\partial t} = \int_{\gamma_2}^{\gamma_1} U_o \frac{\partial}{\partial y} \frac{v'}{\partial y} \, dm = \int_{\gamma_2}^{\gamma_1} \left\{ \phi^2 - k^2 \phi^2 \right\} \, dy = \]

\[ \int_{\gamma_2}^{\gamma_1} \left( \beta - U_o'' \right) \left| U_o - c \right|^2 \phi^2 \, dy \]  \hspace{1cm} (42)

For this to happen, the integral \[ \int_{\gamma_2}^{\gamma_1} \left( \beta - U_o'' \right) \left| U_o - c \right|^2 \phi^2 \, dy \] must be negative. This is certain only if \( U_o (b - U_o'') \) is negative in the interval of integration (Hoilland 1953). Therefore, only certain shapes of velocity profiles of velocity are permissible (Necco 1980). It means that unstable oscillations can exist only if the mean velocity has an absolute maximum within the flow.

**CONCLUSIONS**

The barotropic instability depends on the basic flow. Whenever \( U'' > 0 \) and \( U < 0 \), we would have instability. The same situation holds whenever \( U'' < 0 \) and \( U > 0 \). Let us assume a westerly flow with \( \beta = 0 \) (Fig. 6). This represents an unstable case. The peak of the curve is also where \( U'' < 0 \).

If the \( \beta \)-effect is added, we will have a stabilizing effect. Fig. 7 shows an easterly flow profile \( U'' > 0 \). When \( U < 0 \) we will have instability. This results as in perfect agreement with previous findings.

![Fig 6. Westerly flow showing a typical case of barotropic instability](image)

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Fig 7. Easterly flow showing a typical case of barotropic instability

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