On the Higher Order Edge-Connectivity of Complete Multipartite Graphs

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ABSTRACT

Let G be a connected graph with p ≥ 2 vertices. For k = 1, 2, ..., p - 1, the kth order edge-connectivity of G, denoted by \( \lambda^{(k)}(G) \), is defined as the minimum number of edges whose removal increases the number of components of G by k. As a consequence, we give a necessary and sufficient condition for the graph \( G_n \) to be factored into spanning trees.

1. INTRODUCTION

Let G be a connected simple graph of order p and size q. Denote by V(G) and E(G) the vertex set and edge set of G respectively. The edge-connectivity \( \lambda(G) \) of G is defined to be the smallest number of edges whose removal from G results in a disconnected or trivial graph. This notion has a natural generalization. Following Goldsmith et al. (1980), for each k = 0, 1, ..., p - 1, the kth order edge-connectivity of G, denoted by \( \lambda^{(k)}(G) \), is defined as the minimum number of edges of G whose removal increases the number of components of G by k. Note that \( \lambda^{(0)}(G) = 0 \), \( \lambda^{(1)}(G) = \lambda(G) \) and \( \lambda^{(p-1)}(G) = q \). The properties of \( \lambda^{(k)}(G) \) were studied previously in Boesch and Chen (1978), Goldsmith (1980 and 1981), Goldsmith et al. (1980) and Sampathkumar (1984). It is easy to see that for any tree T, \( \lambda^{(k)}(T) = k \). Furthermore, since any connected graph G contains a spanning tree, \( \lambda^{(k)}(G) \geq k \). It was proved in Peng et al. (1988) that \( \lambda^{(k)}(K_n) = \frac{1}{2} k(2p - k - 1) \) for each k = 0, 1, ..., p - 1. In this note we shall determine the kth order edge-connectivity of a complete n-partite graph and then use the result to derive a necessary and sufficient condition for a complete n-partite graph to be factored into spanning trees.

Throughout this article, we write \( G_n = K_n(m_1, m_2, ..., m_n) \), n ≥ 2, to denote a complete n-partite graph with n partite sets \( V_1, V_2, ..., V_n \) such that \( |V_i| = m_i \geq 1 \) for each i = 1, 2, ..., n. For the sake of convenience, we always assume \( m_1 \leq m_2 \leq ... \leq m_n \).

A graph G is called a complete multipartite graph if \( G \cong G_n \) for some integer n ≥ 2.

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For those graph-theoretic terms used but not defined here we refer to Behzad et al. (1979).

2. EFFICIENT SEPARATION

Let $G$ be a connected graph of order $p$, and $k$ be an integer such that $1 \leq k \leq p-1$. Following Goldsmith et al. (1980) again, by an efficient $k$-separation of $G$, we mean a removal of $\lambda^{(k)}(G)$ edges from $G$ so that $G$ is separated into $k+1$ components. Call a component of a graph trivial if it is a singleton, and non-trivial otherwise.

It was pointed out in Peng et al. (1988) that every efficient $k$-separation of $K_n(1 \leq k \leq p-1)$ always results in at least $k$ trivial components. In this section we shall study the possible situations after performing an efficient separation on $G_n$.

Let $A$ and $B$ be two subsets of $V(G)$. We denote by $E_c(A,B)$ the set of edges of $G$ each joining a vertex of $A$ to a vertex of $B$, and by $e_c(A,B)$ the number of edges in $E_c(A,B)$. In particular, we write $e_c(A)$ for $e_c(A,A)$, and $e_c(v,B)$ for $e_c(\{v\},B)$ where $v \in V(G)$. The minimum degree of $G$ is denoted by $\delta(G)$, i.e. $\delta(G) = \min\{\deg_G(u) \mid u \in V(G)\}$.

First of all, we have

**LEMMA 1.** The number of edges of the graph $G_n$ needed to be removed to separate $G_n$ into two non-trivial components is greater than $\delta(G_n)$, except when $G_n = K_{n-1}$, in which case, the number is equal to $\delta(G_n)$.

**Proof.** We proceed by induction on $n$. For the case $n = 2$, let $G_2$, $G_2 \neq K_2$, be separated into two non-trivial components, and let $e^*$ denote the number of edges removed in this separation. We may assume that both partite sets $V_1$ and $V_2$ of $G_2$ are divided into two sets. Let $V_1'$ be divided into $a$ and $b$ vertices, and $V_2'$ be divided into $c$ and $d$ vertices. (Figure 1(a)) Then $a, b, c, d$ and $e$ are positive. Since $G_2 \neq K_{n-1}$, not all of them are equal to 1. Thus $e^* = ad + be < a + b$. If $ad + be = a + b$, then $c = d = 1$ since $a, b, c$ and $d$ are positive integers. This implies $m_2 = c + d = 2$. Since $m_2 \geq m_1 \geq 2$, $m_2 = 2$. But this contradicts our assumption that $G_n \neq K_{n-1}$. Thus, we have $e^* > a + b = \delta(G_2)$.

Now, suppose that the statement holds for any graph $G_{n-1}$ ($n \geq 3$). We shall show that the statement is also true for any $G_n$. Assume that $G_n \neq K_{n-1}$, and let $G_n$ be separated into two non-trivial components $Q$ and $Q'$. Except for the two cases of separation shown in Figures 1(b) and (c) for $n = 3$ and $n = 4$ respectively, it can be checked that there is always a partite set $V'$ of $G_n$ such that $(Q \cup Q') - V'$ still consists of two non-trivial components $Q = Q - V'$ and $Q' = Q' - V'$ where $V'$ is separated into two sets $V_1$ and $V_2$ in that separation. (Figure 1(d)) Note that $V'$ or $V_1'$ may be empty.

![Figure 1](image_url)

So, the complete $(n-1)$-partite graph $G_n$ = $G_n - V'$ is separated into two non-trivial components $Q_1$ and $Q_2$. Let $e'$ denote the number of edges removed in this separation of $G_n$, and $e^*$ denote the number of edges deleted to separate $G_n$ into $Q$ and $Q'$. Then

$$e^* = e' + e_{c_n}(V_i, V(Q_i)) + e_{c_n}(V_i, V(Q))$$

By induction hypothesis,

$$e' > \delta(G)$$

But

$$\delta(G) = \begin{cases} \delta(G_n) - m_r & \text{if } r \neq n \\ \delta(G_n) - m_{n-1} & \text{if } r = n. \end{cases}$$

and

$$e_{c_n}(V_i, V(Q_i)) + e_{c_n}(V_i, V(Q)) > \begin{cases} m_r & \text{if } r \neq n \\ m_{n-1} & \text{if } r = n. \end{cases}$$

Therefore $e^* > \delta(G_n)$, as required.

It remains to consider the two exceptional cases.

**Case (i).** The separation of $G_3$ as shown in Figure 1(b).

Let the partite set $V_1$ be divided into $a$ and $b$ vertices, and let $e^*$ denote the number of edges removed in this separation. Then

$$e^* = m_1b + m_1m_2 + m_3a$$


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\[ m(b + m_i) + m_a \]
\[ > m_i + m_j \text{ (since } b + m_j \geq 2, a \geq 1) \]
\[ \geq \delta(G_i), \]

as required.

Case (ii). The separation of \( G_i \) as shown in Figure 1(c).

Let \( e^* \) denote the number of edges deleted in this separation. Then

\[ e^* = m_i^*m_i + m_i^*m_j + m_jm_i + m_jm_i \]
\[ > m_i + m_j + m_k \]
\[ \geq \delta(G_i). \]

The proof is now complete. \( \square \)

We are now ready to prove the following main result of this section.

THEOREM 1. Let \( p \) be the order of the graph \( G_n \), and \( k \) be any integer with \( 1 \leq k \leq p-1 \). If \( G_n \) is separated into components by an efficient \( k \)-separation, then

either (i) at least \( k \) of the components are trivial, or
(ii) \( k+1 \) of the components are trivial, and the other two are \( K_2 \).

Proof. Suppose there are two non-trivial components \( Q \) and \( Q' \) of \( G_n \) after the removal of \( \lambda^{(k)}(G_n) \) edges in an efficient \( k \)-separation of \( G_n \). We shall show that the induced subgraph \( H = (Q \cup Q')_{e(G_n)} \) is \( K_2(2,2)\).

We first note that \( H \) is a complete multipartite subgraph of \( G_n \). If \( H \neq K_2(2,2) \), then by Lemma 1, the number of edges removed to separate \( H \) into two components \( Q \) and \( Q' \) is greater than \( \delta(H) \). But \( \delta(H) \) is equal to the number of edges removed to separate \( H \) into a trivial component \( \{u\} \), and a component \( H \) such that \( \deg_{e(G_n)}(u) = \delta(H) \). Thus \( G_n \) can be separated into two components by removing less than \( \lambda^{(k)}(G_n) \) edges. This contradicts the definition of \( \lambda^{(k)}(G_n) \). Therefore \( Q_i = Q_i = K_2 \) and \( H = K_2(2,2) \).

Now, suppose that there is another non-trivial component \( Q_i \) of \( G_n \) after the removal of \( \lambda^{(k)}(G_n) \) edges in an efficient \( k \)-separation of \( G_n \). Then, by the argument above, we conclude that \( H_i = (Q_i \cup Q_i)_{e(G_n)} \) and \( H_i = (Q_i \cup Q_i)_{e(G_n)} \) are all isomorphic with \( K_2(2,2) \). Thus, \( Q_i = K_2 \) and the number of edges removed to separate \( H_i = (Q_i \cup Q_i)_{e(G_n)} \) into three components \( Q_i \), \( Q_i \), and \( Q_i \) is six. However, if we delete all the five edges of \( H_i \) which are incident with the two vertices of \( Q_i \), we also separate \( H_i \) into three components. But this contradicts the minimality of \( \lambda^{(k)}(G_n) \). The result thus follows. \( \square \)

Remark. We note that the result (ii) in Theorem 1 can occur only when \( G_n = K_{2}(m_1, m_2) \), where \( m_1, m_2 \geq 2 \).

3. HIGHER ORDER EDGE-CONNECTIVITY

In this section we shall apply Theorem 1 to determine the \( k \)-th order edge-connectivity of any complete \( r \)-partite graph.

We begin with the following result.

LEMMA 2. Let \( T \supseteq V(G_n) \) such that \( |T| = t \geq 1 \) and \( e_{G_n}(T) + e_{G_n}(T, G_n - T) = \lambda^{(i)}(G_n) \). Then

(i) there exists \( \omega \in T \) such that \( \deg_{e(G_n)}(\omega) = \delta(G_n) \), and
(ii) if \( T' = T - \{u\} \) and \( G_n - u \), where \( u \in T \), then \( e_{G_n}(T') + e_{G_n}(T', G_n' - T') = \lambda^{(i+1)}(G_n) \).

Note. By the assumption of Lemma 2, we are, indeed, given an efficient \( t \)-separation of \( G_n \) which separates it into \( t+1 \) components \( x \) \( (x \in T) \) and \( G_n - T \). The subgraph \( G_n - T \) must be connected as \( \lambda^{(i)}(G_n) < \lambda^{(i+1)}(G_n) \).

Proof. (i) We suppose the contrary. Then no element of \( T \) is in \( V \) or in any other partite set \( V \) of \( G_n \) such that \( |V| = |V'| \). Let \( u \in T \) and \( T' = T - \{u\} \). Consider the graph \( G_n - T' \). Note that \( G_n - T' \) is a complete multipartite subgraph of \( G_n \). So \( V \) is one of its partite sets. Let \( u \in V \) and \( u \in V' \) where \( V' \) is also a partite set in the partition of \( G_n - T' \). Since \( |V'| > |V'| \), we have

\[ e_{G_n}(u, G_n - T') < e_{G_n}(u, G_n - T') \]

Therefore (Figure 2)

\[ e_{G_n}(T' \cup \{u\}) + e_{G_n}(T' \cup \{u\}, G_n - (T' \cup \{u\})) \]
\[ = e_{G_n}(T' \cup \{u\}) + e_{G_n}(T' \cup \{u\}) \]
\[ < e_{G_n}(T', G_n - T') + e_{G_n}(u, G_n - T') \]
\[ e_{G_n}(T', G_n - T') + e_{G_n}(u, G_n - T') \]
\[ = e_{G_n}(T' \cup \{u\}) + e_{G_n}(T' \cup \{u\}, G_n - (T' \cup \{u\})) \]
\[ = e_{G_n}(T, G_n - T) \]
\[ = \lambda^{(i)}(G_n), \]

which contradicts the minimality of \( \lambda^{(i)}(G_n) \). Thus (ii) follows.

(ii) Since \( |T'| = t - 1 \), by the minimality of \( \lambda^{(i)}(G') \), we have

\[ e_{G'}(T') + e_{G'}(T', G' - T') \geq \lambda^{(i-1)}(G'). \]
It is known that a complete graph $K_n$ can be factored into spanning trees (indeed spanning paths) if and only if $n$ is even (see for instance, $\lambda(n) = \sum_{H \in H_{n-1}} \delta(H)$.

Note that $H_k$ is also a complete multipartite graph, and $|S| = k - 1$. Thus by Lemma 2(i), there exists $u \in S$ such that $\deg_{H_k}(u) = \delta(H_k)$. If $k > 1$, by using the same argument as above, we conclude that for $i = 3, 4, \ldots, k$, $S_i$ contains $u$ such that $\deg_{H_k}(u) = \delta(H_k)$ where $H_k = H_k - u$. The proof is now complete. \(\square\)

We are now in a position to establish the main result of this note.

**Theorem 2.** Let $p$ be the order of the graph $G_n$. Then for $k = 1, 2, \ldots, p - 1$,

$$\lambda^{(k)}(G_n) = \sum_{H \in H_{k-1}} \delta(H)$$

for some $H_k$ of $G_n$.

**Proof.** By Lemma 3, there exists $S \subseteq V(G_n)$ such that $|S| = k$ and $\lambda^{(k)}(G_n) = e_{\text{in}}(S) + e_{\text{out}}(S, G_n - S) = \lambda^{(k)}(G_n)$. Furthermore, $S = \{u_1, u_2, \ldots, u_k\} = V(G_n) - V(H_k)$ where $H_k$ is a member of some $H_k$ of $G_n$.

**Lemma 3.** Let $p$ be the order of the graph $G_n$, and $k$ be any integer satisfying $1 \leq k \leq p - 1$. Then there exists a set of vertices $S$ of $G_n$ such that $|S| = k$ and $e_{\text{in}}(S) + e_{\text{out}}(S, G_n - S) = \lambda^{(k)}(G_n)$.

**Proof.** To prove the first part, we show that, for each $k = 1, 2, \ldots, p - 1$, there is an efficient $k$-separation of $G_n$ such that, after performing this separation, at least $k$ of the components are trivial. By Theorem 1, there are at most two non-trivial components $Q$ and $Q'$ in any efficient $k$-separation of $G_n$, and $H = (Q \cup Q') = K_k(2, 2)$. But the number of edges removed to separate $H$ into $Q$ and $Q'$ is equal to the number of edges whose removal separates $H$ into a trivial component and a $K_k(1, 2)$ component. This completes the proof of the first part.

By Lemma 3, there exists $u \in S$ such that $\deg_{G_n}(u) = \delta(G_n)$. If $k > 1$, let us write $S_i = S - \{u_i\}$ and $H_i = G_n - u_i$. By Lemma 2(ii), we have

$$e_{\text{in}}(S_i) + e_{\text{out}}(S_i, H_i) = \lambda^{(k - 1)}(H_i)$$.

The proof is now complete. \(\square\)

We are now in a position to establish the main result of this note.

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for some $H_k$ of $G_n$.

**Proof.** By Lemma 3, there exists $S \subseteq V(G_n)$ such that $|S| = k$ and $\lambda^{(k)}(G_n) = e_{\text{in}}(S) + e_{\text{out}}(S, G_n - S) = \lambda^{(k)}(G_n)$. Furthermore, $S = \{u_1, u_2, \ldots, u_k\} = V(G_n) - V(H_k)$ where $H_k$ is a member of some $H_k$ of $G_n$.

**Proof.** To prove the first part, we show that, for each $k = 1, 2, \ldots, p - 1$, there is an efficient $k$-separation of $G_n$ such that, after performing this separation, at least $k$ of the components are trivial. By Theorem 1, there are at most two non-trivial components $Q$ and $Q'$ in any efficient $k$-separation of $G_n$, and $H = (Q \cup Q') = K_k(2, 2)$. But the number of edges removed to separate $H$ into $Q$ and $Q'$ is equal to the number of edges whose removal separates $H$ into a trivial component and a $K_k(1, 2)$ component. This completes the proof of the first part.

By Lemma 3, there exists $u \in S$ such that $\deg_{G_n}(u) = \delta(G_n)$. If $k > 1$, let us write $S_i = S - \{u_i\}$ and $H_i = G_n - u_i$. By Lemma 2(ii), we have

$$e_{\text{in}}(S_i) + e_{\text{out}}(S_i, H_i) = \lambda^{(k - 1)}(H_i)$$.

The proof is now complete. \(\square\)

We are now in a position to establish the main result of this note.

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$$\lambda^{(k)}(G_n) = \sum_{H \in H_{k-1}} \delta(H)$$

for some $H_k$ of $G_n$.

**Proof.** By Lemma 3, there exists $S \subseteq V(G_n)$ such that $|S| = k$ and $\lambda^{(k)}(G_n) = e_{\text{in}}(S) + e_{\text{out}}(S, G_n - S) = \lambda^{(k)}(G_n)$. Furthermore, $S = \{u_1, u_2, \ldots, u_k\} = V(G_n) - V(H_k)$ where $H_k$ is a member of some $H_k$ of $G_n$.

Thus,

$$\lambda^{(k)}(G_n) = e_{\text{in}}(S) + e_{\text{out}}(S, G_n - S)$$

$$= \sum_{i=0}^{k-1} e_{\text{in}}(u_{i+1}, H_i)$$.

as required. \(\square\)

**4. SPANNING TREE FACTORIZATION**

It is known that a complete graph $K_n$ can be factored into spanning trees (indeed spanning paths) if and only if $p$ is even (see for instance,
Behzad et al. (1979), p. 168). In the following theorem we give a necessary and sufficient condition for the graph $G_n$ to be factored into spanning trees.

**Theorem 3.** The complete n-partite graph $K_n(m_1, m_2, \ldots, m_n)$ can be factored into spanning trees if and only if

$$
\sum_{j=1}^{n} m_j = k \left( \sum_{j=1}^{n} m_j - 1 \right)
$$

for some positive integer $k$.

**Corollary.** The graph $K_n(m_1, m_2, \ldots, m_n)$ is spanning tree factorizable for the following integers $m$ and $n$ ($n \geq m$):

(i) $m = 1$, and $n \geq 1$;

(ii) $m \equiv 1 \pmod{2}$, $m > 1$, and $n = m + 1$;

(iii) $m > 2$, and $n = (m - 1)^2$;

(iv) $m > 4$, and $n = (m - 1)(m - 2)/2$;

(v) $m = ab$ and $n = (ab - 1)(b - 1)$ where $a$ and $b$ are integers $> 2$.

**Corollary 2.** (i) The graph $K_n(m - 1, \ldots, m, m, \ldots, m)$ is spanning tree factorizable if and only if $(n - 1) \equiv 0 \pmod{2}$.

(ii) The graph $K_n(1, \ldots, m, \ldots, m)$ is spanning tree factorizable if and only if $n m \equiv 0 \pmod{2}$.

(iii) The graph $K_n(m, m, \ldots, m)$ is spanning tree factorizable if and only if $m = 1$ and $n$ is even.

Denote by $\omega(G)$ the number of components of $G$. A subset $X$ of $E(G)$ is called an edge-cutset of $G$ if $\omega(G - X) > 1$. Following Peng et al. (1988), the edge-toughness of $G$, denoted by $\tau_1(G)$, is defined as

$$
\tau_1(G) = \min \left\{ \frac{|X|}{\omega(G - X) - 1} \mid X \text{ is an edge-cutset of } G \right\}
$$

The above definition of $\tau_1(G)$ is, as a matter of fact, motivated by the following result due to Nash-Williams (1961) and Tutte (1961) independently.

**Theorem A.** A connected graph $G$ has $s$ edge-disjoint spanning trees if and only if $|X| \geq s(\omega(G - X) - 1)$ for each $X \subseteq E(G)$.

It follows from Theorem A that a connected graph $G$ has $k$ edge-disjoint spanning trees if and only if $\tau_1(G) \geq k$. Thus Theorem 3 is an immediate consequence of the following result.

**Theorem 4.** $\tau_1(K_n(m_1, m_2, \ldots, m_n))$

$$
\sum_{j=1}^{n} m_j = \frac{\sum_{j=1}^{n} m_j - 1}{\sum_{j=1}^{n} m_j - 1}
$$

To prove the above theorem we shall make use of the following result which was obtained in Peng et al. (1988) as a corollary of a more general theorem.

For each $i = 1, 2, \ldots, |V(G)| - 1$, we write

$$\Delta \lambda_i(G) = \lambda^{(i)}(G) - \lambda^{(i-1)}(G).$$

**Theorem B.** Let $G$ be a connected graph of order $p$ and size $q$. If the sequence $(\Delta \lambda_i(G))$ is non-increasing, i.e. $\Delta \lambda_i(G) \geq \Delta \lambda_{i+1}(G)$ for each $i = 1, 2, \ldots, p - 2$, then $\tau_1(G) = q/(p - 1)$.

**Proof of Theorem 4.** By Theorem B, we only need to show that the sequence $(\Delta \lambda_i(G))$ is non-increasing. By Theorem 2, $\Delta \lambda_i(G) = \lambda^{(i)}(G) - \lambda^{(i-1)}(G) = \delta(H_i)$. Note that for $i = 1, 2, \ldots, p - 1$, $H_i = H_{i+1} - v_i$ where $\deg_{H_{i+1}}(v_i) = d(H_i)$ and $v_i$ is adjacent to every vertex of $H_i$ except those in the partite set of the partition of $H_{i+1}$ that $v_i$ belongs to. So, it is clear that for $i = 1, 2, \ldots, p - 1$, $\delta(H_i) \leq \delta(H_{i+1})$. Therefore, the sequence $(\Delta \lambda_i(G))$ is non-increasing.

**References**


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