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The Group Explicit Methods for Parabolic Problems with Cylindrical and Spherical Symmetry

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ABSTRAK

Strategi tak tersirat kumpulan (TK) yang telah digunakan dengan jayanya oleh Evans dan Abdullah (1983) kini diterapkan pula ke atas masalah parabolik yang melibatkan domain sekata dan mempunyai kesimetrian silinder dan sfera. Hasil pengiraan menunjukkan kaedah TK masih boleh digunakan dan mempunyai ciri-ciri kebaikan seperti sebelumnya. Walau bagaimanapun, julat kestabilannya terhad.

ABSTRACT

The group explicit (GE) strategy which has been used successfully to solve parabolic partial differential equations in regular domains involving 1 and 2 space dimensions by Evans and Abdullah (1983) is now applied to parabolic problems involving regular domains that possess both cylindrical and spherical symmetry. The results indicate that the GE methods are still applicable possessing many of their previous advantages but with a reduced stability range.

INTRODUCTION

Consider the following parabolic equation in one-space dimension

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{\alpha}{r}\frac{\partial U}{\partial r} \qquad (1.1)$$

together with the initial-boundary conditions

$$U(r, 0) = f(r), 0 \le t \le T$$

and

$$\frac{\partial U}{\partial r}(0, t) = 0, U(1, t) = 0, 0 \le t \le T \qquad (1.2)$$

Equation (1.1) reduces to the simple diffusion equation when $\alpha = 0$. Evans and Abdullah (1983) have successfully implemented the GE algorithms for this equation. We shall now extend the GE application to parabolic problems that possess cylindrical and spherical symmetry by putting $\alpha = 1$ and 2 respectively. The Group Explicit with Right Ungrouped Point (GER), the Group Explicit with Left Ungrouped Point (GEL), the Single Alternating Group Explicit (SAGE) and the (Double) Alternating Group Explicit (DAGE) schemes will be developed and the stability requirements as well as accuracy established.

2. DERIVATION OF THE GE SCHEMES

Following Evans and Abdullah (1983), the generalised formulae to approximate the derivatives in (1, 1) at the point $(r_i, t_{j+1/2}) = (i\Delta r, (j+1/2)\Delta t)$ are given by

$$\left(\frac{\partial^2 U}{\partial r^2}\right)_{i,j+1/2} \approx (\theta_1 \delta_r u_{i+1/2,j+1} - \theta_2 \delta_r u_{i-1/2,j+1} + \theta_3 \delta_r u_{i+1/2,j+1} - \theta_4 \delta_r u_{i-1/2,j})/(\Delta r)^2, \qquad (2.1)$$

$$\left(\frac{\partial U}{\partial r}\right)_{i,j+1/2} \approx \left(\alpha_1 \Delta_r u_{i,j+1} + \alpha_2 \nabla_r u_{ij} + \alpha_3 \nabla_r u_{i,j+1} + \alpha_4 \Delta_r u_{ij}\right)/2\Delta r \qquad (2.2)$$

and

$$\left(\frac{\partial U}{\partial t}\right)_{i,j+1/2} \approx \left(u_{i,j+1} + u_{ij}\right)/\Delta t \qquad (2.3)$$

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where Δr and Δt are the increments with respect to the r- and t-axes and Δ_r, ∇_r, and δ_r are the usual forward, backward and central difference operators with respect to r. The finite-difference analogue of (1. 1) is therefore given by

$$u_{i,j+1} = u_{ij} + \lambda(\theta_1\delta_r u_{i+1/2,j+1} - \theta_2\delta_r u_{i-1/2,j+1} + \theta_3\delta_r u_{i+\frac{1}{2},j} - \theta_4\delta_r u_{i-1/2,j}) + \frac{\alpha}{2i}\lambda(\alpha_1\Delta_r u_{i,j+1} + \alpha_2\nabla_r u_{ij} + \alpha_3\nabla_r u_{i,j+1} + \alpha_4\Delta_r u_{ij}) \quad (2.4)$$

where $\lambda = \Delta t/(\Delta r)^2$ the mesh ratio.

If we choose $\theta_1 = \theta_4 = 1$; $\theta_2 = \theta_3 = 0$; $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = 0$, equation (2.4) gives us the following right-left (RL) approximation,

$$(1 + q_i)u_{i,j+1} - q_i u_{i+1,j+1} = (1-p_i)u_{ij} + p_i u_{i-1,j} \quad (2.5)$$

whilst with the choice of $\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = 1$; $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$, we get the following LR analogue

$$(1 + p_i)u_{i,j+1} - p_i u_{i-1,j+1} = (1 - q_i)u_{ij} + q_i u_{i+1,j},\ i = 1, 2, ..., m-1 \quad (2.6)$$

where $p_i = \left(1 - \frac{\alpha}{2i}\right)\lambda$ and $q_i = \left(1 + \frac{\alpha}{2i}\right)\lambda$.

By applying the equation (2.5) at the point $(r_{i-1}, t_{j+1/2})$ we find that $-q_{i-1}u_{i,j+1} + (1+q_{i-1})u_{i-1,j+1} = (1-p_{i-1})u_{i-1,j} + p_{i-1}u_{i-2,j}$

which on coupling with (2.6) forms the system

$$\begin{bmatrix}(1+q_{i+1}) & -q_{i-1}\\ -p_i & (1+p_i)\end{bmatrix}\begin{bmatrix}u_{i-1,j+1}\\ u_{i,j+1}\end{bmatrix} = \begin{bmatrix}(1-p_{i-1}) & 0\\ 0 & (1-q_i)\end{bmatrix}\begin{bmatrix}u_{i-1,j}\\ u_{i,j}\end{bmatrix} + \begin{bmatrix}p_{i-1} & u_{i-2,j}\\ q_i & u_{i+1,j}\end{bmatrix} \quad (2.7)$$

i.e. $$A\underset{\sim}{u}_{j+1} = B\underset{\sim}{u}_j + \underset{\sim}{\hat{u}}_j$$

or $$\underset{\sim}{u}_{j+1} = A^{-1} + A^{-1}\underset{\sim}{\hat{u}}_j \quad (2.8)$$

This leads to the following explicit equations for *general points* not on the axis

$$u_{i-1,j+1} = \left[(1+p_i)p_{i-1}u_{i-2,j} + (1+p_{i-1})u_{i-1,j} + q_{i-1}(1-q_i)u_{ij} + q_{i-1}q_i u_{i+1,j}\right]/(1 + p_i + q_{i-1}) \quad (2.9)$$

and

$$u_{i,j+1} = \left[p_i p_{i-1}u_{i-2,j} + p_i(1-p_{i-1})u_{i-1,j} + (1+q_{i-1})(1-q_i)u_{ij} + (1 + q_{i-1})q_i u_{i+1,j}\right]/(1 + p_i + q_{i-1}) \quad (2.10)$$

From the L' Hospital's rule, the following relationships hold on the axis,

$$\left(\frac{\partial u}{\partial t}\right)_{(0,j+1/2)} = (1+\alpha)\left(\frac{\partial^2 u}{\partial r^2}\right)_{(0,j+1/2} \quad (2.11)$$

Choosing $\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = 1$; $\alpha_3 = \alpha_4 = 1$ and $\alpha_1 = \alpha_2 = 0$ in (2.1), (2,2) and (2,3), the approximations of the derivatives $\frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}$ and $\frac{\partial u}{\partial t}$ at the points $(0, j+1/2)$ on the axis can be obtained.

The substitution of these derivatives into (2.11) leads to the approximations,

$$u_{0,j+1} - u_{0j} = (1+\alpha)\lambda(u_{1,j} - u_{0j} - u_{0,j+1} + u_{-1,j+1}) \quad (2.12)$$

and by utilising the boundary condition (1.2) in which $\frac{\partial u}{\partial r}(0, t) = 0$, we arrive at the following formula for the fictitious values $u_{-1,j+1}$,

$$u_{-1,j+1} = u_{0,j+1} + u_{1,j} - u_{0j} \quad (2.13)$$

By inserting (2.13) into (2.12), we therefore obtain the approximations to the *left boundary values.*

$$u_{0,j+1} = (1-2\hat{\alpha})u_{0j} + 2\hat{\alpha}u_{1j} \quad (2.14)$$

where $\hat{\alpha} = (1 + \alpha)\lambda$. The solutions at the single ungrouped point near the right boundary can be obtained from (2.5) by taking $i = m-1$. Hence, the GER scheme is expressed by the following implicit equations

$$u_{0,j+1} = (1-2\hat{\alpha})u_{0j} + 2\hat{\alpha}u_{1j}$$

$$\left.\begin{array}{l}(1+q_{i-1})u_{i,j+1} - q_{i-1}u_{i,j+1} = p_{i-1}u_{i-2,j} + (1-p_{i-1})u_{i-1,j}\\ -p_i u_{i-1,j+1} + (1+p_i)u_{i,j+1} = (1-q_i)u_{ij} + q_i u_{i+1,j}\end{array}\right\}\ \begin{array}{l}i=2, 4, ..,\\ (m-4), (m-2)\\ m \text{ even and}\\ m > 4\end{array}$$

and

$$u_{m-1,j+1} = \frac{q_{m-1}}{(1+q_{m-1})} u_{m,j+1} + \frac{(1-p_{m-1})}{(1+q_{m-1})} u_{m-1,j} + \frac{p_{m-1}}{(1+q_{m-1})} u_{m-2,j}$$

or in matrix form

$$(I+G_1)\underset{\sim}{u}_{j+1} = (I+G_2)\underset{\sim}{u}_j + \underset{\sim}{b}_1$$

i.e.

$$\underset{\sim}{u}_{j+1} = (I+G_1)^{-1}(I+G_2)\underset{\sim}{u}_j + \hat{\underset{\sim}{b}}_1 \qquad (2.15)$$

where I is the identity matrix,

$$\underset{\sim}{u}_j = \left(u_{0j}, u_{1j}, \ldots, u_{m-1,j}\right)^T, \quad \underset{\sim}{b}_1 = \left(0, 0, \ldots, \frac{q_{m-1}}{(1+q_{m-1})} u_{m,j+1}\right)^T,$$

with

$$\hat{\underset{\sim}{b}}_1 = (I+G_1)^{-1}\underset{\sim}{b}_1$$

(ii) The GEL Scheme

By choosing $\theta_1 = \theta_4 = 1; \theta_2 = \theta_3 = 0$, in (2.1), we have as approximation to (2.1) at r = 0, the formula,

$$(1+\hat{\alpha})u_{0,j+1} - \hat{\alpha}\, u_{1,j+1} = (1-\hat{\alpha})u_{0j} + \hat{\alpha}\, u_{-1,j}$$

$$G_1 = \begin{bmatrix} 0 & & & & & \\ & G_1^{(1)} & & & & \\ & & G_1^{(2)} & & & \\ & & & \ddots & & \\ & & & & G^{((1/2)(m-2))} & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}_{(m \times m)}$$

and

$$G_2 = \begin{bmatrix} \begin{matrix} -2\hat{\alpha} & -2\hat{\alpha} \\ p_1 & -p_1 \end{matrix} & & & & \\ & G_2^{(1)} & & & \\ & & \ddots & & \\ & & & G^{((1/2)(m-4))} & \\ & & & & \begin{matrix} q_{m-2} & q_{m-2} \\ \dfrac{p_{m-1}}{(1+q_{m-1})} & \dfrac{-(p_{m-1}+q_{m-1})}{(1+q_{m-1})} \end{matrix} \end{bmatrix}$$

with

$$G_1^{(i)} = \begin{bmatrix} q_{2i-1} & -q_{2i-1} \\ -p_{2i} & p_{2i} \end{bmatrix}, \ i = 1, 2, \ldots, (1/2)(m-2);$$

$$G_2^{(i)} = \begin{bmatrix} -q_{2i} & q_{2i} \\ p_{2i+1} & -p_{2i+1} \end{bmatrix}, \ i = 1, 2, \ldots, (1/2)(m-4);$$

From (2.6), we also obtain, with i = 1, the relation,

$$-p_1 u_{0,j+1} + (1+p_1)u_{1,j+1} = (1-q_1)u_{1j} + q_1 u_{2j} \qquad (2.17$$

Equations (2.16) and (2.17) form the coupled system,

$$\begin{bmatrix} (1+\hat{\alpha}) & -\hat{\alpha} \\ -p_1 & (1+p_1) \end{bmatrix} \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \end{bmatrix} = \begin{bmatrix} (1+\hat{\alpha}) & 0 \\ 0 & (1-q_1) \end{bmatrix} \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} + \begin{bmatrix} \hat{\alpha} u_{-1,j} \\ q_1 u_{2,j} \end{bmatrix}$$

which leads to

$$u_{0,j+1} = \frac{1}{(1+p_1+\hat{\alpha})} \left\{ (1+p_1)(1-\hat{\alpha}) u_{0j} + (1-q_1) \hat{\alpha} u_{1j} + (1+p_1)\hat{\alpha} u_{-1,j} + \hat{\alpha} q_1 u_{2j} \right\}$$

and

$$u_{1,j+1} = \frac{1}{(1+p_1+\hat{\alpha})} \left\{ p_1(1-\hat{\alpha}) u_{0j} + (1+\hat{\alpha})(1-q_1) u_{1j} + p_1 \hat{\alpha} u_{-1j} + (1+\hat{\alpha}) q_1 u_{2j} \right\}$$

By substituting $\alpha_1 = \alpha = 0$ and $\alpha_2 = \alpha_4 = 1$ in (2.2) and utilising the boundary condition $\left(\frac{\partial U}{\partial r}\right)_{(0,j+1/2)} = 0$, we obtain the following approximation $\left(\frac{\partial U}{\partial r}\right)_{(0,j+1/2)} \approx \left(u_{1,j} - u_{-1,j}\right)/2\Delta r = 0$ and hence $-u_{-1,j} = u_{1,j}$.

With these values, the required equations for the left boundary as well as the single ungrouped point adjacent to it are given respectively by,

$$u_{0,j+1} = \frac{1}{(1+p_1+\hat{\alpha})} \left\{ (1+p_1)(1-\hat{\alpha}) u_{0j} + (2+p_1-q_1)\hat{\alpha} u_{1j} + \hat{\alpha} q_1 u_{2j} \right\} \quad (2.18)$$

and

$$u_{1,j+1} = \frac{1}{(1+p_1+\hat{\alpha})} \left\{ p_1(1-\hat{\alpha}) u_{0j} + (1+\hat{\alpha})(1-q_1) + (p_1\hat{\alpha}) u_{1j} + (1+\hat{\alpha}) q_1 u_{2j} \right\} \quad (2.19)$$

These equations together with the system (2.7) and (2.8) for $i = 3,5,\ldots,(m-1)$ describe the GEL scheme which can be written in matrix form as,

$$(I+\hat{G}_2)\, \underset{\sim 1}{b} = \left(I+\hat{G}_1\right) \underset{\sim j}{u} + \underset{\sim 2}{b}$$

i.e

$$\underset{\sim j+1}{u} = (I+\hat{G}_2)^{-1}(I+\hat{G}_1)\, \underset{\sim j}{u} + \underset{\sim 2}{\hat{b}}$$

where

$$\underset{\sim j}{u} = (u_{0j}, u_{1j}, \ldots, u_{m-1,j})^T, \quad \underset{\sim 2}{b} = \left(0,0,\ldots,0,q_{m-1}u_{mj}\right)^T,$$

$$\underset{\sim 2}{\hat{b}} = \left(I+\hat{G}_2\right)^{-1} \underset{\sim 2}{b}$$

(ii) The (S) AGE and (D) AGE Schemes

The alternating group explicit schemes are formed by the application of the GER and GEL processes in their appropriate sequences. Thus, the following formulae constitute the (S) AGE two-step scheme,

$$\left.\begin{aligned} \left(I+G_1\right) \underset{\sim j+1}{u} &= \left(I+G_2\right) \underset{\sim j}{u} + \underset{\sim 1}{b} \\ \left(I+G_2\right) \underset{\sim j+2}{u} &= \left(I+\hat{G}_1\right) \underset{\sim j+1}{u} + \underset{\sim 2}{b} \end{aligned}\right\} j = 0, 2, 4, \ldots$$

$$\hat{G}_1 = \left[\begin{array}{ccc|c|c|c|c|c} \frac{-(2+p_1)\hat{\alpha}}{(1+p_1+\hat{\alpha})} & \frac{(2+p_1-q_1)\hat{\alpha}}{(1+p_1+\hat{\alpha})} & \frac{\hat{\alpha} q_1}{(1+p_1+\hat{\alpha})} & & & & & \\ \frac{p_1(1-\hat{\alpha})}{(1+p_1+\hat{\alpha})} & \frac{p_1(1-\hat{\alpha})-q_1(1+\hat{\alpha})}{(1+p_1+\hat{\alpha})} & \frac{(1+\hat{\alpha})q_1}{(1+p_1+\hat{\alpha})} & & & & & \\ 0 & p_2 & -p_2 & & & & & \\ \hline & & & -G_1^{(2)} & & & & \\ \hline & & & & -G_1^{(3)} & & & \\ \hline & & & & & & & \\ \hline & & & & & & -G_1^{((1/2)(m-2))} & \\ \hline & & & & & & & -q_{m-1} \\ \hline \end{array}\right]$$

and

$$\hat{G}_2 = \left[\begin{array}{cc|c|c|c|c} 0 & 0 & & & & \\ 0 & 0 & & & & \\ \hline & & -G_2^{(1)} & & & \\ \hline & & & -G_2^{(2)} & & \\ \hline & & & & \ddots & \\ \hline & & & & & \\ \hline & & & & & -G_2^{((1/2)(m-2))} \end{array}\right]_{(mxm)}$$

whilst the (D) AGE four-step scheme is given by

$$\left.\begin{array}{l} \left(I+G_1\right) \underset{\sim}{u}_{j+1} = \left(I+G_2\right) \underset{\sim}{u}_{j} + \underset{\sim}{b}_{j}, \\ \left(I+\hat{G}_2\right) \underset{\sim}{u}_{j+2} = \left(I+\hat{G}_1\right) \underset{\sim}{u}_{j+1} + \underset{\sim}{b}_{j+2}, \\ (I+\hat{G}_2)\ \underset{\sim}{u}_{j+3} = (I+\hat{G}_1)\ \underset{\sim}{u}_{j+2} + \underset{\sim}{b}_{2}, \\ \text{and} \\ \left(I+G_1\right) \underset{\sim}{u}_{j+4} = \left(I+G_2\right) \underset{\sim}{u}_{j+3} + \underset{\sim}{b}_{1} \end{array}\right\} j=0,\ 4,\ 8,\ldots$$

3. TRUNCATION ERROR ANALYSIS

(i) Truncation Error (t. e.) for the GER Scheme

Taylor's expansion of $U_{0,j+1}$, $U_{0,j}$ and $U_{1,j}$ about the point $(r_0, t_{j+1/2})$ provides the t.e. for (2.14)

$$\begin{aligned} T_{LB} &= T_{214} \\ &= 2\hat{\alpha}(\Delta r)(\frac{\partial U}{\partial r})_{0,j+1/2} + (\Delta t)(\frac{\partial U}{\partial t})_{0,j+1/2} - \hat{\alpha}(\Delta r)^2 \\ &(\frac{\partial^2 U}{\partial r^2})_{0,j+1/2} + \hat{\alpha}(\Delta r)(\Delta t)(\frac{\partial^2 U}{\partial r\partial t})_{0,j+1/2} - \frac{1}{3}\hat{\alpha}(\Delta r)^3 \\ &(\frac{\partial^3 U}{\partial r^3})_{0,j+1/2} + \frac{1}{2}\hat{\alpha}(\Delta r)^2(\Delta t)(\frac{\partial^3 U}{\partial r^2\partial t})_{0,j+1/2} - \frac{1}{4}\hat{\alpha} \\ &(\Delta r)(\Delta t)^2 (\frac{\partial^3 U}{\partial r\partial t^2})_{0,j+1/2} + \frac{1}{24}(\Delta t)^3 (\frac{\partial^3 U}{\partial t^3})_{0,j+1/2} \\ &-\frac{\hat{\alpha}}{12}(\Delta r)^4 (\frac{\partial^4 U}{\partial r^4})_{0,j+1/2} + \frac{\hat{\alpha}}{6}(\Delta r)^3(\Delta t) \\ &(\frac{\partial^4 U}{\partial r^3\partial t})_{0,j+1/2} - \frac{\hat{\alpha}}{8}(\Delta r)^2(\Delta t)^2(\frac{\partial^4 U}{\partial r^2\partial t^2})_{0,j+1/2} + \frac{\hat{\alpha}}{24} \\ &(\Delta r)(\Delta t)^3(\frac{\partial^4 U}{\partial r\partial t^3})_{0,j+1/2} + \frac{\hat{\alpha}}{192}(\Delta t)^4(\frac{\partial^4 U}{\partial t^4})_{0,j+1/2} + \\ &0((\Delta r)^{\alpha_1}(\Delta t)^{\alpha_2}),\ \alpha_1+\alpha_2 = 5. \end{aligned} \tag{3.1}$$

Similar expansion of (2.4) and (2.10) about the points (ri–1, tj+1/2 and (ri, tj+1/2) lead to the following t.e at the general grouped points,

$$\begin{aligned} &T_{2.9} \\ &=\{(1+p_i)p_{i-1} + (1+q_i)q_{i-1}\}(\Delta r)(\frac{\partial U}{\partial r})_{i-1,j+1/2} + \\ &(1+p_i+q_{i-1})(\Delta t)(\frac{\partial U}{\partial t})_{i-1,j+1/2} \\ &-\frac{1}{2}\{(1+p_i)p_{i-1} + (1+3q_i)\}q_{i-1}(\Delta r)^2(\frac{\partial^2 U}{\partial r^2})_{i-1,j+1/2} \\ &-\frac{1}{2}\{(1+p_i)p_{i-1} - (1+q_i)q_{i-1}\}(\Delta r)(\Delta t)(\frac{\partial^2 U}{\partial r\partial t})_{i-1,j+1/2} \\ &+\frac{1}{6}\{(1+p_i)p_{i-1} - (1+7q_i)q_{i-1}\}(\Delta r)^3(\frac{\partial^3 U}{\partial r^3})_{i-1,j+1/2} \\ &+\frac{1}{4}\{(1+p_i)p_{i-1} + (1+3q_i)q_{i-1}\}(\Delta r)^2(\Delta t)(\frac{\partial^3 U}{\partial r^2\partial t})_{i-1,j+1/2} \\ &+\frac{1}{8}\{(1+p_i)p_{i-1} - (1+q_i)q_{i-1}\}(\Delta r)(\Delta t)^2(\frac{\partial^3 U}{\partial r\partial t^2})_{i-1,j+1/2} \\ &+\frac{1}{24}(1+p_i+q_{i-1})(\Delta t)^3(\frac{\partial^3 U}{\partial t^3})_{i-1,j+1/2} + \frac{1}{24}(1+p_i)p_{i-1} \\ &+\ (1+15q_i)q_{i-1}\}(\Delta r)^4\ (\frac{\partial^4 U}{\partial r^4})_{i-1,j+1/2} \\ &-\frac{1}{12}\{(1+p_i)p_{i-1} - (1+7q_i)q_{i-1}\}(\Delta r^3)(\Delta t) \end{aligned}$$

$$(\Delta r)^4 \ (\frac{\partial^4 U}{\partial r^3 \partial t})_{i-1,j+1/2}$$

$$-\frac{1}{16}\{(1+p_i)+p_{i-1}(1+3q_i)q_{i-1}\}(\Delta r)^2(\Delta t)^2(\frac{\partial^4 U}{\partial r^2 \partial t^2})_{i-1,j+1/2}$$

$$-\frac{1}{48}\{(1+p_i)p_{i-1}-(1+q_i)q_{i-1}\}(\Delta r)(\Delta t)^3(\frac{\partial^4 U}{\partial r \partial t^3})_{i-1,j+1/2}$$

$$+ \ 0(\Delta r)^{\alpha_1}(\Delta t)), \ \alpha_1 + \alpha_2 = 5; \qquad (3.2)$$

and

$$T_{2.10}$$

$$=\{(1+p_i)p_{i-1}-(1+q_{i-1})q_i\}(\Delta r)(\frac{\partial U}{\partial r})_{i,j+1/2}+$$

$$(1+p_i+q_{i-1})(\Delta t)(\frac{\partial U}{\partial t})_{i,j+1/2}$$

$$-\frac{1}{2}\{p_i(1+3q_{i-1})+(1+q_{i-1})q_i\}(\Delta r)^2(\frac{\partial^2 U}{\partial r^2})_{i,j+1/2}$$

$$-\frac{1}{2}\{p_i(1+p_{i-1})-(1+q_{i-1})q_i\}(\Delta r)(\Delta t)(\frac{\partial^2 U}{\partial r \partial t})_{i,j+1/2}$$

$$+\frac{1}{6}\{p_i(1+7p_{i-1})-(1+q_{i-1})q_i\}(\Delta r)^3(\frac{\partial^3 U}{\partial r^3})_{i,j+1/2}$$

$$+\frac{1}{4}\{p_i(1+3p_{i-1})+(1+q_{i-1})q_i\}(\Delta r)^2(\Delta t)(\frac{\partial^3 U}{\partial r^2 \partial t})_{i,j+1/2}$$

$$+\frac{1}{8}\{p_i(1+p_{i-1})-(1+q_{i-1})q_i\}(\Delta r)(\Delta t)^2(\frac{\partial^3 U}{\partial r \partial t^2})_{i,j+1/2}$$

$$+\frac{1}{24}(1+p_i+q_{i-1})(\Delta t)^3(\frac{\partial^3 U}{\partial t^3})_{i,j+1/2}$$

$$-\frac{1}{24}\{p_i(1+15p_{i-1})\}$$

$$+ \ (1+q_{i-1})q_i\}(\Delta r)^4 \ (\frac{\partial^4 U}{\partial r^4})_{i,j+1/2}$$

$$-\frac{1}{12}\{p_i(1+7p_{i-1})-(1+q_{i-1})q_i\}(\Delta r^3)(\Delta t)(\frac{\partial^4 U}{\partial r^3 \partial t})_{i,j+1/2}$$

$$-\frac{1}{16}\{p_i(1+3p_{i-1})+(1+q_{i-1})q_i\}(\Delta r)^2(\Delta t)^2(\frac{\partial^4 U}{\partial r^2 \partial t^2})_{i,j+1/2}$$

$$-\frac{1}{48}\{p_i(1+p_{i-1})-(1+q_{i-1})q_i\}(\Delta r)(\Delta t)^3(\frac{\partial^4 U}{\partial r \partial t^3})_{i,j+1/2}$$

$$+ \ 0((\Delta r)^{\alpha_1}(\Delta t)^{\alpha_2}), \ \alpha_2 + \alpha_2 = 5. \qquad (3.3)$$

Finally, from equation (2.5) with i=m-1, the t.e. of the approximation near to the right boundary is given by

$$T_R$$

$$= \ (p_i-q_i)(\Delta r)(\frac{\partial U}{\partial r})_{m-1,j+1/2}+(\Delta t)(\frac{\partial U}{\partial t})_{m-1,j+1/2}$$

$$-\frac{1}{2}(p_i+q_i)(\Delta r)^2(\frac{\partial^2 U}{\partial r^2})_{m-1,j+1/2}$$

$$-\frac{1}{2}(p_i+q_i)(\Delta r)(\Delta t)(\frac{\partial^2 U}{\partial r \partial t})_{m-1,j+1/2}$$

$$+\frac{1}{6}(p_i-q_i)(\Delta r)^3(\frac{\partial^3 U}{\partial r^3})_{m-1,j+1/2}$$

$$+\frac{1}{6}(p_i-q_i)(\Delta r)^3(\frac{\partial^3 U}{\partial r^3})_{m-1,j+1/2}$$

$$+\frac{1}{4}(p_i-q_i)(\Delta r)^2(\Delta t)(\frac{\partial^3 U}{\partial r^2 \partial t})_{m-1,j+1/2}$$

$$+\frac{1}{8}(p_i-q_i)(\Delta r)(\Delta t)^2(\frac{\partial^3 U}{\partial r \partial t^2})_{m-1,j+1/2}$$

$$+\frac{1}{24}(\Delta t)^3(\frac{\partial^3 U}{\partial t^3})_{m-1,j+1}$$

$$-\frac{1}{24}(p_i+q_i)(\Delta r)^4(\frac{\partial^4 U}{\partial r^4})_{m-1,j+1/2}$$

$$-\frac{1}{12}(p_i+q_i)(\Delta r)^3(\Delta t)(\frac{\partial^4 U}{\partial r^3 \partial t})_{m-1,j+1/2}$$

$$-\frac{1}{8}(p_i+q_i)(\Delta r)^2(\Delta t)^2(\frac{\partial^4 U}{\partial r^2 \partial t^2})_{m-1,j+1/2}$$

$$-\frac{1}{48}(p_i+q_i)(\Delta r)(\Delta t)^3(\frac{\partial^4 U}{\partial r \partial t^3})+0((\Delta r)^{\alpha_1}(\Delta t)^{\alpha_2}),$$

$$\text{where } \alpha_1 + \alpha_2 = 5. \qquad (3.4)$$

(ii) Truncation Error for the GEL Scheme

The t.e. at the left boundary and at the ungrouped point near to it can be derived by expanding equations (2.18) and (2.19) about the points (r_0, $t_{j+1/2}$) and (r_1, $t_{j+1/2}$),

$$T_{LB} = T_{2.18}$$

$$= \ (1+p_1+\hat{\alpha})(\Delta t)(\frac{\partial U}{\partial r})_{0,j+1/2}$$

$$-(2+p_1+q_1)\hat{\alpha}(\Delta r)(\frac{\partial U}{\partial r})_{0,j+1/2}$$

$$-\frac{1}{2}(2+p_1+3q_1)\hat{\alpha}(\Delta r)^2(\frac{\partial^2 U}{\partial r^2})_{0,j+1/2}$$

$$+\frac{1}{2}(2+p_1+q_1)\hat{\alpha}(\Delta r)(\Delta t)(\frac{\partial^2 U}{\partial r \partial t})_{0,j+1/2}$$

$$-\frac{1}{8}\{(1+p_1)(1-\hat{\alpha})+\hat{\alpha}q_1\}(\Delta t)^2(\frac{\partial^2 U}{\partial r^2})_{0,j+1/2}$$

$$-\frac{1}{6}(2+p_1+7q_1)(\Delta r)^3(\frac{\partial^3 U}{\partial r^3})_{0,j+1/2}$$

$$+\frac{1}{4}(2+p_1+3q_1)(\Delta r)^2(\Delta)(\frac{\partial^3 U}{\partial r^3})_{0,j+1/2}$$

$$-\frac{1}{8}(2+p_1+q_1)\hat{\alpha}(\Delta r)^2(\Delta t)(\frac{\partial^3 U}{\partial r \partial t^2})_{0,j+1/2}$$

$$+\frac{1}{24}(1+p_1+\hat{\alpha})(\Delta t)^3(\frac{\partial^3 U}{\partial r^3})_{0,j+1/2}$$

$$-\frac{1}{24}(2+p_1+q_1)\hat{\alpha}(\Delta r)^4(\frac{\partial^4 U}{\partial r^4})_{0,j+1/2}$$
$$+\frac{1}{12}(2+p_1+7q_1)\hat{\alpha}(\Delta r)^3(\Delta t)(\frac{\partial^4 U}{\partial r^3 \partial t})_{0,j+1/2}$$
$$-\frac{1}{16}(2+p_i+3q_i)\hat{\alpha}(\Delta r)^2(\Delta t)^2(\frac{\partial^4 U}{\partial r^2 \partial t^2})_{0,j+1/2}$$
$$+\frac{1}{48}(2+p_1+q_1)(\Delta r)\hat{\alpha}(\Delta t)^3(\frac{\partial^4 U}{\partial r \partial t^3})_{0,j+1/2}$$
$$-\frac{1}{384}(2+p_1)\hat{\alpha}(\Delta t)^4(\frac{\partial^4 U}{\partial t^4})_{0,j+1/2}$$
$$+\ 0((\Delta r)^{\alpha_1}(\Delta t)^{\alpha_2}) \text{ and } \alpha_1+\alpha_2 = 5; \qquad (3.5)$$

and

$$T_L = T_{2.19}$$
$$=\{p_1(1-\hat{\alpha})-q_1(1+\hat{\alpha})\}(\Delta r)(\frac{\partial U}{\partial r})_{1,j+1/2}+(1+p_1+\hat{\alpha})$$
$$(\Delta t)(\frac{\partial U}{\partial t})1,j+1/2$$
$$-\frac{1}{2}\{p_1(1-\hat{\alpha})+(1+\hat{\alpha})q_1\}(\Delta r)^2(\frac{\partial^2 U}{\partial r^2})_{1,j+1/2}$$
$$-\frac{1}{2}\{p_1(1-\hat{\alpha})-(1+\hat{\alpha})q_1\}(\Delta r)(\Delta t)(\frac{\partial^2 U}{\partial r \partial t})_{1,j+1/2}$$
$$+\frac{1}{6}\{p_1(1-\hat{\alpha})-(1+\hat{\alpha})q_1\}(\Delta r)^3(\frac{\partial^3 U}{\partial r^3})_{1,j+1/2}$$
$$+\frac{1}{4}\{p_1(1-\hat{\alpha})+(1+\hat{\alpha})q_1\}(\Delta r)^2(\Delta t)(\frac{\partial^3 U}{\partial r^2 \partial t})_{1,j+1/2}$$
$$+\frac{1}{8}\{p_1(1-\hat{\alpha})-(1+\hat{\alpha})q_1\}(\Delta r)(\Delta t)^2(\frac{\partial^3 U}{\partial r \partial t^2})_{1,j+1/2}$$
$$+\frac{1}{24}(1+p_1+\hat{\alpha})(\Delta t)^3(\frac{\partial^3 U}{\partial t^3})_{1,j+1/2}$$
$$-\frac{1}{24}\{p_1(1-\hat{\alpha})+(1+\hat{\alpha})q_1\}(\Delta r)^4)(\frac{\partial^4 U}{\partial r^4})_{1,j+1/2}$$
$$-\frac{1}{12}\{p_1(1-\hat{\alpha})-(1+\hat{\alpha})q_1\}(\Delta r)^3(\Delta t)(\frac{\partial^4 U}{\partial r^3 \partial t})_{1,j+1/2}$$
$$-\frac{1}{16}\{p_1(1-\hat{\alpha})+(1+\hat{\alpha})q_1\}(\Delta r)^2(\Delta t)^2(\frac{\partial^4 U}{\partial r^2 \partial t^2})_{1,j+1/2}$$
$$-\frac{1}{48}\{p_1(1+\hat{\alpha})(1+\hat{\alpha})q_1\}(\Delta r)(\Delta t)^3(\frac{\partial^4 U}{\partial r \partial t^3})_{1,j+1/2}$$
$$+\ 0((\Delta r)^{\alpha_1}(\Delta t)^{\alpha_2}) \text{ and } \alpha_1+\alpha_2=5. \qquad (3.6)$$

The t.e. of the scheme at the remaining points grouped two at a time are given by $T_{2.9}$ and $T_{2.10}$ respectively for i = 3, 5, ... (m–1)

(iii) Truncation Error for the (S) AGE and (D) AGE Schemes

The t.e. of these alternating group explicit methods are given by the t.e. of the GER and the GEL schemes when they are applied in their correct sequence.

4. STABILITY ANALYSIS

We shall show here the analysis of stability of the GER scheme for the case $\alpha = 2$ (spherical symmetry) only. From (2.15), the explicit form of the GER scheme is

$$\underset{\sim}{u}_{j+1} = r_{GER}\ \underset{\sim}{u}_j + \underset{\sim}{\hat{b}}_1$$

where $r_{GER} = ((I+G_1)^{-1}(I+G_2)$, the amplification matrix which is given by

$$r_{GER} = \begin{bmatrix} (1-2\hat{\alpha}) & 2\hat{\alpha} & & & & & \\ p_{12} & q_{12} & r_{12} & s_{12} & & & \\ \hat{s}_{12} & \hat{r}_{12} & \hat{q}_{12} & \hat{p}_{12} & & & \\ 0 & 0 & p_{3,4} & q_{3,4} & r_{3,4} & s_{3,4} & \\ & & \hat{s}_{3,4} & \hat{r}_{3,4} & \hat{q}_{3,4} & \hat{p}_{3,4} & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & p_{m-3,m-2} & q_{m-3,m-2} & r_{m-3,m-2} & s_{m-3,m-2} & \\ & & \hat{s}_{m-3,m-2} & \hat{r}_{m-3,m-2} & \hat{q}_{m-3,m-2} & \hat{p}_{m-3,m-2} & \\ & & & p_{m-1,m} & q_{m-1,m} & & \end{bmatrix}_{(m \times m)}$$

m even, m > 4

where (with $V_{ij} = 1 + p_j + q_i$ and $p_m = 0$)

$$p_{ij} = (1+p_j)/v_{ij},\ q_{ij} \text{ and } p_m = 0)$$
$$= q_i(1-q_j)/v_{ij},$$
$$S_{ij} = q_j q_i / v_{ij},\ \hat{p}_{ij} = (1+q_i)q_j/v_{ij},\ \hat{q}_{ij} = (1+q_i)(1-q_j)/v_{ij},$$
$$\hat{r}_{ij} = p_j(1-p_i)/v_{ij},\ \hat{s}_{ij} = p_j p_i / v_{ij}$$

We have $\alpha = 3\lambda$, $p_1 = \frac{\lambda}{2}$, $q_1 = 2\lambda$ and $q_2 = \frac{3\lambda}{2}$. The characteristic equation of the matrix r_{GER} is $|D| = \det(r_{GER} = \mu I) = 0$ where μ are the eigen values of r_{GER}. Now, if the expand the determinant by the first column, we get $\mu = (1-6\lambda)$ as an eigen values of r_{GER}. For stability, we require $|\mu| \leq 1$ which leads to the conditions

$$\lambda \leq \frac{1}{3} \tag{4.1}$$

For the second and subsequent even rows of r_{GER}, we have as the sum of the moduli of the elements of the row

$$s_{i+1} = \left|\frac{(1+p_{i+1})\|p_i}{1+p_{i+1}+q_i}\right| + \frac{|(1+p_{i+1})\|1-p_i|}{|1+p_{i+1}+q_i|} + \frac{|q_i\|1-q_{i+1}|}{|1+p_{i+1}+q_i|}$$
$$= +\frac{|q_{i+1}\|q_i|}{1+p_{i+1}+q_i}$$
$$i = 1,\ 3, \ldots, m-3),\ m > 4$$
$$= \frac{(1+p_{i+1})p_i + (1+p_{i+1})|1-p| + q_i\|1-q_{i+1}| + q_{i+1}q_i}{(1+p_{i+1}+q_i)}$$

Since p_i and q_j are non-negative

For i =1 we obtain

$$S_2 = \frac{1+p_2+q_1|1-q_2|+q_2q_1}{(1+p_2+q_1)}$$

If $q_2 \leq 1$ then $\frac{3}{2}\lambda \leq 1$

$$\text{or } \lambda \leq \frac{2}{3}. \tag{4.2}$$

For these values of λ,

$$s_1 = \frac{1+p_2+q_1-q_1q_2 + q_2q_1}{(1+p_2+q_1)}$$
$$= 1.$$

For $i = 3, \ldots, (m-3)$, we find that

if $p_i \leq 1$ then $\lambda \leq \frac{i}{(i-1)}$

and if $q_{i+1} \leq 1$ then $\lambda\ \frac{i+1}{i+2}$

Now for $\lambda \leq \min.\left(\frac{i}{(i-1)}, \frac{i+1}{i+2}\right)$

$$S_{i+1} = \frac{(1+p_{i+1})p_i + (1+p_{i+1})(1-p_i) + q_i(1-q_{i+1}) + q_{i+1}q_i}{(1+p_{i+1}+q_i)}$$
$$= \frac{p_i + p_ip_{i+1} + 1 - p_i + p_{i+1} - p_ip_{i+1} + q_i - q_{i+1}q_i + q_{i+1}q_i}{(1+p_{i+1}+q_i)}$$
$$= (1+p_{i+1} + q_i)(1+p_{i+1}+q_i),$$
$$= 1.$$

For the third row, the sum of the moduli of the elements of the rows is

$$S_3 = \frac{p_2 + (1+q_1)|1-q_2| + (1+q_1)q_2}{(1+p_2+q_1)}$$

Again, if $\lambda \leq \frac{2}{3}$ as in (4.2) we have

$$s_3 = 1$$

For the subsequent odd rows pf r_{GER} we get

$$S_{i+2} = \frac{p_{i+1}p_i + p_{i+1}|1-p_i| + (1+q_i)|1-q_{i+1}| + (1+q_i)q_{i+1}}{(1+p_{i+1}+q_i)},$$
$$i = 3, \ldots, (m-3) \text{ with } m > 4$$

Again using (4.3), if $\lambda\ \frac{(i+1)}{(i+2)}$ then

$$s_{i+2} = 1$$

An application of Brauer's the theorem to the last row leads to

$$\left|\mu - \frac{(1-p_{m-1})}{(1+q_{m-1})}\right| \leq \frac{p_{m-1}}{(1+q_{m-1})},$$

i. e.,

$$\frac{1-2p_{m-1}}{(1+q_{m-1})} \leq \mu \leq \frac{1}{(1+q_{m-1})}.$$

Let $\mu_1 = \frac{1-2p_{m-1}}{(1+q_{m-1})}$ and $\mu_2 = \frac{1}{(1+q_{m-1})}$.

The requirement $|\mu_1| \leq 1$ implies $-1 \leq \frac{1-2p_{m-1}}{(1+)} \leq 1$,

$$\text{or } -1-q_{m-1} \leq 1-2p_{m-1} \leq 1+q_{m-1}.$$

The left - hand side inequality gives

$$2\frac{(m-2)}{(m-1)}\lambda - \frac{m}{(m-1)}\lambda \le 2$$

or

$$\lambda \le \frac{(2m-2)}{(m-4)}.$$

The second requirement

$|\mu_2| \le 1$ implies $\frac{1}{(1+q_{m-1})} \le 1$ which is true for all $\lambda > 0$.

Therefore, for overall stability, using (4.1), (4.2),

$$\lambda \le \min.\left\{\frac{1}{3},\ \frac{2}{3},\ \min\left(\frac{i+1}{i+2}, l=3,\ \ldots,\ (m-3)\right), \frac{2m-2}{m-4}\right\}$$

for $m > 4$

$$= \frac{1}{3}.$$

Hence the GER scheme for the spherically symmetric parabolic problem is conditionally stable for

$$\lambda \le \frac{1}{3}.$$

Results of the stability restrictions of the other schemes in the GE class can be obtained by applying the same procedure as the above and are shown to be

Method	$\alpha = 1$	$\alpha = 2$
GER	$\lambda \le 0.454$	–
GEL	–	$\lambda \le \frac{5}{6}$
S(AGE)	$\lambda \le 0.85587$	$\lambda \le \frac{2}{3}$
D(AGE)	$\lambda \le 1.0006$	$\lambda \le \frac{2}{3}$

5. NUMERICAL EXAMPLES

Experiment 1

This experiment dealt with the solution of the following parabolic problems with cylindrical symmetry (Mitchell and Pearce (1963)

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2},\ (0 \le r \le 1), \tag{5.1}$$

given the auxiliary condition

$$u(r, 0) = J_0(\beta r),\ 0 \le r \le 1,$$

$$\frac{\partial u}{\partial r}(0, t) = 0,\ t > 0,$$

and $u(1, t) = 0,\ t > 0,$

where $J_0(\beta r)$ is *the Bessel function* of the first kind of order 0 and β is the first root of $J_0(\beta) = 0$. The exact solution is

$$u(r, t) = J_0(\beta r)e^{-\beta^2 t} \tag{5.2}$$

and the values of the Bessel function at the grid point are generated using the NAG library subroutine (the first four roots of $J_0(\beta) = 0$ are $\beta_1 = 2.405$, $\beta_2 = 5.520$, $\beta_3 = 8.654$, $\beta_4 = 11.79$). In Tables 1-3 are displayed a comparison of the numerical solutions of the GE schemes with the exact solutions at the appropriate grid points in terms of their absolute errors for various values of the mesh ratio λ. The absolute errors of the solutions of the explicit (EXP) scheme and the Crank-Nicolson (CN) method are also included.

Experiment 2

The following parabolic problem with spherical symmetry (Saulev 1964) is considered

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + k(r, t), \tag{5.3}$$

$$(k(r, t) = e^{-t}\left\{[6 + (1-r^2)\pi^2 t^2 - (1-r^2)]\cos(\pi r t) - [(1-r^2)r + 4rt - 2t(1-r^2)/r]\,\pi\sin(\pi r t)\right\}]$$

subject to the initial-boundary conditions

$$U(r, o) = 1 - r^2,$$

$$\frac{\partial U}{\partial r}(0, t) = 0,$$

and $U(1, t) = 0,$

with the exact solution

$$U(r, t) = (1-r^2)e^{-t}\cos(\pi r t). \tag{5.4}$$

Since our parabolic equation incorporates a source term $k(r,t)$, some modifications on the basic equations governing the GE schemes are therefore required.

The numerical solutions of the above spherical problem using the GE schemes are obtained for various values of λ and to indicate their accuracy, Tables 4-6 provide a comparison with the exact solution in terms of their absolute errors.

It is observed that, presumably due to the term $\frac{\alpha}{r}\frac{\partial U}{\partial r}$ in (5.1) and (5.3) (with $\alpha = 1$ and 2 respectively), the solutions of the GE schemes are slightly less accurate in the vicinity of the axis ($r = 0$) than in the remainder of the field. As we have

TABLE 1

The absolute errors of the numerical solutions to the cylindrical problem t = 0.175, λ = 0.175, Δt = 0.00175, Δr = 0.1

Method \ r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GER	2.15×10^{-3}	2.16×10^{-3}	2.0×10^{-3}	2.01×10^{-3}	1.66×10^{-3}	1.65×10^{-3}	1.17×10^{-3}	1.15×10^{-3}	6.04×10^{-4}	5.67×10^{-4}	1.51×10^{-3}
GEL	1.9×10^{-3}	1.88×10^{-3}	1.89×10^{-3}	1.63×10^{-3}	1.62×10^{-3}	1.19×10^{-3}	1.17×10^{-3}	6.34×10^{-4}	6.07×10^{-4}	6.32×10^{-5}	1.26×10^{-3}
(S)AGE	1.43×10^{-3}	1.42×10^{-3}	1.4×10^{-3}	1.27×10^{-3}	1.21×10^{-3}	9.96×10^{-4}	9.07×10^{-4}	6.34×10^{-4}	5.24×10^{-4}	2.32×10^{-4}	1.0×10^{-3}
(D)AGE	1.47×10^{-3}	1.49×10^{-3}	1.42×10^{-3}	1.36×10^{-3}	1.2×10^{-3}	1.09×10^{-3}	8.66×10^{-4}	7.23×10^{-4}	4.6×10^{-4}	2.99×10^{-4}	1.04×10^{-3}
EXP	3.19×10^{-3}	2.73×10^{-4}	2.54×10^{-4}	2.37×10^{-4}	2.19×10^{-4}	1.99×10^{-4}	1.75×10^{-4}	1.47×10^{-4}	1.15×10^{-4}	7.64×10^{-5}	2.01×10^{-4}
C-N	2.06×10^{-3}	2.04×10^{-3}	1.96×10^{-3}	1.83×10^{-3}	1.64×10^{-3}	1.42×10^{-3}	1.17×10^{-3}	8.95×10^{-4}	6.07×10^{-4}	3.16×10^{-4}	1.39×10^{-3}
EXACT SOLUTION	0.3634169	0.3581809	0.3426988	0.3176383	0.2840764	0.2434480	0.1974773	0.1480959	0.0973516	0.0473117	-

TABLE 2
The absolute errors of the numerical solutions to the cylindrical problem $t = 0.6$, $\lambda = 0.3$, $\Delta t = 0.003$, $\Delta r = 0.1$

r / Method	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GER	7.07×10^{-4}	7.02×10^{-4}	6.62×10^{-4}	6.35×10^{-4}	5.47×10^{-4}	5.03×10^{-4}	3.82×10^{-4}	3.26×10^{-4}	1.9×10^{-4}	1.32×10^{-4}	4.79×10^{-4}
GEL	5.29×10^{-4}	5.22×10^{-4}	5.12×10^{-4}	4.56×10^{-4}	4.31×10^{-4}	3.38×10^{-4}	3.04×10^{-4}	1.91×10^{-4}	1.53×10^{-4}	3.84×10^{-5}	3.47×10^{-4}
(S)AGE	1.07×10^{-4}	1.04×10^{-4}	1.09×10^{-4}	9.18×10^{-5}	9.93×10^{-5}	7.15×10^{-5}	8.06×10^{-4}	4.57×10^{-5}	5.42×10^{-5}	1.72×10^{-5}	7.8×10^{-5}
(D)AGE	2.2×10^{-4}	2.27×10^{-4}	2.12×10^{-4}	2.08×10^{-4}	1.76×10^{-4}	1.68×10^{-4}	1.23×10^{-4}	1.12×10^{-4}	6.06×10^{-5}	4.88×10^{-5}	1.56×10^{-4}
EXP	2.89×10^{-4}	2.91×10^{-4}	2.8×10^{-4}	2.6×10^{-4}	2.32×10^{-4}	1.98×10^{-5}	1.6×10^{-4}	1.19×10^{-4}	$7.65\text{X}10^{-5}$	3.54×10^{-5}	1.94×10^{-4}
C-N	6.2×10^{-4}	6.12×10^{-4}	5.86×10^{-4}	5.44×10^{-4}	4.88×10^{-4}	4.19×10^{-4}	3.42×10^{-4}	2.58×10^{-4}	1.71×10^{-4}	8.49×10^{-5}	4.12×10^{-4}
EXACT SOLUTION	0.0311041	0.030656	0.023309	0.0271860	0.0243135	0.0208362	0.0169017	0.0126752	0.0083321	0.0040493	-

TABLE 3

The absolute errors of the numerical solutions to the cylindrical problem $t = 0.6$, $\lambda = 0.6$, $\Delta t = 6.003$, $\Delta r = 0.1$

Method \ r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
(S)AGE	1.39×10^{-3}	1.39×10^{-3}	1.3×10^{-3}	1.25×10^{-3}	1.05×10^{-3}	9.58×10^{-4}	6.93×10^{-4}	5.82×10^{-4}	2.93×10^{-4}	1.84×10^{-4}	9.09×10^{-4}
(D)AGE	2.88×10^{-4}	2.72×10^{-4}	2.86×10^{-4}	2.18×10^{-4}	2.5×10^{-4}	1.43×10^{-4}	1.87×10^{-4}	6.33×10^{-5}	1.2×10^{-4}	2.35×10^{-5}	1.85×10^{-4}
C-N	9.75×10^{-4}	9.63×10^{-4}	9.23×10^{-4}	8.57×10^{-4}	7.69×10^{-4}	6.62×10^{-4}	5.4×10^{-4}	4.08×10^{-4}	2.71×10^{-4}	1.35×10^{-4}	6.50×10^{-4}
EXACT SOLUTION	0.0311041	0.030656	0.0293309	0.0271860	0.0243135	0.0208362	0.0169017	0.0126752	0.0083321	0.0040493	-

TABLE 4

The absolute errors of the numerical solutions to the cylindrical problem $t = 0.175$, $\lambda = 0.175$, $\Delta t = 0.00175$, $\Delta r = 0.1$

Method \ r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GER	7.86×10^{-4}	7.88×10^{-4}	7.91×10^{-4}	7.64×10^{-4}	7.5×10^{-4}	7.23×10^{-4}	6.41×10^{-4}	6.45×10^{-4}	4.15×10^{-4}	9.9×10^{-4}	6.79×10^{-4}
GEL	6.23×10^{-4}	6.27×10^{-4}	6.01×10^{-4}	5.89×10^{-4}	5.43×10^{-4}	4.84×10^{-4}	4.45×10^{-4}	2.75×10^{-4}	2.82×10^{-4}	9.27×10^{-5}	4.56×10^{-4}
(S)AGE	5.53×10^{-4}	5.57×10^{-4}	5.37×10^{-4}	5.15×10^{-4}	4.78×10^{-4}	4.3×10^{-4}	3.91×10^{-4}	3.0×10^{-4}	2.77×10^{-4}	1.16×10^{-4}	4.16×10^{-4}
(D)AGE	5.73×10^{-4}	5.69×10^{-4}	5.56×10^{-4}	5.29×10^{-4}	4.94×10^{-4}	4.52×10^{-4}	3.88×10^{-4}	3.42×10^{-4}	2.31×10^{-4}	1.85×10^{-4}	4.32×10^{-4}
EXACT SOLUTION	0.839457	0.828068	0.8010120	0. 07535391	0.6881618	0.6059549	0.5082867	0.3968087	0.2734428	0.1403673	-

TABLE 5
The absolute errors of the numerical solutions to the cylindrical problem $t = 0.6$, $\lambda = 0.3$, $\Delta t = 0.003$, $\Delta r = 0.1$

r / Method	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GER	1.04×10^{-2}	1.03×10^{-2}	9.88×10^{-3}	8.97×10^{-3}	7.99×10^{-3}	6.66×10^{-3}	5.24×10^{-3}	3.91×10^{-3}	2.28×10^{-3}	1.39×10^{-3}	6.71×10^{-3}
GEL	9.85×10^{-3}	9.77×10^{-3}	9.19×10^{-3}	8.48×10^{-3}	7.32×10^{-3}	6.11×10^{-3}	4.74×10^{-3}	3.19×10^{-3}	2.06×10^{-4}	5.12×10^{-4}	6.12×10^{-3}
(S)AGE	9.71×10^{-3}	9.64×10^{-3}	9.04×10^{-3}	8.23×10^{-3}	7.06×10^{-3}	5.79×10^{-3}	4.44×10^{-3}	3.0×10^{-3}	1.92×10^{-3}	7.11×10^{-4}	5.96×10^{-3}
(D)AGE	9.87×10^{-3}	9.7×10^{-3}	9.2×10^{-3}	8.3×10^{-3}	7.23×10^{-3}	5.92×10^{-3}	4.51×10^{-3}	3.2×10^{-3}	1.8×10^{-4}	9.28×10^{-4}	6.07×10^{-3}
EXACT SOLUTION	0.5488116	0.5336998	0.4898613	0.4216731	0.3360558	0.2419376	0.1495505	0.0696068	0.0124057	-0.013068	-

TABLE 6

The absolute errors of the numerical solutions to the cylindrical problem $t = 0.6$, $\lambda = 0.6$, $\Delta t = 0.006$, $\Delta r = 0.1$

Method \ r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
(S)AGE	7.91×10^{-3}	8.45×10^{-3}	7.55×10^{-3}	6.74×10^{-3}	5.28×10^{-3}	4.01×10^{-3}	2.79×10^{-3}	1.37×10^{-3}	1.17×10^{-3}	1.35×10^{-5}	4.53×10^{-3}
(D) AGE	9.01×10^{-3}	9.16×10^{-3}	8.8×10^{-3}	7.64×10^{-3}	6.69×10^{-3}	5.24×10^{-3}	3.82×10^{-3}	2.73×10^{-3}	9.33×10^{-4}	1.12×10^{-3}	5.52×10^{-3}
EXACT SOLUTION	0.5488116	0.5336998	0.4898613	0.4216731	0.3360558	0.2419376	0.1495505	0.0696068	0.0124057	-0.013068	-

already seen, special equations have to be formulated to cope with this difficulty at the point of singularity. In fact, an examination of the truncation errors of the GE schemes at r = 0 indicates the presence of the term $\frac{\Delta t}{\Delta r}$(equations (3.1) and (3.5) and it is therefore essential that to attain consistency, Δt approaches 0 faster than does Δr. For our cylindrical problem, it is also found that the GE class of methods is more accurate than the other schemes under investigation. From Table 2, the (S) AGE and (D) AGE methods in that order are more superior whilst in Table 3, (D) AGE has the edge on other different formulae. This is to be expected since the truncation error expressions of the constituent GER and GEL formulae possess terms of different signs and hence the correct alternate applications of these formulae to constitute the (S) AGE and (D) AGE schemes can lead to cancellations of error terms. The same observation also applies for our spherical problem although comparative results from other schemes are not available. However it suffices to say that among the GE class of methods, the (S) AGE method gives a more satisfactory result.

We conclude that despite the limited stability of the GE schemes, their stability ratios are not so restricted as to be impractical for implementation for special geometries when compared with other schemes. Being explicit, they are simple and incur low computational load and above all exhibit better accuracy. The use of the alternating schemes in particular, i.e. (S) AGE and (D) AGE, is highly recommended.

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