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Convergence Proof of Some Generalised Backward Differentiation (GBDF) Methods to Solve the General Second Order Ordinary Differential Equations (ODE).

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RINGKASAN

Kaedah-kaedah bagi penyelesaian masalah itlak peringkat kedua persamaan pembezaan biasa dibincangkan. Kekakuan dan penumpuan bagi masalah tersebut juga ditakrifkan. Akhir sekali dibuktikan penumpuan bagi beberapa kaedah Pembezaan Kebelakang Itlak untuk masalah-masalah kaku.

SUMMARY

Methods for the solution of the general second order ODE are discussed. Stiffness and convergence are also defined. Finally proof of convergence is given for certain cases of the GBDF methods for stiff problems.

1. INTRODUCTION

The general system of second order initial value problem is given by

$$y'' = f(x, y, y'),$$

$$y(a) = \eta,\ y'(a) = \eta',\ a < x < b. \quad (1)$$

In this discussion f is assumed to satisfy the conditions of the theorem for the existence of solutions.

The most common technique of solving (1) is to reduce it to a system of first order equations, viz:

$$z = f^*(x, z),\ z(a) = \eta^*, \quad (2)$$

where $z = [y, y']^T$,

$$f^*(x, z) = [y', f(x, z)]^T,\ \eta^* = [\eta, \eta']^T,$$

and the methods commonly being used to solve (2) are either the Adams-Moulton or the Runge-Kutta classes of methods if (2) is non-stiff or the Backward Differentiation (BDF) class of methods if it is stiff. However works by Krogh (1969, 1973) and Suleiman (1979) have shown that for many problems in (1), it is more efficient to solve them using Direct Integration (DI) methods if they are non-stiff and a combination of the DI and BDF methods which shall be called the Generalised Backward Differentiation (GBDF) methods if they are stiff.

2. STIFFNESS

Stiffness is somewhat associated with absolute stability (A-stability). The A-stability region of a method for solving the first order system in (2) is the set of values of $h\lambda_i$ in the left half of the complex plane, where h is the stepsize used and λ_i the eigenvalues of the Jacobian $\frac{\partial f^*}{\partial y}$, for which the moduli of the roots of the A-stability polynomial of the method are less than one. This is in fact the A-stability of the method to the linearised problem of (2) to

$$y' = Ay + \phi(x) \quad , \quad \text{where} \quad A = \frac{\partial f^*}{\partial y}.$$

In the same way, the A-stability of a method when solving directly the second order system

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in (1) is considered by linearising the problem to

$$y'' = By' + Ay + \phi(x)$$

where $A = \frac{\partial f}{\partial y}$ and $B = \frac{\partial f}{\partial y'}$. Here the eigenvalues λ_i are given by

$$| \lambda_i^2 I - \lambda_i B - A | = 0$$

The set of values of $\lambda_i h$ in the left half of the complex plane for which the moduli of the roots of the A-stability polynomial are less than one is again the region of A-stability. The region is only method dependent in the first order case, while Gear (1978) and Hall and Suleiman (1981) have shown that for the second order case, it is both method and problem dependent. Methods for which the A-stability region is the whole of the left half plane are said to be A-stable.

Stiffness has been defined only for a system of first order equations and there are various definitions of stiffness, but the most widely known is given in Lambert (1973), which is dependent only on the large ratio of the magnitudes of the negative real parts of the largest eigenvalue to that of the smallest one. Ehle (1972) attempts to define stiffness in terms of the stability restriction on the stepsize, thus making it method dependent, while Söderlind's (—) definition is problem, method and interval dependent. The definition given below for the second order system is more agreeable to that of Söderlind's.

Definition 1: The system in (1) is said to be stiff in the interval [a, b] if all the eigenvalues of the Jacobian $\frac{\partial f^*}{\partial y}$ have negative real parts and explicit methods are restricted by stability to relatively smally step-sizes to such an extent that (near A-stable) implicit methods would be more efficient.

A problem is either stiff or not stiff. It is mildly stiff if only small gains in efficiency are possible by switching to implicit (near A-stable) methods; severely stiff if great gains in efficiency are possible.

The single equation $y'' + 8y' + 12y = c$, c-constant, with $y(0) = 2 + \frac{c}{12}$, (3)

$$y'(0) = -8,$$

when reduce to the equivalent first order system will have eigenvalues −6, −2. Then such a system will not normally be considered stiff by Lambert's definition. However it may be stifff from a certain point along the x-axis depending on the explicit method being used. If the Adams two-step explicit method is used, then the local truncation error τ is given by

$$\tau = \frac{5}{12} y'''(x)\, h^3 + 0(h^4).$$

Normally the stepsize strategy chosen is such that

$$\mathrm{TOL} \simeq \text{(tolerance required)},$$

$$\text{i.e., } h_{ac} \cong \left(\frac{\mathrm{TOL}*2.4}{y'''(x)}\right)^{1/3} \quad (4)$$

This is the step limitation due to accuracy. Since the interval of absolute stability of the two-step Adams explicit method is (−1 , 0), therefore the stepsize limitation due to stability for the above problem is

$$h_{st} \simeq \frac{1}{|\lambda|} = 0.17, \text{ where } \lambda \text{ is the eigenvalue}$$

with the largest magnitude.

The solution to (3) is given by

$$y(x) = e^{-2x} + e^{-6x} + \frac{c}{12}$$

Hence along the point of integration for TOL = 10^{-4} at x = 1,

$$h_{ac} \simeq 0.04.$$

Since $h_{st} > h_{ac}$ the solution is still stable.

At x = 3,

$$h_{ac} \simeq 0.23,$$

and at x = 5,

$$h_{ac} \simeq 5.29. \quad (\simeq 31 \text{ times } h_{st}).$$

Hence from the point x = 3 onwards, $h_{st} < h_{ac}$, indicating stiffness. Clearly at x = 5, there is a large gain in stepsize if near A - stable method is chosen, i.e., the method becoming more stiff as x increases. For higher order methods, the point of instability are generally earlier as h_{st} are much smaller and h_{ac} larger.

The same arguments apply qualitatively, even when (3) is not reduced to the first order system, rather solved directly by the DI methods. In fact the explicit DI methods become stiff earlier than the Adams methods since their interval of absolute stability are smaller and the orders of their local truncation errors are higher for the same k.

3. DI AND GBDF METHODS

Let $P_k(x) = \sum_{i=o}^{k} \theta_i y^{(2-j)}_{n-i}$, $j = 0, 1, 2$;

be the k degree polynomial interpolating $y^{(2-j)}_{n-i}$, $i = 0, 1, \ldots, k$. For $j = 0$, $y^{(2)}_{n-i} \equiv f_{n-i}$, i.e.,

$$P_k(x) = \sum_{i=o}^{k} \theta_i f_{n-i}\ .$$

Replacing f with $P_k(x)$ in (1) and integrating twice, we have

$$\begin{aligned} y_n &= y_{n-1} + hy'_{n-1} + h^2 \sum_{i=o}^{k} \beta_i^* f_{n-i}, \\ y'_{n} &= y_n + h \sum_{i=o}^{k} \beta_i f_{n-i}, \end{aligned} \qquad (5)$$

the DI methods.

For $j = 1$, integrating $P_k(x)$ and then differentiating and equating $P'_k(x)$ with f, respectively, obtaining

$$\begin{aligned} y_n &= y_{n-1} + h \sum_{i=o}^{k} \alpha_i^* y'_{n-i}, \\ h f_n &= \sum_{i=o}^{k} \alpha_i y'_{n-i} \end{aligned} \qquad (6)$$

For $j = 2$, differentiating twice and equating $P''_k(x)$ with f, we have

$$\begin{aligned} hy'_n &= \sum_{i=o}^{k} \gamma_i^* y_{n-i}, \\ h^2 f_n &= \sum_{i=o}^{k} \gamma_i y_{n-i}\ . \end{aligned} \qquad (7)$$

The case $j = 1$ and $j = 2$ are the GBDF class of methods for stiff problems.

For any of the above methods to be of any practical use they must be convergent.

Definition 2: The above multistep methods are said to be convergent, if for all sequences $\{y_n\}$ $\{y'_n\}$ with starting values satisfying the conditions

$$\lim_{h \to o} \eta_\mu (h) = \eta\ , \quad \mu = 0, 1, \ldots, k-1,$$

$$\lim_{h \to o} \eta'_\mu (h) = \eta'\ , \quad \mu = 1, \ldots, k-1,$$

then

$$\lim_{\substack{h \to o \\ x_n = x}} y_n = y(x)$$

$$\lim_{\substack{h \to o \\ x_n = x}} y'_n = y'(x)$$

where $y(x)$ is the solution to (1).

The convergence of the DI methods have been proven in Suleiman (1979) and its stability discussed by Gear (1978) and Hall and Suleiman (1981). Henrici (1962) gives the proof of a special second order problem $y'' = f(x, y)$, while Gear (1971) gives the proof of convergence of certain methods for non-stiff higher order problems, which can be transformed into the Nordsieck's array.

4. CONVERGENCE OF THE GBDF METHODS FOR J = 1

For the case $j = 1$, the following are defined.

Let $\rho(z) = \sum_{i=o}^{k} \alpha_i z^{k-i}$ be the associated polynomial to the method in (6).

Consistency is defined in the usual way using the operator

$$\begin{aligned} \mathcal{L}\,[y(x), h] &= (\sum_{i=o}^{k} \alpha_i y'(x - ih)) - hy''(x) \\ &= c_o y'(x) + c_1 hy''(x) + c_2 h^2 y'''(x) \\ &\quad + \ldots + c_p h^p y^{(p+1)}(x) + \ldots \end{aligned}$$

where the order is p, if $c_o = c_1 = \ldots = c_p = 0$ and $c_{p+1} \neq 0$, and it is consistent if $p \geq 1$.

Henceforth in proving the theorems, we assume (1) to be a single equation. The proofs for the case when (1) is a system are done by taking norms instead of moduli.

The following theorem is the necessary conditions for convergence.

Theorem 1:

The necessary conditions for convergence of the method in (6) are that

(i) the modulus of no root of $\rho(z)$ exceeds 1 and that roots of moduli 1 are simple (condition of stability),

(ii) the method is consistent.

The proof of both the conditions (i) and (ii) follow the line of the proof in Henrici (1962) where for (i) the equation considered is $y''(x) = 0$, $y(a) = y'(a) = 0$ and considering the solution of y'_n instead of y_n. For condition (ii) the equation considered is $y''(x) = 1$, $y(a) = y'(a) = 1$ togehter with the solution of y'_n.

The following lemma is proved in Henrici (1962).

Lemma. Let the polynomial $\rho(z)$ satisfy the condition of stability, and let the coefficients a_ℓ, ($\ell = 0, 1, 2, \ldots$) be defined by

$$\frac{1}{\sum_{i=o}^{k} \alpha_i z^i} = a_o + a_1 z + a_2 z^2 + \ldots\ldots \qquad (8)$$

Then $\Gamma = \sup_{\ell=0,1,\ldots} |a_\ell| < \infty$.

There exists positive constants d_1, d_2 such that

$$\Lambda_1 = d_1 h^{k+2} \text{ and } \Lambda_2 = d_2 h^{k+1}.$$

Let $L = \max_m (|G_m|, |G'_m|)$, $A = \{ (|\alpha_o| + |\alpha_1| + \ldots + |\alpha_k|) + (|\alpha^*_o| + |\alpha^*_1| + \ldots + |\alpha^*_k|) \}$.

Theorem 2:

Let the starting values satisfy the conditions of definition 2 and further, let $|e'_i| \leq Z \leq d_3 \delta(h)$, $i = o, 1, \ldots, k\text{-}1$ and $|e_{k-1}| \leq d_4 \delta(h)$ where Z, d_3, d_4 are positive constants, $\delta(h) > 0$ and $\delta(h) \to o$ with h. Then the necessary and sufficient conditions for convergent of (6) are that it is consistent and stable.

The necessary part is proven in theorem 1. The fact that $P_k(x)$ is of degree k giving the local truncation error $o_m = c_m h^{k+1}$ implies that the order of the method is at least one for $k \geq 1$. Hence the method is consistent.

Consider the left hand side of (11) and multiply it by a_ℓ, $\ell = 0, 1, \ldots, n - k$ and $m = n - \ell$. Add the resulting equations and denote it by S_n. Using the identity in (9), we have

$$S_n = e'_n + (\alpha_o a_1 + \alpha_1 a_o)e'_{n-1} + \ldots + (\alpha_o a_{n-k} + \alpha_1 a_{n-k-1} + \ldots + \alpha_k a_{n-2k})e'_k + (\alpha_1 a_{n-k} + \ldots + \alpha_o a_{n-2k+1}) e'_{k-1} + \ldots + \alpha_k a_{n-k} e'_o.$$

Then,

$$| S_n - e'_n | \leq \Gamma A Z k. \qquad (13)$$

From the right hand side of (11), we have

$$S_n = h(\sum_{m=k}^{n} G_m e_m a_{n-m} + \sum_{m=k}^{n} G'_m e'_m a_{n-m} + \sum_{m=k}^{n} \delta_m a_{n-m}). \qquad (14)$$

From (10), for $m = n, n-1, \ldots, k$ add the resulting equations,

$$\therefore | e_n | = | e_{k-1} + h[\alpha^*_c e'_n + (\alpha^*_1 + \alpha^*_o) e'_{n-1} + \ldots + (\alpha^*_{k-1} + \alpha^*_{k-2} + \ldots + \alpha^*_o)e'_{n-k+1} + (\sum_{i=o}^{k} \alpha^*_i)e'_{n-k} + \ldots + (\sum_{i=o}^{k} \alpha^*_i)e'_k + \sum_{i=o}^{k-1} \alpha^*_i e'_{k-i}] + \sum_{m=k}^{n} \delta'_m |$$

$$\leq | e_{k-1} | + hA \sum_{m=o}^{n} | e'_m | + \Lambda_1 n. \qquad (15)$$

$$\therefore \sum_{m=n}^{k} | e_m | \leq n (|e_{k-1}| + hA \sum_{m=o}^{n} |e'_m| + \Lambda_1 n).$$

$$\therefore \left| \sum_{m=k}^{n} G_m e_m a_{n-m} \right| \leq L\Gamma \sum_{m=k}^{n} |e_m| \leq L\Gamma n(|e_{k-1}| + hA \sum_{m=o}^{n} |e'_m| + \Lambda_1 n). \qquad (16)$$

$$\left| \sum_{m=k}^{n} G'_m e'_m a_{n-m} \right| \leq L\Gamma \sum_{m=o}^{n} | e'_m |. \qquad (17)$$

$$\left|\sum_{m=k}^{n} \delta_m \, a_{n-m}\right| \leq \Gamma\Lambda_2 n . \quad (18)$$

Equating (12) and (14); and using the inequalities (13), (16), (17) and (18), we have

$$|e'_n| < \Gamma AZk + h\,[L\Gamma n(|e_{k-1}| + hA\sum_{m=o}^{n}|e'_m| + \Lambda_1 n)$$

$$+ L\Gamma\sum_{m=o}^{n}|e'_m| + \Gamma\Lambda_2 n] .$$

Bringing the terms involving $|e'_n|$ from the right to the left and simplifying,

$$|e'_n| \leq R\sum_{m=o}^{n-1} e'_m + K^* , \quad (19)$$

where $R = \dfrac{h^2 nAL\Gamma + hL\Gamma}{B} = \dfrac{h(x_n - a)AL\Gamma + hL\Gamma}{B}$,

$$K^* = \frac{\Gamma AZk + (x_n - a)L\Gamma|e_{k-1}|}{B} + \frac{h^{-1}L\Gamma\Lambda_1(x_n - a)^2 + \Gamma\Lambda_2(x_n - a)}{B},$$

$$B = (1 - hL\Gamma - hL\Gamma A\,(x_n - a)) ,$$

$$nh = (x_n - a), \text{ fixed.}$$

We now proceed by induction. Since $A\Gamma \geq 1$ and hence $K^* \geq Z$, therefore the inequality $|e'_m| \leq K^*(1 + R)^m$ is true for $m = 0, 1, \ldots, k-1$. Assuming its truth for $m = 0, 1, \ldots, n-1$ and using on the right of (19, we have

$$|e'_n| \leq RK^* \frac{(1 + R)^n - 1}{R} + K^*$$

$$= K^*(1 + R)^n$$

Therefore $|e'_n| \leq K^* \exp(nR) = K^* \exp((x_n - a)h^{-1}R)$

$$\exp((x_n - a)h^{-1}R) \rightarrow \text{a constant as } h \rightarrow 0,$$

but $K^* \rightarrow 0$ as $h \rightarrow 0$,

Therefore $|e'_n| \rightarrow 0$ as $h \rightarrow 0$, implying convergence of e'_n. Convergence of e'_n implies

$$e_m = K_m\,\chi(h) \text{ , where } \chi(h) > 0 \text{ and } \chi(h) \rightarrow 0 \text{ as } h \rightarrow 0.$$

Therefore, $|e'_m| \leq K\chi(h)$ for some K and $\forall$ m.

From inequality (15), we have,

$$|e_n| \leq |e_{k-1}| + hAKn\chi(h) + \Lambda_1 h^{-1}(x_n - a).$$

Since $hn = (x_n - a)$ is fixed,

we have $|e_n| \rightarrow 0$ as $h \rightarrow 0$, implying convergence of e_n. Hence the GBDF method for $j = 1$ is convergent.

The condition $f(x, y)$ being continuous is actually enough for the canvergence of (6). Then we could show that by the condition of consistency δ_m and δ'_m are at least of $0(h\,\delta(h))$.

5. CONVERGENCE OF A SPECIAL CASE OF THE GBDF METHOD FOR J = 2

We now consider only a special case of (7) for $k = 2$ where the method is reduced to

$$y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = \frac{2}{3}hy'_n . \quad (20a)$$

$$y_n - 2y_{n-1} + y_{n-2} = h^2 f(x_n, y_n, y'_n). \quad (20b)$$

Again if e_n, e'_n are the global errors of y_n, y'_n. respectively, then

$$e_n - \frac{4}{3}e_{n-1} + \frac{1}{3}e_{n-2} = \frac{2}{3}he'_n + \delta_{n-2} , \quad (21a)$$

$$e_n - 2e_{n-1} + e_{n-2} = h^2(G_n e_n + G'_n e'_n) + \bar{\delta}_{n-2} , \quad (21b)$$

where G_n, G'_n as defined before,

$$|\delta_{n-2}| \leq \theta_{n-2}\,Kh^2\chi(h) , \ |\bar{\delta}_{n-2}| \leq \bar{\theta}_{n-2}\bar{K}h^2\chi(h),$$

$$\chi(h) > 0, \ \lim_{h\rightarrow o}\chi(h) = 0 , \ 0 < \theta_{n-2} \leq 1, 0 < \bar{\theta}_{n-2} \leq 1$$

and δ_{n-2}, $\bar{\delta}_{n-2}$ are the local truncation errors of (20a, b) respectively, assuming only $y(x) \in C^2[a, b]$.

Theorem 3:

If the starting values satisfy the conditions of definition 2 and further, let $e_o = \delta_o(h)$, $e_1 = \delta_1(h)$, $e_1 - e_o = h\delta_2(h)$ where $\delta_o(h)$, $\delta_1(h)$, $\delta_2(h)$ tends to zero with h, then the method defined in (20a, b) for solving (1) is convergent.

Proof: Solving the difference equation (21a) gives

$$e_n = \frac{3}{2}e_1 - \frac{1}{2}e_o + \frac{3}{2}(e_o - e_1)(\frac{1}{3})^n + \frac{1}{1-t}((1-t)\mu_{n-2} + (1-t^2)\mu_{n-3} + \ldots + (1-t^{n-1})\mu_o), \quad (22)$$

where $\mu_i = \frac{2}{3}he'_{i+2} + \delta_i$ and $t = \frac{1}{3}$.

Using (22) on the left hand side of (21b) gives

$$S_n = \frac{3}{2}(e_o - e_1)((\frac{1}{3})^n - 2(\frac{1}{3})^{n-1} + (\frac{1}{3})^{n-2}) + \frac{1}{1-t}\{(1-t)\mu_{n-2} + [(1-t^2) - 2(1-t)]\mu_{n-3} + \{(1-t^3) - 2(1-t^2) + (1-t)\}\mu_{n-4} + \ldots + \{(1-t^{n-1}) - 2(1-t^{n-2}) + (1-t^{n-3})\}\mu_o\};$$

Simplifying and substituting for μ_i, δ_i and $(e_1 - e_o)$ we get

$$S_n = -6(\frac{1}{3})^n h\delta_2(h) + \frac{2}{3}h\{e'_n - (1-t)e'_{n-1} - t(1-t)e'_{n-2} \ldots - t^{n-3}(1-t)e'_2\} + Kh^2\chi(h)w, \quad (23)$$

where $w = (\theta_{n-2} - (1-t)\theta_{n-3} - t(1-t)\theta_{n-4} \ldots - t^{n-3}(1-t)\theta_o)$. (24)

Using (22) on the right hand side of (21b) and substituting $\bar{\mu}_i$ and $\bar{\delta}_i$ gives

$$S_n = h^2 G_n\{\frac{3}{2}e_1 - \frac{1}{2}e_o - \frac{3}{2}(e_1 - e_o)(\frac{1}{3})^n\} + \frac{\frac{2}{3}h^3G_n\{(1-t)e'_n + (1-t^2)e'_{n-1} + \ldots + (1-t^{n-1})e'_2\}}{1-t} + \frac{h^4G_nK\chi(h)w^*}{1-t} + h^2G'_ne'_n + \bar{\theta}_{n-2}\bar{K}h^2\chi(h), \quad (25)$$

where $w^* = \{(1-t)\theta_{n-2} + (1-t^2)\theta_{n-3} + \ldots + (1-t^{n-1})\theta_o\}$. (26)

Multiplying (23) and (25) by $\frac{3}{2}$ and equating them, gives

$$e'_n = (1-t)e'_{n-1} + t(1-t)e'_{n-2} + \ldots + t^{n-3}(1-t)e'_2 + \frac{h^2G_n((1-t)e'_n + (1-t^2)e'_{n-1} + \ldots + (1-t^{n-1})e'_2\}}{1-t} + \frac{3}{2}hG'_ne'_n + \frac{i}{3^{n-2}}\delta_2(h) + \frac{3}{2}hG_n\{\frac{3}{2}e_1 - \frac{1}{2}e_o - \frac{3}{2}(e_1 - e_o)(\frac{1}{3})^n\} + \frac{\frac{3}{2}G_nKh^3\chi(h)}{1-t}w^* - \frac{3}{2}Kh\chi(h)w + \frac{3}{2}\bar{\theta}_{n-2}\bar{K}h\chi(h). \quad (27)$$

From (24),

$$|w| \leq \{1 + (1-t) + t(1-t) + \ldots + t^{n-3}(1-t)\} = \{1 + (1-t)\frac{(1-t^{n-2})}{(1-t)}\} = (2 - t^{n-2})$$

Hence $|w| \leq 2$.
From (26), $|w^*| \leq n$. (28)

Hence from (27) and using (28)

$$|e'_n| \leq \frac{2}{3}\{|e'_{n-1}| + \frac{1}{3}|e'_{n-2}| + \ldots + \frac{1}{3^{n-3}}|e'_2|\} + \frac{3}{2}hL|e'_n| + \frac{3}{2}h^2L\sum_{i=2}^{n}|e'_i| + K^*, \quad (29)$$

where $K^* = \frac{1}{3^{n-2}}|\delta_2(h)| + \frac{3}{2}hL\{\frac{3}{2}|\delta_1(h)| + \frac{1}{2}|\delta_o(h)| + \frac{3}{2}h|\delta_2(h)|(\frac{1}{3})^n\} + \frac{9}{4}LKh^3\chi(h)n + 3Kh\chi(h) + \frac{3}{2}\bar{K}h\chi(h)$

and $L = \max_n\{|G_n|, |G'_n|\}$.

Let $|E'_i| = \max\{|e'_2|, \ldots, |e'_i|\}$, then for sufficiently small h (29) becomes,

$$A^*|e'_n| \leq B^*E'_{n-1} + K^*, \quad (30)$$

where $A^* = (1 - \frac{3}{2}(h^2L + hL))$ and

$B^* = (1 + \frac{3}{2}Lnh^2)$.

Since nh is fixed, hence $\frac{B^*}{A^*} \leq 1 + Dh$ for some $D > 0$.

Hence from (30),

$$E'_n \leq (1 + Dh)E'_{n-1} + \frac{K^*}{A^*},$$

i.e. $E'_n \leq (1 + Dh)^{n-2} (|e'_2| + \frac{K^*}{A^*}) + \frac{K^*}{A^*}$

Since $\frac{K^*}{A^*} \to 0$, $|e'_2| \to 0$ with h and nh fixed,

$E'_n \to 0$ i.e. the method of (20a,b) is convergent in y'.

Finally we prove the method of (20a, b) is convergent in y.

Writing $\bar{\mu}_{n-2} = \bar{\delta}_{n-2} + h^2 G'_n e'_n$ then (21b) gives

$$e_n - 2e_{n-1} + e_{n-2} = h^2 G_n e_n + \bar{\mu}_{n-2}$$

$$2e_{n-1} - 4e_{n-2} + 2e_{n-3} = 2h^2 G_{n-1} e_{n-1} + 2\bar{\mu}_{n-3}$$

$$3e_{n-2} - 6e_{n-3} + 3e_{n-4} = 3h^2 G_{n-2} e_{n-2} + 3\bar{\mu}_{n-4}$$

$$\vdots \qquad\qquad \vdots$$

$$(n-1)e_2 - 2(n-1)e_1 + (n-1)e_o$$
$$= (n-1)h^2 G_2 e_2 + (n-1)\bar{\mu}_o.$$

Adding the left hand side, we have

$$\bar{S}_n = e_n - ne_1 + (n-1)e_o$$
$$= e_n - n(e_1 - e_o) - e_o. \qquad (31)$$

The sum of the right hand side

$$\bar{S}_n \leq nh^2 L \sum_{i=2}^{n} |e_i| + n \sum_{i=o}^{n-2} |\mu_i|. \qquad (32)$$

Hence (31) and (32) implies,

$$|e_n| \leq Bh \sum_{i=o}^{n-1} |e_i| + \hat{K}, \qquad (33)$$

where $B \geq r\,n\,h\,L$, nh fixed, $r = \frac{1}{1 - nh^2 L}$,

$$\hat{K} = E^*_o + nr \sum_{i=o}^{n-2} |\bar{\mu}_i|. \qquad (34)$$

$$E^*_o \geq \max \{ r(|\delta_o(h)| + nh\,|\delta_2(h)|), \delta_1(h) \}$$

We now proceed by induction. Since $E^*_o \geq \max \{\delta_o(h), \delta_1(h)\}$ therefore

$$|e_m| \leq \hat{K}(1 + Bh)^m$$

is true for m = 0 , 1.

Assuming its truth for m = 0 , 1 , . . . , n-1 and using on the right of (33), again it is easily seen that

$$|e_n| \leq \hat{K}(1 + Bh)^n.$$

Hence $|e_n| \leq \hat{K}\, e^{nhB}$

Now, $\bar{\mu}_i = \bar{\delta}_i + h^2 G'_{i+2} e'_{i+2}$

$$= h^2(\bar{\theta}_i \bar{K}\chi(h) + G'_{i+2} e'_{i+2})$$

$$\leq b_i h^2 \chi(h)$$, b_i is some positive constant

Let $b^* = \max \{b_o, b_1, \ldots, b_{n-2}\}$,

then $nr \sum_{i=o}^{n-2} \mu_i \leq r\,n^2 h^2 b^* \chi(h) \to 0$ with h and nh fixed

Since $E^*_o \to 0$, hence from (34) we have $\hat{K} \to 0$.

Hence $|e_n| \to 0$, implying the method defined in (20a, b) is convergent.

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