Malaysian Journal of Mathematical Sciences 7(2): 283-295 (2013)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

On Commutativity of Completely Prime Gamma-Rings

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ABSTRACT

In this paper we prove that any completely prime Γ-ring M satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ ($a,b,c\in M$ and $\alpha,\beta\in\Gamma$) with nonzero derivation, is a commutative integral Γ-domain if its characteristic is not two. We also show that if the characteristic of M is 2 the Γ-ring M is either commutative or is an order in a simple 4-dimensional algebra over its center. We give necessary condition in terms of derivations for belongings of an element of the Γ-ring M to the center of M when the characteristic of M is not two. If char $M = 2$, and $a \notin Z(M)$, then we show that the derivation is the inner derivation.

Keywords: Γ-ring, completely prime Γ-ring, centroid of Γ-rings, annihilator, extended centroid of Γ-rings.

1. INTRODUCTION

The notion of a gamma ring was introduced as an extensive generalization of the concept of a classical ring. From its first appearance, the extensions and the generalizations of various important results in the theory of classical rings to the theory of gamma rings have attracted a wider attention as an emerging field of research to the modern algebraists to enrich the world of algebra. A good number of prominent mathematicians have worked out on this interesting area of research to develop many basic characterizations of gamma rings and have extended numerous significant results in this context in the last few decades. We start with the following necessary introductory definitions and examples.

Definition 1.1. Let M and Γ be additive abelian groups. If there exists a tri-additive mapping $M\times\Gamma\times M\to M:(x,\alpha,y)\mapsto x\alpha y$ satisfying the condition $(a\alpha b)\beta c=a\alpha(b\beta c),$ for all $a,b,c\in M$ and $\alpha\beta\in\Gamma,$ then M is said to be a Γ-ring.

Examples.

1. Every associative ring is a Γ-ring. In this case Γ is a single element set.

2. Let R be an integral domain with the identity element 1. Take $M=M_{1\times 2}(R)$ and $\Gamma=\left\{\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix}\middle| n \text{ is an integer}\right\}$. Then M is a Γ-ring. If we assume that $N=\{(a,a):a\in R\}\subset M,$ then it is easy to verify that N also is a Γ-ring (in fact, N is a subring of M).

3. Let M be a Γ-ring. We denote the sets of $m\times n$ matrices with entries from M and set of $n\times m$ matrices with entries from Γ by $M_{m\times n}$ and $\Gamma_{n\times m}$, respectively, then $M_{m\times n}$ is a $\Gamma_{n\times m}$ ring with the multiplication defined by
$$(a_{ij})(\gamma_{ij})(b_{ij})=(c_{ij}),\text{ where } c_{ij}=\sum_p\sum_q a_{ip}\gamma_{pq}b_{qj}.$$
If $m=n,$ then M_n is a Γ_n-ring.

Definition 1.2. Let M be a Γ-ring. A subring I of M is an additive subgroup which is also a Γ-ring. A right ideal of M is a subgroup I such that $I\Gamma M\subset I.$ The concept of a left ideal is defined similarly. If I is both the right and the left ideal then we say that I is an ideal.

Definition 1.3. The ring M is called a prime Γ-ring if $a\Gamma M\Gamma b=0$ implies $a=0$ or $b=0.$ The ring M is called a completely prime Γ-ring if $a\Gamma b=0$ implies $a=0$ or $b=0.$

Let M be a Γ-ring and $N=\{(a,a):a\in R\}\subset M$, then it is easy to verify that N is also a Γ-ring (in fact, N is a subring of M). It is clear that N is a completely prime Γ-ring satisfying the condition $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in N$ and $\alpha,\beta\in\Gamma$.

It is easy to see that every completely prime Γ-ring is a prime Γ-ring. Let M be a Γ-ring and I be a subset of M. The subset $Ann_l(I)=\{a\in M \mid a\Gamma I=0\}$ of M is called a left annihilator of I. A right annihilator $Ann_r(I)$ is defined similarly. If I is a nonzero ideal of M then $Ann_l(I)=Ann_r(I)$ and we denote it by $Ann(I)$.

Definition 1.4. Let M be a Γ-ring. An additive mapping $d:M\to M$ is called a derivation if $d(a\alpha b)=d(a)\alpha b+a\alpha d(b)$ holds for every $a,b\in M$ and $\alpha\in\Gamma$.

We write $[a,b]_\alpha=a\alpha b-b\alpha a$ (the commutator of a and b). Let $a\in M$. Define $d:M\to M$ by $d(x)=[a,x]_\alpha$ for every $\alpha\in\Gamma$. Then it is clear that d is a derivation. This derivation is called an inner derivation. We consider an assumption $(*) a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$. By the definition of the commutator it is easy to see that the following basic commutator identities hold

$$[a\beta b,c]_\alpha=a\beta[b,c]_\alpha+[a,c]_\alpha\beta b+a[\beta,\alpha]_c b$$

$$[a,b\beta c]_\alpha=b\beta[a,c]_\alpha+[a,b]_\alpha\beta c+b[\beta,\alpha]_a c.$$

Taking into account the above assumption $(*)$ the basic commutator identities can be reduced to $[a\beta b,c]_\alpha=a\beta[b,c]_\alpha+[a,c]_\alpha\beta b$ and $[a,b\beta c]_\alpha=b\beta[a,c]_\alpha+[a,b]_\alpha\beta c$, respectively. In other words, the later shows that the left and right multiplication operators are inner derivations. This fact we extensively use in this paper.

We refer to Ozturk and Jun (2000, 2001) for the definitions of centroid, extended centroid of Γ-rings. The Γ-rings were first introduced by Nobusawa (1964). These are generalizations of classical rings. Later, Barnes (1966) generalized the definition of Nobusawa (1964) which is given above. Many researchers worked on Γ-rings and their derivations. The importance of study of derivations was noticed by Herstein (1978, 1979). The end 1990th is considered the renaissance of the Γ-ring's theory. Ceven (2002) investigated the Jordan left derivations of completely prime Γ-rings and showed that every Jordan left derivation of a 2 torsion-free completely prime Γ-rings is a Jordan left derivation. The notion of k-derivation first is introduced by Kandamar (2000) and he proved that every prime Γ-ring of Nobusawa (1964) is commutative by the help of k-derivations. Chakraborty and Paul (2010) proved that every Jordan k-left derivations of a 2-torsion free completely prime Γ-rings is a k-left derivation. Sapanci and Nakajima (1997) showed that every Jordan derivation of a 2-forsion free prime Γ-rings is a derivation. The commutativity in prime Γ-rings was investigated with the help of two derivations by Soyturk (1994). He proved that M is commutative when the two non-zero derivations are contained in the center of M. We remind that commutativity conditions for Γ-near rings with one and two derivations have been considered in Rakhimov (2013a) and Rakhimov (2013b).

In this paper, we prove that a completely prime Γ-ring M with a nonzero derivation is either a commutative integral Γ-domain or is an order in a simple algebra of dimension four over its center, when char $M \neq 2$ or char $M = 2,$ respectively. We also prove the following

(i) If $a \in M$ and $[a, d(x)]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma,$ then $\alpha \in Z(M),$ provided char $M \neq 2,$ where $Z(M)$ is the center of M.

(ii) If M is of characteristic 2, then it is proven that $a\alpha a$ is an element of the center of M for all $\alpha \in \Gamma.$ Moreover, if a belongs to the center of M, then there is an element c of the centroid of M and such that $d(x) = [c\alpha a, x]_\beta,$ that is d is an inner derivation of M.

2. COMMUTATIVITY OF Γ-RINGS WITH DERIVATIONS.

In this section first we prove a few subsidiary lemmas (Lemmas 2.1-2.4) to use them in the proving of the main results (Theorems 2.5-2.7).

Lemma 2.1 Let M be a prime Γ-ring and suppose that $a \in M$ centralizes a non-zero right ideal of M. Then $a \in Z(M)$.

Proof. Suppose that a centralizes a non-zero right ideal A of M. If $x \in M, r \in A,$ then $r\alpha x \in A$ for every $\alpha \in \Gamma,$ hence $a\alpha(r\beta x) = r\beta x\alpha a,$ for

$\alpha,\beta\in\Gamma$. But $a\alpha r=r\alpha a$, for $\alpha\in\Gamma$, we thus get that $r\alpha(a\beta x-x\beta a)=0$, which is to say that $a\alpha[a,x]_\beta)=0$ for all $x\in M$ and $\alpha,\beta\in\Gamma$.
Since M is prime and $A\neq 0$, we conclude that $[a,x]_\beta=0$ for all $x\in M,\beta\in\Gamma$, hence $a\in Z(M)$.

Lemma 2.2 Let M be any Γ-ring satisfying the condition $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ and let $u\in M$. Then the set $V=\{a\in M: a\alpha[u,x]_\beta=0,$ for all $x\in M$ and $\alpha,\beta\in\Gamma\}$ is an ideal of M.

Proof. It is clear that V is a left ideal of M. Now, we show that V also is the right ideal. Let $a\in V$ and $x,r\in M$. For all $\alpha,\beta,\delta\in\Gamma$, we have $a\alpha[u\beta(r\delta x)-(r\delta x)\beta u]=0$. The Jacobi identity for the commutators gives, $a\alpha[u\beta r\delta x-r\delta x\beta u]=(u\beta r-r\beta u)\delta x+r\beta(u\delta x-x\delta u)$, then using the condition, we get,

$$0=a\alpha[u\beta(r\delta x)-(r\delta x)\beta u)=a\alpha[u,r]_\delta\beta x+a\alpha r\beta[u,x]_\delta,$$

that is, $a\alpha r\beta[u,x]_\delta=0$, for any $\alpha,\beta,\delta\in\Gamma$. Hence $a\alpha r\in V$ and V is a right ideal of M.

Lemma 2.3 Let M be a prime Γ-ring satisfying the condition $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ and suppose that $0\neq u\in M$ satisfies $a\alpha[u,x]_\beta=0$, for all $x\in M,\alpha,\beta\in\Gamma$. Then $u\in Z(M)$.

Proof. By Lemma 2.2,

$$V=\{a\in M: a\alpha[u,x]_\beta=0,\text{ for all } x\in M \text{ and } \alpha,\beta\in\Gamma\}$$

is a nontrivial ideal of M. Since, M is prime and $u\alpha x-x\alpha u\in Ann_rV$ we have $u\alpha x-x\alpha u=0$, for all $x\in M,\alpha\in\Gamma$, hence $u\in Z(M)$.

Lemma 2.4 Let M be a semiprime Γ-ring satisfying the condition $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ and suppose that $a\in M$ centralizes all the commutators $[u,x]_\alpha$, for all $x,y\in M$ and $\alpha\in\Gamma$. Then $u\in Z(M)$.

Proof. If $x, y \in M$ and $\alpha, \beta \in \Gamma$, then since $x\alpha(y\beta a) - (y\beta a)\alpha x$ is a commutator, a must commute with $x\alpha(y\beta a) - (y\beta a)\alpha x.$ But $x\alpha(y\beta a) - (y\beta a)\alpha x = [x, y]_\alpha \beta a + y\alpha[x, y]_\beta.$ By the hypothesis, a commutes with the left sides and the first term of the right sides of the last relations. Thus the element a must commute with $y\alpha[x,a]_\beta$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. This gives $[y,a]_\alpha \delta [x,a]_\beta = 0,$ for all $x, y \in M$ and $\alpha, \beta, \delta \in \Gamma$. This implies that $[y,a]_\alpha \in V$ for all $y \in M$ and $\alpha \in \Gamma$. Now due to Lemma 2.2, V is an ideal of M and $[y,a]_\alpha \in Ann_r(V)$, hence in $[y,a]_\alpha \in V \cap Ann_r(V)$. But M is semiprime therefore $V \cap Ann_r(V) = 0,$ hence $[y,a]_\alpha = 0,$ for all $y \in M$ and $\alpha \in \Gamma$. So we conclude that $u \in Z(M)$.

Theorem 2.5 Let M be a Γ-ring, d be a nonzero derivation of M such that $d^3 \neq 0.$ Then the subring A of M generated by all $d(a\alpha b),$ with $\alpha \in \Gamma$ and $a, b \in M$, contains a nonzero ideal of M.

Proof. Since $d^3 \neq 0$ and $d(M\Gamma M) \subset A$ we have $d^2(M\Gamma M) \neq 0.$ Take $y \in A$ such that $d^2(y) \neq 0.$ If $x \in M$, then $d(x\alpha y) = d(x)\alpha y + x\alpha d(y) \in A.$ Since y and $d(x)$ are in $A,$ then $x\alpha d(y) \in A.$ This implies that $M\Gamma d(y) \subset A.$ Similarly $d(y)\Gamma M \subset A.$ If $r, s \in M$, then $d(r\alpha d(y)\beta s) \in A.$ But

$$d(r\alpha d(y)\beta s) = d(r)\alpha d(y)\beta s) + r\alpha d^2(y)\beta s + r\alpha d(y)\beta d(s).$$

Since $d(y)\beta s \in A$ and $r\alpha d(y) \in A$ this implies that $r\alpha d^2(y)\beta s \in A$ for all $r, s \in M$ and $\alpha, \beta \in \Gamma$. Hence we get $M\Gamma d^2(y) \subset A,$ $d^2(y)\Gamma M \subset A$ and $M\Gamma d^2(y)\Gamma M \subset A.$ Therefore we conclude that the ideal of M generated by $d^2(y) \neq 0$ must be in $A.$ This proves the theorem.

If $d^3 = 0,$ the result need not be true. Let M be any completely prime Γ-ring having nilpotent elements, and let $a \neq 0 \in M$ be such that $a\alpha a = 0$ for every $\alpha \in \Gamma$. Let $d: M \to M$ be defined by $d(x) = [a, x]_\alpha$ for all $\alpha \in \Gamma.$ Then $B = a\Gamma M - M\Gamma a$ is a subring of M (since $a\alpha a = 0$) and contains $d(M)$. Also $d^3 = 0,$ $d^2 \neq 0$ (if char of $M \neq 2$). Yet B contains no nonzero ideal of M, for

$$a\Gamma B\Gamma a = a\Gamma(a\Gamma M - M\Gamma a)\Gamma a = a\Gamma a\Gamma M\Gamma a + a\Gamma M\Gamma a\Gamma a = 0.$$

Theorem 2.6 Let M be a completely prime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$. Let d be a nonzero derivation of M such that $[d(x), d(y)]_\alpha = 0$ for all $\alpha \in \Gamma$ and $x, y \in M$. Then

(i) If characteristic of $M \neq 2$ then M is a commutative integral Γ-domain.

(ii) If characteristic of $M = 2$ then M either is commutative or is an order of 4-dimentional simple algebra over its center.

Proof. Let A be the subring of M generated by all $d(x)$ for $x \in M$. By the hypothesis the A is a commutative subring of M. If $a \in A$ and $x \in M$ then $d(a)\alpha x + a\alpha d(x) = d(a\alpha x) \in A$, hence centralizes A. Therefore, if $b \in A$ then

$$0 = b\beta d(a\alpha x) - d(a\alpha x)\beta b = d(a)\alpha(b\beta x - x\beta b) = d(a)\alpha[b,x]_\beta.$$

If $A \not\subset Z$, we must have $d(a) = 0$ for every $a \in A$ and for the annihilator of all $[b,x]_\beta = 0$, $x \in M$, $\beta \in \Gamma$, is an ideal of M. Suppose that $A \not\subset Z$. By the above $d(A) = 0$, hence $d(d(M)) \subset d(A) = 0$, that is, $d^2(x) = 0$ for all $x \in M$. We obtain $2d(x)\alpha d(y) = 0$, so, if char $M \neq 2$ we get $d(x)\alpha d(y) = 0$. Using $y = z\beta x$ leads to $d(x)\Gamma d(x) = 0$ and hence $d(x) = 0$ for all $x \in M$. This contradicts $d \neq 0$. Thus if $A \not\subset Z$, we must have char $M = 2$.

Let $T = \{x \in M \mid d(x) = 0\}$. By the same argument above, we get $T \supset A$. If $t \in T$ and $x \in M$, then $d(t\alpha x) = d(t)\alpha x + t\alpha d(x) = t\alpha d(x)$, hence $t\alpha d(x)$ centralizes A for all $\alpha \in \Gamma$. If $a \in A$ then a commutes with $d(x) \in A$ and $t\alpha d(x)$, therefore, $(a\alpha t - t\alpha a)\beta d(x) = 0$ for all $x \in M$ and $\alpha, \beta \in \Gamma$. Since M is a completely prime Γ-ring, this gives that $a\alpha t - t\alpha a = [a,t]_\alpha = 0$, hence A centralizes T. The T is clearly a Γ-subring of M. Moreover, it is a Lie ideal of M. If $t \in T$ and $x \in M$, then

$$d(t\alpha x - x\alpha t) = t\alpha d(x) - d(x)\alpha t = 0 \text{ (since } d(t) = 0\text{)}$$

and $d(x)\in A$ centralizes T. Therefore, $\alpha x - x\alpha t \in T$.

Since M is a completely prime Γ-ring, T is a subring and it is a Lie ideal of M. Then either T contains a nonzero ideal of M or T is commutative and $t\alpha t\in Z(M)$ for $t\in T$. If T contains a nonzero ideal of M then A must centralize this ideal, since A centralizes all of T in a completely prime Γ-ring, this forces A to be in $Z(M)$, contrary to supposition. Hence, we have that T is commutative and $t\alpha t\in Z(M)$ for all $t\in T, \alpha\in\Gamma$. Let $a\in A$ but $a\notin Z(M)$, then $\alpha\in T$ and $a\alpha x - x\alpha a\in T$, thus $a\alpha a\in Z(M)$ and

$$(a\alpha x - a\alpha a)\gamma(a\alpha x - x\alpha a)\in Z(M) \text{ for all } x\in M \text{ and } \alpha,\gamma\in\Gamma.$$

It follows easily that M must be an order of 4-dimentional simple algebra over its center. Hence if $A\not\subset Z(M)$.

Now suppose that $A\not\subset Z(M)$. Thus $d(x)\in Z(M)$ for all $x\in M$. Hence if $x,y\in M$ then $d(x\alpha y) = x\alpha d(y) + d(x)\alpha y\in Z(M)$. Commuting this with x and using the fact that $d(x), d(y)$ are in $Z(M)$, we obtain $d(x)\alpha(x\beta y - y\beta x) = 0$ for all $\alpha,\beta\in\Gamma$. However if $d(x)\neq 0$, since it is in $Z(M)$ we get that it is not a zero divisor. Thus in this case we have $x\beta y = y\beta x$ for all $y\in M$ and $\beta\in\Gamma$. In short, if $d(x)\neq 0$, since $d(M)\neq 0$, pick x_0 such that $d(x_0)\neq 0$, then $x_0\in Z(M)$ by the above. Also $d(x+x_0) = d(x)+d(x_0) = d(x_0)\neq 0$, hence $x+x_0\in Z(M)$. This leaves us with $x\in Z(M)$. In the other words, if $A\subset Z(M)$, then M is commutative and since M is a completely prime Γ-ring, it must be an integral Γ-domain. The theorem is now completely proved.

Theorem 2.7 Let M be a completely prime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ and let d be an nonzero derivation of M. Suppose that $a\in M$ is such that $[a,d(x)]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$. Then

(i) If M is not of char 2, then a is in the center of M.

(ii) If M is of char 2, then $a\alpha a\in Z(M)$. Moreover, if a belongs to the center of M, then there is an element c of the centroid of M and $\alpha,\beta\in\Gamma$ such that $d(x) = [c\alpha a, x]_\beta$, that is d is an inner derivation of M.

Proof. Suppose that $a \in Z(M)$. According to the hypothesis one has $[a, d(x\beta y)]_\alpha = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Since $d(x\beta y) = d(x)\beta y + x\beta d(y)$ substituting it into the former identity we get $[a, d(x)\beta y + b\beta d(y)]_\alpha = 0$ and this along with the condition $(*)$ gives

$$d(x)\beta[a, y]_\alpha + [a, x]_\alpha \beta d(y) = 0. \tag{1}$$

If $y \in M$ commutes with a then $[a, y]_\alpha = 0$, hence the equation (1) reduces to $[a, x]_\alpha \beta d(y) = 0$ for all $x \in M$, $\alpha, \beta \in \Gamma$. Because of $a \notin Z(M)$ we have $d(y) = 0$. In the other words d vanishes on the centralizer $C_M(a) = \{y \in M \,|\, [a, y]_\alpha = 0,\ \alpha \in \Gamma\}$ of a in M. But for any $x \in M$, $d(x) \in C_M(a)$ by the hypothesis, thus we get $d^2(x) = 0$.

However, if M is a completely prime Γ-ring of char M is not 2 and d is a derivation of M such that $d^2 = 0$, then $d = 0$. Since we have supposed that $d \neq 0$ therefore if the char M is not 2, by the results above, we are led to the conclusion that $a \in Z(M)$. This settles the situation when the char M is not 2. So, from this point on we assume that M is of char 2 and that $a \notin Z(M)$. In this case the equation (1) becomes

$$[a, x]_\alpha \beta d(y) = d(x)\beta[a, y]_\alpha \tag{2}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Thus if $d(y) = 0$ then from the equation (2) we obtain $d(x)\alpha[a, y]_\beta = 0$ for all $x \in M$. Since M is a completely prime Γ-ring, we must have $[a, y]_\alpha = 0$, that is $y \in C_M(a)$. Combined with the previous result as we have said earlier d vanishes on $C_M(a)$. Now one knows that $C_M(a)$ coincides with $T = \{y \in M \,|\, d(y) = 0\}$. We return to the equation (2) and substitute in it $x\gamma w$ for x, where x, w are arbitrary elements of M and $\gamma \in \Gamma$. Hence we get

$$[a, x]_\alpha \gamma w \beta d(y) = d(x)\gamma w \beta[a, y]_\alpha \tag{3}$$

If $[a, x]_\alpha \neq 0$, using a result of [7] we have $d(x) = c(x)\beta[a, x]_\alpha$, where $c(x)$ is in the extended centroid of M. Moreover, since

$$C_M(a) = \{y \in M \,|\, d(y) = 0\},$$

we must have that $c(x) \neq 0.$ Also, if $[a,x]_\alpha = 0,$ then $d(x) = 0,$ hence $0 = d(x) = 0\beta[a,x]_\alpha$. Thus for all $x \in M$, $d(x) = c(x)\gamma[a,x]_\alpha$ where $c(x)$ is in the extended centroid of *M*.

We claim that $a\beta[a,x]_\alpha - [a,x]_\alpha \beta a = 0$ i.e., $[a,[a,x]_\alpha]_\beta = 0$ for all $x \in M$ and $\alpha, \beta \in \Gamma.$ It is clear that if $[a,x]_\alpha = 0$ then $[a,[a,x]_\alpha]_\beta = 0.$ On the other hand, if $[a,x]_\alpha \neq 0,$ then by the observation above

$$d(x) = c(x)\gamma[a,x]_\alpha \text{ where } c(x) \neq 0,$$

so since $[a,d(x)]_\alpha = 0,$ one has $c(x)\gamma[a,[a,x]_\alpha]_\beta = 0.$ Because $c(x) \neq 0$ is invertible, we end up with $[a,\ [a,x]_\alpha]_\beta = 0$ for all $x \in M$. Writing this out

$$a\beta(x\alpha a + a\alpha x) = (a\alpha x + x\alpha a)\beta a,$$

we conclude that $a\alpha a \in Z(M)$.

Now to the final part of the theorem, we just see that if $a \notin Z(M)$ then $d(x) = c(x)\gamma[a,x]_\alpha$ with $c(x)$ is in the extended centroid for all $x \in M$. We want to prove that $c(x)$ is a constant. Let $x, y \in M$, then $d(x\alpha y) = c(x\alpha y)\gamma[a,\ x\alpha y]_\beta$, that is

$$d(x)\alpha y + x\alpha d(y) = c(x\alpha y)\gamma[a,x]_\beta \delta y + c(x\alpha y)\gamma x\delta[a,y]_\beta,$$

by the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma.$ Because of $d(x) = c(x)\gamma[a,x]_\beta$ and $d(y) = c(y)\gamma[a,y]_\beta$. We get

$$\begin{aligned} & c(x\alpha y)\gamma[a,x]_\beta \delta y + c(x\alpha y)\gamma x\delta[a,y]_\beta \\ & = c(x\alpha y)\gamma[a,x]_\beta \delta y + c(x\alpha y)\delta[a,y]_\beta. \end{aligned} \quad (4)$$

Hence if $p = c(x) + c(x\alpha y)$ and $q = c(y) + c(x\alpha y)$ then we have

$$(c(x) + c(x\alpha y))\delta[a,x]_\beta \gamma y = (c(y) + c(x\alpha y))\delta x\gamma[a,y]_\beta$$

for all $x, y \in M$ and for all $\alpha, \beta, \gamma, \delta \in \Gamma.$

Since $a\alpha a \in Z(M)$ commuting it with a we obtain $[a,[a,x]_{\alpha}]_{\beta} = 0.$

Also due to (4) we have $(p+q)\delta[a,x]_{\beta}\gamma[a,x]_{\beta} = 0,$ for all $x, y \in M$ and $\delta, \gamma \in \Gamma.$ If $[a,x]_{\beta}\gamma[a,y]_{\beta} \neq 0$ then $p+q=0$ i.e.,

$$c(x)+c(x\alpha y)+c(y)+c(x\alpha y)=0$$

and so $c(x)=c(y).$ This shows that c should be a constant on all elements failing to commute with $a.$

Further

$$C_M(a) = \{y \in M \,|\, d(y)=0\},$$

would imply that $d(x)=[c\alpha a, x]_{\beta}$ for $x \in M$ and $\alpha, \beta \in \Gamma$ for some c in the extended centroid. This is the desired result. So to finish, we must show the existence of a such that $w \in M$. In fact, we shall show a little more namely that there is an element $w \in M$ such that

$$[a,x]_{\alpha}\beta[a,w]_{\alpha}\gamma[a,y]_{\alpha} \neq 0.$$

If this is not true then

$$[a,x]_{\alpha}\beta[a,z]_{\alpha}\gamma[a,y]_{\alpha} = 0 \text{ for all } z \in M \text{ and } \alpha,\beta,\gamma \in \Gamma,\ \alpha,\beta,\gamma \in \Gamma,$$

that is, we get

$$[a,x]_{\alpha}\beta[a,z]_{\alpha}\gamma[a,y]_{\alpha} = [a,x]_{\alpha}\beta a\alpha z\gamma[a,y]_{\alpha} - [a,x]_{\alpha}\beta z\alpha a\gamma[a,y]_{\alpha}.$$

because of $[a,x]_{\alpha}\beta a = p\beta[a,x]_{\alpha}$ where p is in the extended centroid of M. Since $a\alpha a = b \in Z(M)$ we get $p\alpha p = b,$ the extended centroid is a field and is of char 2, the element p is uniquely determined by $b,$ hence does not depend on $x.$ But then $[a,x]_{\alpha}\beta(a+q)=0$ for all x such that $[a,x]_{\alpha} \neq 0.$ If $[a,x]_{\alpha}=0,$ this relation is certainly true. So $[a,x]_{\alpha}\beta(a+p)=0,$ for all $x \in M, \alpha, \beta \in \Gamma.$ But then this carries over to all x in central closure T of M which is itself a completely prime Γ-ring. Since $a \notin Z(M)$ and $[a,x]_{\alpha}\beta(a+p)=0,$ for all $x \in T$ and $\alpha, \beta \in \Gamma$ we deduce that $a + p = 0$ and so $a \in Z(M)$. We have a contradiction that completes the proof.

ACKNOWLEDGMENT

The research was supported by grant 05-02-12-2188RU Universiti Putra Malaysia (UPM), Malaysia.

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