Automorphisms of Fuchsian Groups of Genus Zero

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ABSTRACT

Every automorphism in Fuchsian group is induced by some automorphism of a free group. This paper gives a presentation of a automorphism group of Fuchsian group of genus zero via braid groups. We also obtained the pure mapping class groups and the Seifert Fibre Groups.

INTRODUCTION

A co-compact Fuchsian group, \( \Gamma \), is known to have the following presentation:

\[
\Gamma = \langle a_1, b_1, \ldots, a_r, b_2, x_1, x_2, \ldots, x_r \mid x_i \rangle
\]

where \( g \geq 0, r \geq 0, m_i \geq 2 \) and \( [a, b] = aba^{-1}b^{-1} \) (See [5]). The integers \( m_1, m_2, \ldots, m_r \) are called the periods and \( g \) is called the genus. We say \( \Gamma \) has signature \( (g; m_1, m_2, \ldots, m_r) \). If \( g = 0 \), we simply write \( (m_1, m_2, \ldots, m_r) \) for \( (0; m_1, m_2, \ldots, m_r) \). If \( g = 0 \), \( r = 3 \), we call \( (\ell, m, n) \) the triangle group.

\( \Gamma \) is the fundamental group of some surface. By Nielsen’s theorem, every automorphism in the fundamental group of a surface is induced by a self-homeomorphism of the surface. With abuse of language, we call those automorphisms induced by the orientation-preserving self-homeomorphisms of the surface, the orientation-preserving automorphisms, denoted by \( \text{Aut}^+ \). In this paper, we will give a presentation of \( \text{Aut}^+ \Gamma \), for \( \Gamma \) a Fuchsian group of genus zero.

1. BRAID GROUPS

Artin (1925, 1947) defined the braid group (the full braid group) of the plane, \( B_r \), with \( r \) strings as:

Generators: \( a_i, \quad 1 \leq i \leq r - 1 \).

Defining relations:

\[
a_i a_{i+1} a_i = a_{i+1} a_{i+1} a_i, \quad 1 \leq i \leq r - 2
\]

(1.1)

\[
a_i a_j = a_j a_i, \quad |i - j| \geq 2
\]

The braid group, \( B_r \), can be looked upon as the subgroup of the automorphism group of a free group of rank \( r \). We will adopt the convention of operating from right to left, that is

\[
a_i a_j(x) = a_j a_i(x).
\]

Let \( u: B_r \to \Sigma_r \) be defined by \( u(a_i) = (i + 1) \), for \( 1 \leq i \leq r - 1 \), where \( \Sigma_r \) is a symmetric group on \( r \) letters. Let \( P_r = \ker u \). Then \( P_r \) is called the pure braid group and is known to have the following presentation: Generators:
Defining relations:

\[
A^{-1}_{ij} A_{st} = A^{-1}_{ij} A_{sj}, \text{ if } t = i
\]

\[
A^{-1}_{ij} A_{tj} A_{ij}, \text{ if } s = i < j < t
\]

\[
A^{-1}_{tj} A_{sj} A_{tj} A_{ij}, A^{-1}_{tj} A_{sj},
\]

\[
\text{if } s < i < t < j
\]

As a representation of the automorphism of the free group \(F_r = \langle x_1, x_2, ..., x_r \rangle\), we have.

\[
c_i : x_i \mapsto x_i x_i x_i^{-1}
\]

\[
x_i + 1 \mapsto x_i
\]

\[
x_j \mapsto x_j, \quad \text{for } j \neq i, i+1.
\]

and

\[
A_{st} : x_i \mapsto x_i
\]

\[
x_i + 1 \mapsto x_i
\]

\[
x_j \mapsto x_j, \quad \text{for } j \neq i, i+1.
\]

Note:

\[
(\sigma_1 \sigma_2 \ldots \sigma_{r-1})^T = (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_1)^T
\]

\[
= I(x_1 x_2 \ldots x_r)
\]

\[
(A_{r-1}, r A_{r-2}, r \ldots A_{r} A_{r-1}) (A_{r-2}, r-1 A_{r-3}, r-1)
\]

\[
= (A_{r-2}, A_{r-3} A_{r-1}) (A_{r-1} A_{r-2})
\]

\[
= (A_{r-1} A_{r-2}) (A_{r-3} A_{r-1}) \ldots (A_{r} A_{r-1})
\]

\[
= I(x_1 x_2 \ldots x_r).
\]

The centre of \(B_r, r \geq 3\), is the infinite cyclic subgroup generated by

\[
a^r = (\sigma_1 \sigma_2 \ldots \sigma_{r-1})^r = (A_{12} A_{23} A_{13} \ldots A_{r-1} A_{r-2} r \ldots A_{1r}).
\]

\[
\ldots \ldots (1.7)
\]

(See Birman, 1974 and Chow, 1948)

We now state the well-known necessary and sufficient condition for an automorphism of a free group to be an element of the braid group \(B_r\).

**Theorem 1**

Let \(F_r = \langle x_1, x_2, ..., x_r \rangle\). Then \(\beta \in B_r \subset \text{Aut } F_r\)

if and only if \(\beta\) satisfies:

\[
\beta(x_i) = \lambda_i x_{\mu_i} x_i^1, \quad 1 \leq i \leq r
\]

\[
\beta(x_1 x_2 \ldots x_r) = x_1 x_2 \ldots x_r
\]

where \((1 2 \ldots r)\) is a permutation and

\[
\mu_1, \mu_2, ..., \mu_r
\]

\[
\lambda_i = \lambda_i (x_1, x_2, ..., x_r).
\]

(Artin, 1925 and Birman, 1974) (See [1], [3])

The mapping class groups are closely related to the braid groups and the automorphism groups of the Fuchsian groups. (See [3], [7]). The mapping class group (full mapping class group), \((M(o, r),\) is known to have the following presentation:

Generators: \(\xi_i, \quad 1 \leq i \leq r - 1\).

Defining relations:

\[
\xi_i \xi_{i+1} \xi_i^{-1} = \xi_{i+1} \xi_i, \quad 1 \leq i \leq r - 2
\]

\[
\xi_i \xi_{j} = \xi_{j} \xi_i, \quad |i - j| \geq 2
\]

\[
\xi_1 \xi_2 \ldots \xi_{r-2} \xi_{r-1} \xi_{r-2} \ldots \xi_2 \xi_1 = 1
\]

\[
(\xi_1 \xi_2 \ldots \xi_{r-1})^T = 1
\]

2. AUTOMORPHISM GROUPS

We now state a restricted version of Zieschang’s theorem (1966):
Theorem 2.1.

Let $\Gamma = < x_1, x_2, \ldots, x_r | x_i^{m_i} = 1 >$ be a Fuchsian group of genus zero and $\Gamma = < \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_r >$ be a free group of rank $r$. Then every $\phi \in \text{Aut}^+ \Gamma$ is induced by some $\phi \in \text{Aut} \Gamma$ satisfying:

$$\phi(x_i) = \lambda_i \mu_i^{-1}, \quad 1 \leq i \leq r$$

(2.1)

where $(\mu_1, \mu_2, \ldots, \mu_r)$ is a permutation with $m_i = \lambda_i$, $1 \leq i \leq r$ and $\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda \in \Gamma$.

Let $\psi: \hat{\Gamma} \rightarrow \Gamma$, $\psi(\hat{x}_i) = x_i$, $1 \leq i \leq r$, be the natural homomorphism. If $\phi \in \text{Aut} \hat{\Gamma}$ satisfies (2.1), then there is a unique $\phi \in \text{Aut}^+ \Gamma$ defined by:

$$\hat{\phi}(x_i) = \lambda_i \mu_i^{-1}, \quad 1 \leq i \leq r$$

(2.2)

Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ where $k$$\sum_{i=1}^{k} \alpha_i = r$, be the symmetric group corresponding to the permutation of the periods. Then we have:

$$v: B_r \rightarrow \Sigma_1 \supset \pi_i \Sigma_i \supset \{1\}$$

We are interested in the structure of the groups $v^{-1}$

$$B_r \leftarrow \eta v^{-1}(\pi_i \Sigma_i) \leftarrow P_r \leftarrow \eta$$

and

$$B_r \leftarrow v^{-1}(\pi_i \Sigma_i) \leftarrow P_r \leftarrow v$$

(2.3)

Then the defining relations of $\Gamma$ with signature (2.4) are:

$$\alpha_1, \alpha_2, \ldots, \alpha_k \quad (m_1, m_2, \ldots, m_k)$$

(2.4)

where $\sum_{i=1}^{k} \alpha_i = r$, to mean that the first $\alpha_1$ generators have period $m_1$, the next $\alpha_2$ generators have period $m_2$, ..., and the last $\alpha_k$ generators have period $m_k$. We set $\alpha_0 = 0$, the significance of which will become clear later for the simplicity of notation. Let $\ell_n = n$, $0 \leq n \leq k$. Then

$$\ell_0 = 0, \ell_1 = \alpha_1, \ell_2 = \alpha_1 + \alpha_2, \ldots, \ell_k = r.$$
\( a_i \) for \( 1 \leq i \leq r - 1, i \neq \ell, 1 \leq n \leq k - 1. \) (2.6)

\( A_{ij} \) for \( 1 \leq i < j \leq r. \) (2.7)

From the definition of \( A_{ij} \) in terms of \( a_i \)'s, we see that it suffices to substitute (2.7) with:

\[ a_i^{n+1} \text{ for } 1 \leq i \leq n, 1 \leq n \leq k - 1. \] (2.8)

Hence, (2.6) and (2.8) form a sufficient set of generators of \( \nu^{-1}(\prod_{i=1}^{k} \alpha_i). \)

The defining relations are those of the braid group and the pure braid group wherever definable corresponding to the symmetric group \( \pi \sum \alpha_i. \)

We then have the following:

**Theorem 2.2.**

\[ \nu^{-1}(\prod_{i=1}^{k} \alpha_i) \text{ admits a presentation with generators:} \]

\( \alpha_i, \text{ for } 1 \leq i \leq r - 1, i \neq \ell, 1 \leq n \leq k - 1 \)

\( A_{ij}, \ell n + 1, \text{ for } 1 \leq i \leq \ell, 1 \leq n \leq k - 1 \)

and defining relations: (1.2)

\[ \alpha_i \alpha_j = \alpha_j \alpha_i, \text{ for } |i-j| \geq 2 \]

\[ A_{ij} A_{kl} = A_{ij} A_{kl}, \text{ for } t \neq i - 1, i, j. \]

**Theorem 2.3**

Let \( \Gamma \) be a Fuchsian group with signature (2.4). Then.

\[ \text{Aut}^+ \Gamma = \eta \nu^{-1}(\prod_{i=1}^{k} \alpha_i). \]

**Lemma 2.1.**

\[ I(\Gamma) \subset \mathbb{P}^* \subset \eta \nu^{-1}(\prod_{i=1}^{k} \alpha_i) \subset B^*_r. \]

**Proof:**

If we denote the inner automorphisms

\[ x_j \rightarrow (x_1 x_2 \ldots x_{\ell}) x_j (x_1 x_2 \ldots x_{\ell})^{-1}, 1 \leq j \leq r \]

by \( \theta_{\ell}. \) \( 1 \leq \ell \leq r, \) then we have the following:

\[ \theta_1 = (\sigma_1 \sigma_2 \ldots \sigma_2)^{1-r} \]

\[ \theta_i = (\sigma_{i-1} \sigma_{i-2} \ldots \sigma_i)^{-r} \]

where \( \sigma_i \)'s now are the elements of \( B^*_r. \) Since each element \( x_i \) is mapped on a conjugate, it follows then by definition that \( I(\Gamma) \subset \mathbb{P}^* \).

**Remarks 2.1.**

1. Note that with the action on \( \Gamma \) (that is, considering \( \sigma_i \)'s as the elements of \( B^*_r \))

\[ \sigma_1 \sigma_2 \ldots \sigma_{r-2} \sigma_{r-1} \sigma_{r-2} \ldots \sigma_1 \]

\[ = (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_3)_{r-1}^{r-r} (\sigma_{r-1} \ldots \sigma_2)_{r}^{1-r} \]

\[ = (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_2)^{1-r} \]

II. If \( x_i \) and \( x_j \) have equal periods, then their inner automorphisms are conjugate of each other. Since the periods are equal, there is an automorphism

\[ \gamma: x_i \rightarrow x_j \]

such that for each \( k, 1 < k < r, \)

\[ [\gamma I(x_i) \gamma^{-1}] (x_k) = \gamma I(x_i) (\gamma^{-1} (x_k)) \]

\[ = \gamma(x_i) \gamma^{-1} (x_k) (x_{k^{-1}}) \]

\[ = \gamma(x_i) x_k \gamma(x_{k^{-1}}) = x_j x_k x_j^{-1} \]

\[ = [I(x_j)] (x_k). \]

Therefore, \( I(x_j) = \gamma I(x_i) \gamma^{-1}. \)

**Proof of Theorem 2.3.**

By Zieschang's theorem, every \( \phi \in \text{Aut}^+ \Gamma \) is induced by \( \tilde{\phi} \in \tilde{\Lambda}(\Gamma) \) which satisfies (2.1.). Then

\[ \tilde{\Lambda}(\Gamma) = I(\Gamma) \nu^{-1}(\prod_{i=1}^{k} \alpha_i) \text{ and } \text{Aut}^+ \Gamma = I(\Gamma) \eta \nu^{-1}. \]
Stage 1:
Let \( \pi \Sigma \alpha_i \). By Lemma 2.1, then we have the result.

**Corollary 2.1.**

I. If all the periods are equal, then \( \text{Aut}^* \Gamma = B_r^* \).

II. If all the periods are distinct, then \( \text{Aut}^* \Gamma = P_r^* \).

Our aim now is to find the structure of these groups \( \nu^{-1}(\pi \Sigma \alpha_i) , B_r^* , P_r^* \). We will do this in two stages.

**Stage 1:**

Let \( N_i \) be the normal closure of \( \{ x_1 x_2 \ldots x_r \} \) in \( \hat{\Gamma} \) and \( \Gamma_i = \hat{\Gamma}/N_i \). Let \( \eta_1 \nu^{-1}(\pi \Sigma \alpha_i) \) be the group of automorphisms in \( \Gamma_i \) induced by \( \nu^{-1}(\pi \Sigma \alpha_i) \). Correspondingly, let \( \eta_1(B_r^*) = B_r^* \) and \( \eta_1(P_r^*) = P_r^* \).

Then by Magnus's theorem, (Maclachlan, 1973 and Magnus 1934), \( \ker \eta_1 = \text{center} \). Hence we have:

**Theorem 2.4.**

\[ \eta_1 \nu^{-1}(\pi \Sigma \alpha_i) \text{ is isomorphic to } \nu^{-1}(\pi \Sigma \alpha_i) \text{ modulo the center.} \]

Hence we can find the presentation of \( \eta_1 \nu^{-1}(\pi \Sigma \alpha_i) \) by expressing \( \hat{(x_1 x_2 \ldots x_r)} \), which is the generator of the center by (1.7), in terms of the generators \( \nu^{-1}(\pi \Sigma \alpha_i) \).

**Corollary 2.2.**

\( B_r^1 \cong B_r/\text{center} \). Therefore \( B_r^1 \) is generated by \( \sigma_i \), \( 1 \leq i \leq r - 1 \), with defining relations (1.1) and

\[ (\sigma_1 \sigma_2 \ldots \sigma_{r-1})^r = 1 \]

**Corollary 2.3.**

\[ P_r^1 \cong P_r/\text{center} \]. Therefore \( P_r^1 \) is generated by \( A_{ij} \), \( 1 \leq i < j \leq r \), with defining relations (1.2) and

\[ (A_{12}^r)(A_{23}^rA_{13}) \ldots (A_{r-1}^rA_{r-2}^r \ldots A_1^r) = 1 \]

**Remarks 2.2.**

Maclachlan, (1973), gives the presentation of \( B_r^1 \). By the same argument as Theorem 2.3., \( \text{Aut}^* \Gamma_i = B_r^1 \).

**Stage 2:**

Let \( \Omega : \text{Aut}^* \Gamma_i \to \text{Aut}^* \Gamma_i \) be the natural homomorphism with \( \Omega(B_r^1) = B_r^* \), \( \Omega(P_r^1) = P_r^* \), \( \Omega(\eta_1 \nu^{-1}(\pi \Sigma \alpha_i)) = \eta_2 \nu^{-1}(\pi \Sigma \alpha_i) \). We will first find \( B_r^* \).

Let \( K \) be the normal closure of \( \{ I(x_i^m) : 1 \leq i \leq r \} \) in \( B_r^1 \). We will now prove the following:

**Theorem 2.5.**

\[ B_r^* \cong B_r^1/K. \]

**Proof:**

By Maclachlan & Harvey (1975) we have:

\[ B_r^1/I(\Gamma_i) \cong \text{Aut}^* \Gamma_i/I(\Gamma_i) \cong M(0, r) \cong \text{Aut}^* \Gamma/I(\Gamma) \cong B_r^* /I(\Gamma). \]

with \( \ker \psi_1 = I(\Gamma_i) \), \( \ker \psi_2 = I(\Gamma) \). So, \( \Omega^{-1}(\ker \psi_2) = I(\Gamma_i) \). Therefore \( \ker \Omega \subset I(\Gamma_i) \). Hence, \( \ker \Omega \subset K \). Clearly, \( K \subset \ker \Omega \). Thus \( \ker \Omega = K \) proving our theorem.
The extra relations that we have to add in $B^*_r$ are those of \( \{I(x^m_i); 1 \leq i \leq r\} \) By remark 2.1, it suffices to add only:

\[
I(x^m_i) = (u_1 u_2 \cdots u_{r-1}) u_{r-2} \cdots u_1 u_{r-1} u_{r-2} \cdots u_2 u_1 = 1.
\]

Hence we have shown.

**Theorem 2.6**

If $\Gamma$ is a Fuchsian group of genus zero with $r$ equal periods, $m$, then $\text{Aut}^+ \Gamma$ is generated by $\sigma_i$, $1 < i < r - 1$, with defining relations:

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq r - 2
\]

\[
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2
\]

\[
(\sigma_1 \sigma_2 \cdots \sigma_{r-1})^r = 1
\]

\[
(\sigma_1 \sigma_2 \cdots \sigma_{r-2} \sigma_{r-1} \sigma_{r-2} \cdots \sigma_2 \sigma_1)^m = 1
\]

We will next find $\eta w^{-1} \left( \sum_{i=1}^{k} \alpha_i \right)$. Let $K$ now be the normal closure of $\{I(x^m_{k+n+1}); \sigma \leq \eta \leq k-1\}$, $\eta + 1 < \eta \leq \eta + 1$ in $\Gamma$. By a similar argument to (2.5), with the 'mapping class group' corresponding to $i = 1, \sum \alpha_i$, then $K = K$. Hence we have the following.

**Theorem 2.7**

If $\Gamma$ is a Fuchsian group with signature (2.4.), then

\[
\text{Aut}^+ \Gamma = \eta w^{-1} \left( \sum_{i=1}^{k} \alpha_i \right) \text{ is isomorphic to } \eta_1 w^{-1} \left( \sum_{i=1}^{k} \alpha_i \right) \text{ modulo } K.
\]

**Remark 2.3.**

Our problem of finding the presentation is reduced to expressing $\{I(x^m_{k+n+1}); 0 \leq n \leq k-1, \eta + 1 \leq n \leq \eta + 1 \}$ in terms of the generators of $w^{-1} \left( \sum_{i=1}^{k} \alpha_i \right)$, which depend on the signature of $\Gamma$.

**Corollary 2.4.**

If $\Gamma$ is a Fuchsian group of genus zero and all the periods are distinct, then $\text{Aut}^+ \Gamma = P^*_r$ is isomorphic to $P^1$ modulo $K$, where $K$ is the normal closure of $\{I(x^m_1); 1 \leq i \leq r, m_1 \neq m_j \text{ for all } i \neq j\}$.

**Examples**

1. $\Gamma = < x_1, x_2, x_3, x_4 | x_1 x_2 x_3 x_4 = x_1^m = 1 >$ \quad $1 \leq i < 4, m_i \neq m_i$ for $i \neq j$.

$\text{Aut}^+ \Gamma$ is generated by $A_{ij}$, $1 \leq i < j \leq 4$, with defining relations (1.2) and

\[
(A_{34} A_{24} A_{14} A_{12} A_{34}^{-1})^m = 1
\]

\[
(A_{34} A_{24} A_{14} A_{12} A_{34}^{-1})^m = 1
\]

\[
(A_{23} A_{24} A_{25} A_{35} A_{12} A_{35}^{-1} A_{45} A_{34}^{-1})^m = 1
\]

\[
(A_{23} A_{24} A_{25} A_{35} A_{12} A_{35}^{-1} A_{45} A_{34}^{-1})^m = 1
\]

$\text{Aut}^+ \Gamma$ is generated by $A_{ij}$, $1 \leq i < j \leq 5$, with defining relations (1.2) and

\[
(A_{23} A_{24} A_{25} A_{35} A_{12} A_{35}^{-1} A_{45} A_{34}^{-1})^m = 1
\]

$\text{Aut}^+ \Gamma$ is generated by $A_{ij}$, $1 \leq i < j \leq 6$, with defining relations (1.2) and

\[
(A_{23} A_{24} A_{25} A_{35} A_{12} A_{35}^{-1} A_{45} A_{34}^{-1})^m = 1
\]
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\[ A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{45}A_{35}A_{25}A_{15}A_{56} \]
\[ A_{46}A_{36}A_{26}A_{16} = 1 \]
\[ (A_{56}A_{46}A_{36}A_{26}A_{16})^m = 1 \]
\[ (A_{56}A_{46}A_{36}A_{26}A_{16}A_{12}A_{23}A_{13}A_{34}A_{24})^m = 1 \]
\[ A_{56}^{-1}A_{46}^{-1}A_{36}^{-1}A_{26}^{-1}A_{16}^{-1}A_{12}^{-1}A_{23}^{-1}A_{13}^{-1}A_{34}^{-1}A_{24}^{-1} \]
\[ (A_{34}A_{45}A_{35}A_{56}A_{46}A_{36}A_{26}A_{16}A_{12}A_{23}A_{13}A_{34}A_{24})^m = 1 \]
\[ (A_{34}A_{45}A_{35}A_{56}A_{46}A_{36}A_{26}A_{16}A_{12}A_{23}A_{13}A_{34}A_{24})^{-1}A_{56}^{-1} \]
\[ (A_{23}A_{34}A_{24}A_{45}A_{35}A_{25}A_{56}A_{46}A_{36}A_{26})^m = 1 \]

**Remarks 2.4.**

I. We are unable to find the general formulae for \( m \), since our technique is iterative. However, given a particular \( r \), one can calculate \( I(x_i) \).

II. If \( \Gamma \) is a triangle group with distinct periods, then

\[ \text{Aut}^* \Gamma = P \]

3. **PURE MAPPING CLASS GROUPS**

The mapping class group can be looked upon as the quotient group of the orientation-preserving automorphisms, \( \text{Aut}^+ \Gamma \), of a Fuchsian group, \( \Gamma \), by its normal subgroup of inner automorphisms, \( \text{Int}(\Gamma) \) (Maclachlan and Harvey, 1975). Corresponding to the Fuchsian group of genus zero with \( r \) distinct periods, we can get the pure mapping class group, denoted by \( \text{PM}(0, r) \). So much has been said in the past about the full mapping class groups, (Birman, 1974), but we cannot find much information about the pure mapping class groups.

In this section, we will give the presentations of \( \text{PM}(0, r) \), based on the calculations in the examples. The technique is to set the terms within the brackets, that is the terms with periods, equal to one, since they are either \( I(x_i) \) or \( I(x_i^{-1}) \). Then we reduce these relations to the simplified form.

**3.1. PM(0, 3) = 1.**

(Trivial form remark 2.4.)

PM(0, 4) is generated by \( A_{ij} \), \( 1 \leq i < j \leq 4 \), with defining relations (1.2) and

\[ A_{34}A_{23}A_{13} = 1 \]
\[ A_{34}A_{24}A_{14} = 1 \]
\[ A_{12}A_{34} = 1 \]
\[ A_{23}A_{34}A_{24} = 1 \]

3.3 PM(0, 5) is generated by \( A_{ij} \), \( 1 \leq i < j \leq 5 \), with defining relations (1.2) and

\[ A_{45}A_{34}A_{24}A_{14} = 1 \]
\[ A_{45}A_{35}A_{25}A_{15} = 1 \]
\[ A_{12}A_{23}A_{13}A_{45}^{-1} = 1 \]
\[ A_{34}A_{45}A_{35}A_{12} = 1 \]
\[ A_{23}A_{34}A_{24}A_{45}A_{35}A_{25}A_{15}A_{25}A_{36}A_{26} = 1 \]

3.4 PM(0, 6) is generated by \( A_{ij} \), \( 1 \leq i < j \leq 6 \), with defining relations (1.2) and

\[ A_{56}A_{45}A_{35}A_{25}A_{15} = 1 \]
\[ A_{56}A_{46}A_{36}A_{26}A_{16} = 1 \]
\[ A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{56}^{-1} = 1 \]
\[ A_{23}A_{13}A_{12}A_{46}A_{36}A_{16}^{-1}A_{56}^{-1}A_{45}^{-1} = 1 \]
\[ A_{34}A_{45}A_{35}A_{56}A_{46}A_{36}A_{12}^{-1} = 1 \]
\[ A_{23}A_{34}A_{25}A_{45}A_{35}A_{25}A_{56}A_{46}A_{36}A_{26} = 1 \]
Remarks 3.1.

If $\Gamma$ is a Fuchsian group with signature $(2,4)$, then $\text{Aut}^+ \Gamma / I(\Gamma)$ is isomorphic to the mapping class group corresponding to the symmetric group $k = \prod_{i=1}^{k} \alpha_i$.

\[ \{ (x_i): \xi n + 1 \leq i \leq \xi n+1, 0 \leq n \leq k - 1 \} \]

in terms of the generators of $\eta_1^{-1} \left( \prod_{i=1}^{k} \alpha_i \right)$, we can determine the presentation of this mapping class group. This mapping class group lies in between the pure mapping class group and the full mapping class group.

4. SEIFERT FIBRE GROUPS

Let $\Gamma$ be a Fuchsian group:

\[ \Gamma = \langle a_1, b_1, \ldots, a_g, b_g, x_1, x_2, \ldots, x_r, z \rangle \]

\[ = \prod_{i=1}^{r} x_i, \pi \prod_{i=1}^{r} [a_i, b_i] = 1 \]

Let $G$ be a central extension, by $\Gamma$, of $Z$.

\[ 1 \to (z) \to G \xrightarrow{\psi} \Gamma \to 1 \quad (4.1) \]

such that:

\[ G = \langle a_1, b_1, \ldots, a_g, b_g, x_1, x_2, \ldots, x_r, z \rangle \]

\[ m_i, n_i, \quad x_i, z \mid \quad i = 1, \]

\[ r \quad \pi \prod_{i=1}^{r} [a_i, b_i] = z^n, \]

\[ \text{where} \leftarrow \text{denotes commutativity.} \]

In Orlik's notation, (1972), we restrict ourselves to the case $0_1 : e_1 = 1$ for all $i$. If for each $i, 1 \leq i \leq r$, $(m_i, n_i)$ are relatively prime positive integers and $0 < n_i < m_i$, then $G = \pi_1(M)$, where $M$ is a Seifert manifold. We call $G$ a Seifert fibre group.

Theorem 4.1.

Let $M$ and $M'$ be large $0_1$ - Seifert manifolds. If $\phi : G' = \pi_1(M') \to G = \pi_1(M)$ is an isomorphism with $z' \to z$, then $g' = g$, $r' = r$, $m_i' = m_i$, $n_i' = n_i$, $(\lambda = 0)$ for all $i$, and $\phi(x_i') = \xi x_i' \xi_i^{-1}$, $1 \leq i \leq r$.

Corollary 4.1.

Let $M$ be a large $0_1$ - Seifert manifold with signature $\{ n; 0; (m_1, n_1), (m_2, n_2), \ldots, (m_r, n_r) \}$. Then an automorphism $A^*: G \to G$ such that $A^*(z) = z$ satisfies:

\[ A^*(x_i) = \xi x_i \xi_i^{-1}, 1 \leq i \leq r, \]

where

\[ (1 \quad 2 \quad \ldots \quad r) \quad \mu_1 \mu_2 \ldots \mu_r \]

is a permutation, $m_i = m_i' \xi_i \in G$.

Proof

Set $M' = M$ in Theorem 4.1, for $g = 0$.

We denote those automorphisms which satisfy Corollary 4.1. by $\text{Aut}^* G$, which form a subgroup of $\text{Aut} G$. We call the element $A^* \in \text{Aut}^* G$, a regular automorphism.

Theorem 4.2

Suppose $G$ and $\Gamma$ are as (4.1) and (4.2), respectively, for $g = 0$. Then $\text{Aut}^* G \cong \text{Aut}^* \Gamma$. 
AUTOMORPHISMS OF FUCHSIAN GROUPS OF GENUS ZERO

Proof

By Zieschang's theorem (1966) $A \in \text{Aut}^{+} \Gamma$ satisfies:

$$A(x) = x_{1}^{\ell}x_{2}^{\ell^{-1}}, \quad 1 \leq i < r,$$

where

$$\begin{pmatrix} 1 & 2 & \cdots & r \\ \mu_{1} & \mu_{2} & \cdots & \mu_{r} \end{pmatrix}$$

is a permutation, $m_{i} = m_{i}', \ell_{i} \in \Gamma$

Let $\psi: G \to \Gamma$. Then $\psi$ induces $\psi_{*}: \text{Aut}^{+} G \to \text{Aut}^{+} \Gamma, \psi_{*}(A^{+}) = A$ and ker $(\psi_{*})$ trivial. Hence, $\text{Aut}^{+} G \cong \text{Aut}^{+} \Gamma$

Corollary 4.2.

$\text{Out}^{+} G = \text{Aut}^{+} G/\text{I}(G) \cong \text{Aut}^{+} \Gamma/\text{I}(\Gamma)$

$\cong$ Mapping class group of a closed orientable surface $X_{0}$ of genus zero such that $X_{0} = \pi_{1}(\Gamma)$.

Proof:

Observe that $\text{I}(G) \cong G/(z) \cong \Gamma \cong \text{I}(\Gamma)$ and $\psi_{*}(\text{I}(G)) = \text{I}(\Gamma)$. Therefore, $\text{Out}^{+} G = \text{Aut}^{+} \Gamma/\text{I}(\Gamma)$ has ker $\psi_{*} = \text{I}(G)$. Hence the results follow.

REFERENCES


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