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Automorphisms of Surface Groups of Genus Two

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ABSTRAK

Kumpulan kelas pemetaan bagi suatu permukaan ialah hasil bahagi kumpulan automorfisma $Aut^+\Gamma$ bagi kumpulan permukaan oleh subkumpulan normal automorfisma terkedalam. Penentuan $Aut^+\Gamma$ boleh dilihat sebagai masa*lah perluasan. Kertas ini memberikan suatu persembahan bagi $Aut^+\Gamma$ untuk Γ kumpulan permukaan genus dua. Perluasan kepada kumpulan gentian Seifert juga diberikan.*

ABSTRACT

The mapping class group of a surface is a quotient of the automorphism group $Aut^+\Gamma$ of the surface group by its normal subgroup of inner automorphism. The determination of $Aut^+\Gamma$ may be looked upon as an extension problem. This paper will give a presentation of $Aut^+\Gamma$ where *Γ is a surface group of genus two. The extension to Seifert fibre group will be given.*

INTRODUCTION

In this paper, we consider the surface group

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1. \rangle$$

Γ is a fundamental group $\pi_1(T_g)$ of some surface of genus g. By Nielsen's theorem, every automorphism in the fundamental group of a surface is induced by a self-homeomorphism of the surface. With abuse of language, we call those automorphisms induced by the orientation-preserving self-homeomorphisms of the surface, the orientation-preserving automorphisms, denoted by Aut^+.

The mapping class group of a surface is the quotient of the automorphism group, $\mathrm{Aut}^+\Gamma$, of the surface group, Γ, by its normal subgroup of inner automorphisms, (Hatcher & Thurston, 1980; Maclachlan & Harvey, 1975). Hence the determination of $\mathrm{Aut}^+\Gamma$ may be looked upon as an extension problem.

Although the presentation of the mapping class group of a closed orientable surface of genus $g \geqslant 2$ have been completed theoretically by Hatcher and Thurston, (1980), the presentation is not laid out explicitly. Hence the determination of $\mathrm{Aut}^+\Gamma$ would be very technical. Here we will give an explicit presentation of $\mathrm{Aut}^+\Gamma$, where Γ is a surface group of genus two.

1. PRELIMINARY

In order to obtain our answer, we need the following presentation of the mapping class group M(2, 0).

Theorem: 1.1

The mapping class group of a surface of genus two, M(2, 0), admits the presentation with generators:

$\omega_i \qquad ,1 \leqslant i \leqslant 5.$

and defining relations:

$$\omega_i\,\omega_{i+1}\,\omega_i = \omega_{i+1}\,\omega_i\,\omega_{i+1} \qquad ,1 \leqslant i \leqslant 4$$

$$\omega_i\,\omega_j = \omega_j\,\omega_i \qquad , |i-j| \geqslant 2$$

$$(\omega_1\,\omega_2\,\omega_3\,\omega_4\,\omega_5)^6 = 1$$

$$(\omega_1\,\omega_2\,\omega_3\,\omega_4\,\omega_5^2\,\omega_4\,\omega_4\,\omega_3\,\omega_2\,\omega_1)^2 = 1$$

$$(\omega_1\,\omega_2\,\omega_3\,\omega_4\,\omega_5^2\,\omega_4\,\omega_3\,\omega_2\,\omega_1) \longleftrightarrow \omega_1$$

where $\longleftrightarrow$ denotes commutativity.
(For a proof see Birman, 1974).

Remarks: 1.1

I. The generators ω_i are the isotopy class of Dehn twists about the simple closed curves on the surface T_2. (Birman, 1969 ; Birman, 1974).

II. Throughout this paper, we will use the convention of operating from right to left, that is $f_g(x) = f(g(x))$.

We also need the following well-known lemma to find our presentation.

Lemma: 1.1

Let A, B, C be groups which satisfy the exact sequence:

$$1 \to A \to B \to C \to 1$$

If $A = \langle x_1, x_2, \ldots, x_n \mid S_i(x_1, x_2, \ldots, x_n);\ 1 \leq i \leq \ell \rangle$

and $C = \langle w_1, w_2, \ldots, w_m \mid R_i(w_1, w_2, \ldots, w_m);\ 1 \leq i \leq k \rangle$

are finitely presented, then B is finitely presented with Generators:

$$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$$

where

$$\upsilon(y_i) = w_i,\ 1 \leqslant i \leqslant m$$

Defining relations:

$$R_i(y_1, y_2, \ldots, y_m) = T_i(x_1, x_2, \ldots, x_n),\quad 1 \leqslant i \leqslant k,$$

$$S_i(x_1, x_2, \ldots, x_n) = 1,\ 1 \leqslant i \leqslant \ell,$$

$$y_i x_j y_i^{-1} = T_{ij}(x_1, x_2, \ldots, x_n),\quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,$$

for some words $T_1(x_1, x_2, \ldots, x_n)$,

$T_{ij}(x_1, x_2, \ldots, x_n)$.

2. THE RESULT

Let $\Gamma = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle$

Then $M(2, 0) \cong \mathrm{Aut}^+\Gamma / I(\Gamma)$ where $I(\Gamma)$ is the inner automorphism group of Γ.

Let ξ_i, $1 \leqslant i \leqslant 5$, be the automorphisms in Γ induced by ω_i, that is $\phi(\xi_i) = \omega_i$, $1 \leqslant i \leqslant 5$, in the sequence:

$$1 \longrightarrow I(\Gamma) \rightarrowtail \mathrm{Aut}^+\Gamma \xrightarrow{\ \phi\ } M(2,0) \longrightarrow 1$$

Then we have:

$$\xi_1 : \begin{array}{l} a_1 \longrightarrow a_1 \\ a_2 \longrightarrow a_2 \\ b_1 \longrightarrow b_1 a_1^{-1} \\ b_2 \longrightarrow b_2 \end{array} \qquad \xi_2 : \begin{array}{l} a_1 \longrightarrow a_1 b_1 \\ a_2 \longrightarrow a_2 \\ b_1 \longrightarrow b_1 \\ b_2 \longrightarrow b_2 \end{array}$$

$$\xi_3 : \begin{array}{l} a_1 \longrightarrow a_1 \\ a_2 \longrightarrow a_1^{-1} b_2 a_2 \\ b_1 \longrightarrow a_1^{-1} b_2 b_1 \\ b_2 \longrightarrow b_2 \end{array} \qquad \xi_4 : \begin{array}{l} a_1 \longrightarrow a_1 \\ a_2 \longrightarrow a_2 \\ b_1 \longrightarrow b_1 \\ b_2 \longrightarrow b_2 a_2^{-1} \end{array}$$

$$
\begin{aligned}
\xi_5 : \; a_1 &\longrightarrow a_1 \\
a_2 &\longrightarrow a_2 b_2 \\
b_1 &\longrightarrow b_1 \\
b_2 &\longrightarrow b_2
\end{aligned}
$$

Theorem: 2.1

If Γ is a surface group of genus two, then $\mathrm{Aut}^+\Gamma$ is generated by:

$$\xi_i \, , \; 1 \leqslant i \leqslant 5$$

with defining relations:

$$\xi_i \, \xi_{i+1} \, \xi_i = \xi_{i+1} \, \xi_i \, \xi_{i+1} \qquad , \; 1 \leqslant i \leqslant 4$$

$$\xi_i \, \xi_j = \xi_j \, \xi_i \qquad , \; i - j \geqslant 2$$

$$(\xi_1 \xi_2 \xi_3 \xi_4 \xi_5)^6 = 1$$

$$(\xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_4 \xi_3 \xi_2 \xi_1)^2 = 1$$

Hence, $\mathrm{Aut}^+\Gamma = \mathrm{Aut}^+\Gamma_1$, where

$$\Gamma_1 = \langle x_1, x_2, x_3, x_4, x_5, x_6 \quad x_i^2 = x_1 x_2 x_3 x_4 x_5 x_6 = 1 : 1 \leqslant i \leqslant 6 \rangle$$

is a Fuchsian group of genus zero with six equal periods, 2.

Proof:

Since Γ is centerless, $I(\Gamma) \cong \Gamma$ and hence is finitely presented. We denote the inner automorphisms $I(a_i) = \alpha_i$, $I(b_i) = \beta_i$ for $i = 1,2$. Then by Lemma 1.1. $\mathrm{Aut}^+\Gamma$ is finitely generated by: $\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_i$ for $1 \leqslant i \leqslant 5$

Our problem now is reduced to checking the defining relations as in the lemma. We then have the following:

$$
\left.
\begin{aligned}
&\xi_i \xi_{i+1} \xi_i = \xi_{i+1} \xi_i \xi_{i+1} && , i \leqslant i \leqslant 4 \\
&\xi_i \xi_j = \xi_j \xi_i && , |i - j| \geqslant 2 \\
&(\xi_1 \xi_2 \xi_3 \xi_4 \xi_5)^6 = 1 && \\
&(\xi_1 \xi_2 \xi_3 \xi_4 \xi_5^2 \xi_4 \xi_3 \xi_2 \xi_1)^2 = 1 &&
\end{aligned}
\right\} (2\ 1)
$$

$$
\left.
\begin{aligned}
&\alpha_1 (\xi_1 \xi_2 \xi_3 \xi_4 \xi_5^2 \xi_4 \xi_3 \xi_2 \xi_1) \xi_1 = \\
&\xi_1 (\xi_1 \xi_2 \xi_3 \xi_4 \xi_5^2 \xi_4 \xi_3 \xi_2 \xi_1) \\
&\alpha_1 \longleftrightarrow \xi_i \quad , \quad i = 1, 3, 4, 5 \\
&[\xi_2, \alpha_1] = \beta_1 \\
&\alpha_2 \longleftrightarrow \xi_i \quad , \quad i = 1,2,4 \\
&[\alpha_2, \xi_3] = \beta_2^{-1} \alpha_1 \\
&[\xi_5^{-1}, \alpha_5^{-1}] = \beta_2 \\
&\beta_1 \longleftrightarrow \xi_i \, , \quad i = 2, 4, 5 \\
&[\beta_1^{-1}, \xi_1^{-1}] = \alpha_1 \\
&[\beta_1, \xi_3] = \beta_2^{-1} \alpha_1 \\
&\beta_2 \longleftrightarrow \xi_i \quad , \quad i = 1, 2, 3, 5 \\
&[\beta_2^{-1}, \xi_4^{-1}] = \alpha_2 \\
&\pi = [\alpha_1, \beta_1] \, [\alpha_2, \beta_2] = 1.
\end{aligned}
\right\} (2.2)
$$

where $[a, b] = aba^{-1}b^{-1}$

Now let $\gamma = \xi_1 \xi_2 \xi_3 \xi_4 \xi_5^2 \xi_4 \xi_3 \xi_2 \xi_1$, then we have:

$$
\left.
\begin{aligned}
\alpha_1 &= \xi_1 \gamma \xi_1^{-1} \gamma^{-1} \\
\beta_1 &= \xi_2^{-1} \alpha_1^{-1} \xi_2 \alpha_1 \\
\beta_2 &= \alpha_1 \xi_3 \beta_1 \xi_3^{-1} \beta_1^{-1} \\
\alpha_2 &= \beta_2^{-1} \xi_4 \beta_2 \xi_4
\end{aligned}
\right\} (2.3)
$$

Hence $\mathrm{Aut}^+\Gamma$ is generated only by ξ_i, $1 < i < 5$. The relations (2.2) can be shown to be redundant and thus can be eliminated giving the result.

The last part of the theorem follows from the presentation of $\mathrm{Aut}^+\Gamma_1$. (See Abu Osman,).

Maclachlan, (1974), proved the following:

Theorem: 2.2

If Γ has a signature (g; –; s) and Γ^0 is a uniformizing group of ${}^{IH}/\Gamma \setminus x_0$, then $Aut^+\Gamma$ is isomorphic to a subgroup of index (s + 1) in $Aut^+\Gamma / I(\Gamma)$, where IH is the upper half of the complex plane.

In our case, Γ is such that g = 2 and s = 0, and hence $Aut^+\Gamma = {}^{Aut^+\Gamma^0}/_{I(\Gamma^0)} = M(2,1)$ the mapping class group of a surface of genus two with one puncture. Hence we have shown that:

Theorem: 2.3

It Γ is a surface group of genus two, then

$$Aut^+\Gamma \cong M(2,1)$$

Remarks: 2.1

By the above theorem, we now know the presentation of M(2,1).
If $\Gamma^* = \langle a_1, b_1, a_2, b_2 \mid (a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1})^n = 1 \rangle$

then $M(2,1) \cong Aut^+\Gamma^* /_{I(\Gamma)^*}$. Using the same procedure, we can get the presentation of $Aut^+\Gamma^*$. (Abu Osman, 1984).

3. THE EXTENSION TO SEIFERRT FIBRE GROUP

Let Γ be a Fuchsian group of genus two as in 2. Let G be a central extension group of Z, by Γ such that:

$$G = \langle a_1, b_1, a_2, b_2, z \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = z^n, \ z \longleftrightarrow a_i, b_i \rangle \quad (3,1)$$

where $\longleftrightarrow$ denoted commutativity.

$$1 \longrightarrow (z) \rightarrowtail G \xrightarrow{\Psi} \Gamma \longrightarrow 1.$$

G is called the *Siefert fibre group* which is the fundamental π_2 (M) of the Seifert manifold. (Orlik, 1972)

We denote Aut^+G those automorphisms in G induced by the automorphisms in $Aut^+\Gamma$ and map z $\longrightarrow$ z. Ψ induced $\Psi_*: Aut^+G \longrightarrow Aut^+\Gamma$ in a natural way. Let $N = \ker \Psi_*$. We then have:

$$1 \longrightarrow N \longrightarrow Aut^+G \xrightarrow{\Psi_*} Aut^+\Gamma \longrightarrow 1.$$

It is easy to see that N is guaranted by τ_i, $1 < i \leqslant 4$, defined as:

$$\tau_1 : \begin{array}{l} a_1 \longrightarrow za_1 \\ a_2 \longrightarrow a_2 \\ b_1 \longrightarrow b_1 \\ b_2 \longrightarrow b_2 \end{array} \qquad \tau_2 ; \begin{array}{l} a_1 \longrightarrow a_1 \\ a_2 \longrightarrow za_2 \\ b_1 \longrightarrow b_1 \\ b_2 \longrightarrow b_2 \end{array}$$

$$\tau_3 : \begin{array}{l} a_1 \longrightarrow b_1 \\ a_2 \longrightarrow a_2 \\ b_1 \longrightarrow zb_1 \\ b_2 \longrightarrow b_2 \end{array} \qquad \tau_4 : \begin{array}{l} a_1 \longrightarrow a_1 \\ a_2 \longrightarrow a_2 \\ b_1 \longrightarrow b_1 \\ b_2 \longrightarrow zb_2. \end{array}$$

Observe that for every i and j, $\tau_i \tau_j = \tau_j \tau_i$. Hence N is a free abelian group Z^4. By Lemma, we can find the presentation of Aut^+G; since we know the presentation $Aut^+\Gamma$.

For each i, $1 \leqslant i \leqslant 5$, pick δ_i Aut^+G, the lift ξ_i $Aut^+\Gamma$, in such a way that δ_i map a_j, b_j in in exactly the same manner as ξ_i and map z into z. Then $\Psi_*(\delta_i) = \xi_i$, $1 \leqslant i \leqslant 5$. Hence,

$$\left.\begin{array}{lll} \delta_i & , & 1 \leqslant i \leqslant 5 \\ \tau_j & , & 1 \leqslant j \leqslant 4 \end{array}\right\} \quad (3.2)$$

generate Aut^+G. The defining relations induced by the defining relations of $Aut^+\Gamma$ are exactly the same due to our choice of δ_i. The relations arising from N are $\tau_i \tau_j = \tau_j \tau_i$ for every i and j. The other relations can easily be calculated. Hence we have the following:

Theorem: 3.1

If G is a Seifert fibre group with presentation (3-1), then Aut^+G is generated by:

$$\delta_i \quad , \quad 1 \leqslant i \leqslant 5$$

$$\tau_j \quad , \quad 1 \leqslant j \leqslant 4$$

with defining relations:

$$\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1} \quad , \quad 1 \leqslant i \leqslant 4$$

$$\delta_i \delta_j = \delta_j \delta_i \quad , \quad |i - j| \geqslant 2$$

$$(\delta_1 \delta_2 \delta_3 \delta_4 \delta_5)^6 = 1$$

$$(\delta_1 \delta_2 \delta_3 \delta_4 \delta_5^2 \delta_4 \delta_3 \delta_2 \delta_1)^2 = 1$$

$$\tau_i \longleftrightarrow \tau_j \quad \text{, for all i and j.}$$

$$\delta_1 \longleftrightarrow \tau_2, \tau_3, \tau_4$$

$$[\delta_1, \tau_2] = \tau_3$$

$$\delta_2 \longleftrightarrow \tau_1, \tau_2, \tau_4$$

$$[\delta_2^{-1}, \tau_3^{-1}] = \tau$$

$$\delta_3 \longleftrightarrow \tau_2, \tau_3, \tau_4$$

$$[\delta_3, \tau_1] = \tau_3 \tau_2$$

$$\delta_4 \longleftrightarrow \tau_1, \tau_3, \tau_4$$

$$[\delta_4, \tau_2] = \tau_4$$

$$\delta_5 \longleftrightarrow \tau_1, \tau_2, \tau_3$$

$$[\tau_4^{-1}, \delta_5^{-1}] = \tau_2$$

where denoted commutativity and $[a, b] = aba^{-1}b^{-1}$.

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