The Rate of Mixing of Two-dimensional Markov Shifts

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ABSTRACT

The definitions of a two-dimensional Markov shift and the associated Markov measure are given. Then a sufficient condition for the exponential mixing of such shifts is provided. This generalizes the well-known result in the one-dimensional case.

Keywords: two-dimensional Markov shifts, Markov measure, mixing rate

INTRODUCTION

Unlike the theory of one-dimensional Markov shifts, the theory of higher dimensional analogue of these dynamical systems is filled with anomalies and difficulties (see Schmidt 1990 for a brief survey). In this paper, we make a small contribution towards a better understanding of these higher-dimensional Markov shifts by studying the rate of mixing of such shifts.

A standard result in the ergodic theory of one-dimensional Markov shifts is as follows: Let \((T, X, C, \mu)\) be a (one-dimensional) Markov shift where the Markov measure is given by some transition probability \((p, P)\). Suppose \(A\) and \(B\) are arbitrary cylinder sets in \(C\). Then the sequence \((\mu(A \cap T^{-n}B))_{n \geq 0}\) converges to \(\mu(A)\mu(B)\) at an exponential rate as \(n\) tends to infinity, when the matrix \(P\) is irreducible and aperiodic (i.e., there exists some integer \(N > 0\) such that all the entries of \(P^N\) are strictly positive). This result follows from the crucial matrix fact that when \(P\) is irreducible and aperiodic, then the sequence \((P^n(i,j))_{n \geq 0}\) converges exponentially fast to \(p(j)\) as \(n\) tends to infinity, for all \(i,j\). Note that an immediate corollary to the above result is that \(T\) is strong-mixing.
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Our purpose in this short note is to generalize the aforementioned results to the case of a two-dimensional Markov shift. Observe that the dynamical system in question consists of two commuting (invertible) measure-preserving transformations acting on the measurable space of functions from $\mathbb{Z}^2$ to some fixed finite set together with the Markov measure. Here the Markov measure is defined by two commuting stochastic matrices $P$ and $Q$ such that they share a common stationary probability vector $p$ (see below for details). Working analogously with the one-dimensional case, we need to look at the rate of convergence of the sequence $(P^m Q^n)(i,j)_{m,n\geq 0}$ to $p(j)$ as $m,n$ tends to infinity, for all $i,j$. We show that if either $P$ or $Q$ is irreducible and aperiodic then the convergence rate of the aforementioned sequence is exponentially fast. This in turn implies the exponential convergence of measures on rectangle sets for the corresponding two-dimensional Markov shift (see Theorem 1). An immediate corollary to the above is that the two-dimensional Markov shift is strong-mixing.

**DEFINITIONS AND RESULTS**

Let $Y$ be the finite set \{1, 2, \ldots, $k$\} equipped with the $\sigma$-algebra $2^Y$. The measurable space $(Y^{\mathbb{Z}^2}, B)$ is defined as the space of all functions $x : \mathbb{Z}^2 \to Y$ endowed with the product $\sigma$-algebra $B$. Recall that this means, $B$ is the smallest $\sigma$-algebra such that the collection of all projection maps $\pi_F : Y^{\mathbb{Z}^2} \to Y^F$ which is given by $\pi_F(x) = x_{[F]}$ for each finite subset $F$ of $\mathbb{Z}^2$, is measurable. Of course, the set $Y^F$ here is equipped with the product $\sigma$-algebra $\prod_{c \in F} 2^Y$. Given $x \in Y^{\mathbb{Z}^2}$, then we shall write $x_c$ for the value of the function $x$ at $c \in \mathbb{Z}^2$.

We shall be interested in the following subsets of $Y^{\mathbb{Z}^2}$. First, let $F$ be the set \{$c = (c_1, c_2) \in \mathbb{Z}^2 : a_t \leq c_t \leq a_t + u_t, t = 1, 2$\} for some given $a = (a_1, a_2) \in \mathbb{Z}^2$, $u = (u_1, u_2) \in (\mathbb{Z}^+)^2$. Then, an (elementary) rectangle $R_{a,u}$ is any subset of $Y^{\mathbb{Z}^2}$ which takes the form

$$R_{a,u} = \{x \in Y^{\mathbb{Z}^2} : x_{(c_1, c_2)} = i_{(c_1, c_2)}, \forall a_t \leq c_t \leq a_t + u_t, t = 1, 2\}$$

for some fixed elements $i_c$ of $Y$, for each $c \in F$. It is clear that such subsets are measurable. Moreover, it is not difficult to see that the collection of such rectangles generates the product $\sigma$-algebra $B$.

We shall now move on to the notion of a Markov measure on $(Y^{\mathbb{Z}^2}, B)$. For this, assume that we are given two $k \times k$-matrices $P$ and $Q$ satisfying the following three properties:

1. $P, Q$ are stochastic matrices such that $PQ = QP$.
2. There exists a probability vector $p = (p(1), \ldots, p(k))$ such that $pP = p$ and $pQ = p$.
3. If $P^0, Q^0$ denotes the 0–1 matrices which are compatible with $P$ and $Q$ respectively, then we require $P^0 Q^0 = Q^0 P^0$ and $P^0 Q^0$ is also a 0–1 matrix.
Let

\[ R_{a,u} = \{ x \in Y^{Z^2} : x_{(c_1,c_2)} = i_{(c_1,c_2)} , \forall a_t \leq c_t \leq a_t + u_t , t = 1,2 \} \]

be a rectangle, for some \( a = (a_1, a_2) \in Z^2 \) and \( u = (u_1, u_2) \in (Z^+)^2 \). We shall call \( R_{a,u} \) an allowable rectangle if, in addition,

\[ P^0(x_{(c_1,c_2)}, x_{(c_1+1,c_2)}) = Q^0(x_{(c_1,c_2)}, x_{(c_1,c_2+1)}) = 1 \]

for all \( x \in R_{a,u} \) and \( a_t \leq c_t \leq a_t + u_t - 1 , t = 1,2 \). We are now ready to define the Markov measure \( m \) on \((Y^{Z^2}, B)\) associated with the matrices \( P \) and \( Q \). Let \( R_{a,u} = \{ x \in Y^{Z^2} : x_{(c_1,c_2)} = i_{(c_1,c_2)} , \forall a_t \leq c_t \leq a_t + u_t , t = 1,2 \} \) be an allowable rectangle. Then the measure of \( R_{a,u} \) is taken to be

\[ m(R_{a,u}) = p(i_{(a_1,a_2)}) \prod_{e=0}^{u_1-1} P(i_{(a_1+e,a_2)}, i_{(a_1+e+1,a_2)}) \times \prod_{f=0}^{u_2-1} Q(i_{(a_1+u_1,a_2+f)}, i_{(a_1+u_1,a_2+f+1)}) \]

For non-allowable rectangles \( R \), we take \( m(R) \) to be zero. By using the Kolmogorov consistency theorem (see, e.g., Parthasarathy 1967), \( m \) extends uniquely to a probability measure on the product \( \sigma \)-algebra \( B \). In analogy with the one-dimensional case, we shall call this measure \( m \) the Markov measure defined by the matrices \( P \) and \( Q \).

We shall define the horizontal shift \( \sigma : Y^{Z^2} \rightarrow Y^{Z^2} \) and the vertical shift \( \tau : Y^{Z^2} \rightarrow Y^{Z^2} \) by

\[
(\sigma x)_{(c_1,c_2)} = x_{(c_1+1,c_2)} \quad \text{and} \quad (\tau x)_{(c_1,c_2)} = x_{(c_1,c_2+1)}
\]

for all \( x \in Y^{Z^2} \) and \((c_1, c_2) \in Z^2\). Then, it is clear that \( \sigma \) and \( \tau \) commute. Moreover, since each \( \sigma \) and \( \tau \) preserve the measure \( m \) on the algebra \( A \) of finite disjoint union of rectangles then they are measure-preserving on the smallest \( \sigma \)-algebra containing \( A \), which is, of course, the product \( \sigma \)-algebra \( B \). Thus \( \sigma \) and \( \tau \) are two commuting measure-preserving automorphisms acting on \((Y^{Z^2}, B, m)\). We shall call the resulting (invertible) measure-preserving dynamical system \((Y^{Z^2}, B, m, \sigma, \tau)\) a (two-dimensional) Markov shift with transition probability \((p, P, Q)\).

**DISCUSSION**

By working on rectangle sets, it can be shown that the assumption that \( P^0Q^0 \) is also a 0–1 matrix is needed to check consistency of the Markov measure (c.f. Kolmogorovs theorem).

A second implication of the 0–1 assumption on the matrix \( P^0Q^0 \) is that for allowable rectangles \( R_{a,u}, m(R_{a,u}) \) is also given by

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\[ m(R_{a,u}) = p \left( i_{(a_1,a_2)} \right) \prod_{e=0}^{u_2-1} P(i_{(a_1,a_2+e')}, i_{(a_1,a_2+e'+1)}) \times \]
\[ \prod_{f'=0}^{u_1-1} Q(i_{(a_1+f',a_2+u_2)}, i_{(a_1+f'+1,a_2+u_2)}) \]

Suppose we give the set \( Y \) the discrete topology. Then the Markov measure \( m \) is supported by the subshift of finite type

\[ X = \{ x \in Y^Z : P^0(x_{(e,f)}, x_{(e+1,f)}) = Q^0(x_{(e,f)}, x_{(e,f+1)}) = 1, \forall e, f \in Z \} \]

when we assume that the stationary probability vector \( p \) is a strictly positive vector. Note that the fact that \( X \) is non-empty follows from the commuting assumption on \( P^0 \) and \( Q^0 \).

Using well-known methods from the theory of one-dimensional Markov shifts, we prove:

**Lemma 1**

Suppose \( P \) and \( Q \) are two commuting \( k \times k \)-stochastic matrices such that there exists some probability vector \( p = (p(1), \ldots, p(k)) \) satisfying \( pP = p = pQ \). If either \( P \) or \( Q \) is irreducible and aperiodic then the sequence \( (P^m Q^n(i, j))_{m,n \geq 0} \) converges to \( p(j) \) at an exponential rate as \( m, n \) tends to infinity, for all \( i, j = 1, \ldots, k \), i.e., there exists constants \( C > 0 \), \( 0 \leq \alpha, \beta \leq 1 \) with \( \alpha \beta < 1 \) such that

\[ |P^m Q^n(i, j) - p(j)| \leq C \alpha^m \beta^n \quad \text{for all } m, n \geq 0 \]

and for all \( i, j = 1, \ldots, k \).

**Proof**

Without loss on generality, we shall assume that \( P \) is irreducible and aperiodic. Thus by the Perron-Frobenius theorem (see, for instance, Seneta 1981), the dominant eigenvalue 1 is a simple eigenvalue for \( P \). Now let \( V \) be the subspace \( \{ v \in C^k : < p, v > = 0 \} \). (Here \( <,> \) denotes the standard inner-product in \( C^k \)). Then, it is easy to see that \( P \) leaves \( V \) invariant, i.e., \( PV \subseteq V \). Moreover, if we denote the vector \( (1,1,\ldots,1) \) by \( \mathbf{1} \), then each \( w \in C^k \) can be uniquely written as

\[ w = w - < p, w > 1 + < p, w > 1 \]

such that \( w - < p, w > 1 \in V \) and \( < p, w > 1 \in U \), where \( U \) is the one-dimensional subspace generated by the vector \( \mathbf{1} \). Thus we have

\[ C^k = V \oplus U. \]

Hence, by virtue of the simplicity of the eigenvalue 1 of \( P \), we deduce that
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the spectral radius of $P_V$ (the restriction of $P$ to the invariant subspace $V$) is strictly less than 1. The spectral radius formula then implies there exists some $0 \leq \alpha < 1$ such that

$$|| P_V^m || \leq C_1 \alpha^m$$

for all $m \geq 0$ and some constant $C_1 \geq 0$. (Here the matrix norm is the usual operator norm.) Now, if $Q$ also has 1 as a simple eigenvalue then using the same argument as above, it gives us

$$|| Q_V^n || \leq C_2 \beta^n$$

for all $n \geq 0$ and some constants $0 \leq \beta < 1, C_2 > 0$. Thus, in this case, we have

$$|| P_V^m Q_V^n || \leq C \alpha^m \beta^n$$

for all $m, n \geq 0$ and some constant $C > 0$.

On the other hand, when the eigenvalue 1 of $Q$ is no longer simple the spectral radius of $Q_V^n$ may equal 1. In this case, it suffices to note that $Q_V^n$ is bounded so that

$$|| Q_V^n || \leq C_3$$

for all $n \geq 0$ and for some constant $C_3 > 0$. This in turn implies that

$$|| P_V^m Q_V^n || \leq C' \alpha^m$$

for all $m, n \geq 0$, and some constant $C' > 0$. Hence, in either case, we have

$$|| P_V^m Q_V^n || \leq C \alpha^m \beta^n \quad \forall m, n \geq 0$$

and for some constants $C > 0, 0 \leq \alpha, \beta \leq 1$ with $\alpha \beta < 1$. From this last inequality we deduce that

$$|| P^m Q^n || \leq C \alpha^m \beta^n || v || \quad \forall m, n \geq 0, \forall v \in V.$$  

(Here the vector norm is the usual $l_1$ norm.) Recall that given $w \in C^k$, then $w- < p, w > 1 \in V$. Thus, since $P^m Q^n$ is stochastic for all $m, n \geq 0$, we have

$$|| P^m Q^n w- < p, w > 1 || \leq C \alpha^m \beta^n || w- < p, w > 1 ||$$

for all $m, n \geq 0$ and all $w \in C^k$. Furthermore, by taking $w = (0, \ldots, 0, 1, 0, \ldots, 0)$, the $j$-th unit vector, it is easy to see that

$$|P^m Q^n(i, j) - p(j)| \leq || P^m Q^n w- < p, w > 1 ||$$

for all $m, n \geq 0$ and $i, j = 1, 2, \ldots, k$. Thus

$$|P^m Q^n(i, j) - p(j)| \leq C' \alpha^m \beta^n \quad \forall m, n \geq 0,$$
and for all $i, j = 1, 2, \ldots, k$ where $C' = C \max\{|w - p, w > 1|: w = j-$th unit vector, $j = 1, 2, \ldots, k\}$. This gives us the required result.

Observe that the essential ingredient in the above proof is the fact that either $P$ or $Q$ has a simple dominant eigenvalue 1 and the rest of the spectrum has a modulus strictly less than 1. Of course, a similar observation also holds in the case of a one-dimensional Markov shift. The following theorem is the main result of this note.

**Theorem 1**

Let $(Y_{Z^2}, B, m, \sigma, \tau)$ be a Markov shift with transition probability $(p, P, Q)$. Suppose either $P$ or $Q$ is irreducible and aperiodic. Then given any rectangle $A, B \in B$, there exists an integer $N > 0$ and constants $C > 0, 0 \leq \alpha, \beta < 1$ with $\alpha\beta < 1$ such that

$$|m(A \cap \sigma^{-m}\tau^{-n}B) - m(A)m(B)| \leq C\alpha^m\beta^n$$

for all integers $m, n \geq N$.

**Proof**

Let $A, B \in B$ be two arbitrary rectangles. Then, by definition, there exists $a = (a_1, a_2), b = (b_1, b_2) \in Z^2, u = (u_1, u_2), v = (v_1, v_2) \in (Z^+)^2$ such that $A = R_{a, u}$ and $B = R_{b, v}$ where

$$R_{a, u} = \{x \in Y_{Z^2}: x(c_1, c_2) = i(c_1, c_2), \forall a_t \leq c_t \leq a_t + u_t, t = 1, 2\}$$

and

$$R_{b, v} = \{x \in Y_{Z^2}: x(d_1, d_2) = i'(d_1, d_2), \forall b_t \leq d_t \leq b_t + v_t, t = 1, 2\}$$

Hence

$$A \cap \sigma^{-m}\tau^{-n}B = \{x \in Y_{Z^2}: x(c_1, c_2) = i(c_1, c_2), \forall a_t \leq c_t \leq a_t + u_t$$

and $x(d_1 + m, d_2 + n) = i'(d_1, d_2), \forall b_t \leq d_t \leq b_t + v_t, t = 1, 2$.

Now, let $m > a_1 + u_1 - b_1$ and $n > a_2 + u_2 - b_2$. Then, in particular, $A \cap \sigma^{-m}\tau^{-n}B$ is a finite disjoint union of elementary rectangles $R_1, R_2, \ldots, R_k$, say. If either $A$ or $B$ is non-allowable, then each of the $R_i$'s are non-allowable. Thus, in this case, we have $m(A \cap \sigma^{-m}\tau^{-n}B) = \sum_{i=1}^k m(R_i) = 0$. Moreover, since $m(A)m(B)$ is also zero, then the required result holds trivially in this case. We are now left with the case when the elementary rectangles $A$ and $B$ are both allowable. Then, by the assumption $P^0Q^0$ is also a 0–1 matrix (see above), it is straightforward (but tedious) to check that
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\[ m(A \cap \sigma^{-m}T^{-n}B) = p(i_{(a_1,a_2)}) \prod_{e=0}^{u_1-1} P(i_{(a_1+e,a_2)}, i_{(a_1+e+1,a_2)}) \times \]
\[ \prod_{f=0}^{u_2-1} Q(i_{(a_1+u_1,a_2+f)}, i_{(a_1+u_1,a_2+f+1)})P^{m'} Q^{n'}(i_{(a_1+u_1,a_2+u_2)}, i'_{(b_1,b_2)}) \times \]
\[ \prod_{e=0}^{v_1-1} P(i'_{(b_1+e,b_2)}, i'_{(b_1+e+1,b_2)}) \prod_{f=0}^{v_2-1} Q(i'_{(b_1+v_1,b_2+f)}, i'_{(b_1+v_1,b_2+f+1)}) \]

where \( m' = b_1 + m - (a_1 + u_1) > 0 \), \( n' = b_2 + n - (a_2 + u_2) > 0 \). Observe that since one of \( P \) or \( Q \) is aperiodic, then the stationary probability vector \( p \) is strictly positive. Hence

\[ m(A \cap \sigma^{-m}T^{-n}B) = \frac{m(A)m(B)}{p(i'_{(b_1,b_2)})} p^{m'} Q^{n'}(i_{(a_1+u_1,a_2+u_2)}, i'_{(b_1,b_2)}). \]

So that

\[ \left| \frac{m(A \cap \sigma^{-m}T^{-n}B)}{m(A)m(B)} - 1 \right| = \left| \frac{1}{p(i'_{(b_1,b_2)})} p^{m'} Q^{n'}(i_{(a_1+u_1,a_2+u_2)}, i'_{(b_1,b_2)}) - p(i'_{(b_1,b_2)}) \right| \]

Thus, by combining the previous line and Lemma 1, we gather that there exists \( 0 \leq \alpha, \beta \leq 1 \) with \( \alpha \beta < 1 \) such that

\[ |m(A \cap \sigma^{-m}T^{-n}B) - m(A)m(B)| \leq C \alpha^{m'} \beta^{n'} \]

for all \( m', n' > 0 \) and for some constant \( C > 0 \). Finally, by taking \( N = \max((a_1 + u_1) - b_1, (a_2 + u_2) - b_2, 1) \), we deduce that

\[ |m(A \cap \sigma^{-m}T^{-n}B) - m(A)m(B)| \leq C' \alpha^m \beta^n \]

for all \( m, n \geq N \) and some constant \( C' > 0 \). This then gives us the required result.

Let \( T_1, T_2 \) be two commuting measure-preserving transformations acting on the probability space \((Z, D, \nu)\). Then the resulting dynamical system is said to be strong-mixing if

\[ \lim_{m,n \to \infty} m(A \cap T_1^{-m}T_2^{-n}B) = m(A)m(B) \]

for all \( A, B \in D \).

Recall that for Markov shifts, disjoint unions of rectangles form an algebra that generates the product \( \sigma \)-algebra. Hence, by using a standard approximation theorem (see, e.g., Walters 1981), we have an immediate corollary to Theorem 1.
Corollary 1
Let $(Y^Z, B, m, \sigma, \tau)$ be a Markov shift with transition probability $(p, P, Q)$. If either $P$ or $Q$ is irreducible and aperiodic then the Markov shift is strong-mixing.

Observe that if $P$, say, is irreducible and aperiodic and $Q$ is the identity matrix then we can identify the two-dimensional Markov shift with the one-dimensional Markov shift with transition probability $(p, P)$. Thus we can retrieve the well-known mixing result for one-dimensional Markov shifts from the above corollary. As a final remark, note that all the above results can be generalized for Markov shifts of arbitrary dimensions.

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