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A Geometric Look At Repeated Measures Design with Missing Observations

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RINGKASAN

Di dalam kertas ini kami akan memberi gambaran geometri bagi Rekabentuk Sukatan Berulang untuk bilangan subjek yang tak sama serawatan yang mempunyai kehilangan cerapan. Untuk pembentukan geometri, kami menghadkan rekabentuk ini kepada tiga tahap bagi faktor A dan empat tahap bagi faktor B. Tujuan kertas ini ialah untuk membentuk ujian statistik bagi hipotesis yang dikehendaki iaitu tiada kesan utama A, tiada kesan utama B, dan tiada tindakan bersaling AB.

SUMMARY

In this paper, we will provide a geometric view of Repeated Measures Design for unequal number of subjects per treatment that has missing observations. For our geometric development we restrict our design to three levels of factor A and four levels of factor B. The purpose of this paper is to develop a test statistics for hypotheses of interest i.e. no main effect A, no main effect B, and no AB interaction.

1. INTRODUCTION

The data for a two-factor Repeated Measures Design is collected and tabulated in a data table as shown in Figure 1. Let Y_{ijk} be the measurement made on subject i ($1 \leq i \leq n_j$) at level j ($1 \leq j \leq a$) of factor A and level k ($1 \leq k \leq b$) of factor B. For illustrative purposes, we let a=3, b=4, $n_1=3$, $n_2=2$, $n_3=4$.

	B_1	B_2	B_3	B_4
A_1	Y_{111}	Y_{112}	Y_{113}	Y_{114}
	Y_{211}	Y_{212}	Y_{213}	Y_{214}
	Y_{311}	Y_{312}	Y_{313}	Y_{314}
A_2	Y_{121}	Y_{122}	Y_{123}	Y_{124}
	Y_{221}	Y_{222}	Y_{223}	Y_{224}
A_3	Y_{131}	Y_{132}	Y_{133}	Y_{134}
	Y_{231}	Y_{232}	Y_{233}	Y_{234}
	Y_{331}	Y_{332}	Y_{333}	Y_{334}
	Y_{431}	Y_{432}	Y_{433}	Y_{434}

Figure 1: Data table for observations.

We arbitrarily set the observations Y_{113}, Y_{312}, Y_{123}, Y_{232}, Y_{233} and Y_{334} as missing. We model our experiment as:

$$Y_{ijk} = U_{jk} + S_{ij} + E_{ijk} \qquad (1.1)$$

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TABLE 1
Set of vectors that span the cell means space, C

w_{11}	w_{12}	w_{13}	w_{14}	w_{21}	w_{22}	w_{23}	w_{24}	w_{31}	w_{32}	w_{33}	w_{34}
1	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1

where $\{ S_{ij}, E_{ijk} \}$ are 9+30 = 39 mutually independent normal random variables each having mean zero, with $Var(S_{ij}) = \sigma_S^2$, $Var(E_{ijk}) = \sigma_E^2$. Alternatively we can write the model as

$$Y_{ijk} \text{ is } N(U_{jk}, \sigma_S^2 + \sigma_E^2) \qquad (1.2)$$

and

$$Cov(Y_{ijk}, Y_{i'j'k'}) = \begin{cases} 0 & i \neq i' \text{ or } j \neq j' \\ \sigma_E^2 & i=i' \text{ and } j=j' \text{ and } k \neq k' \end{cases}$$

2. GEOMETRIC DEVELOPMENT

The observational vector is written as:

$Y_{ij} = [Y_{ij1}, Y_{ij2}, Y_{ij3}, Y_{ij4}]'$ for ij=21.22,13,43

$Y_{11} = [Y_{111}, Y_{112}, Y_{114}]'$

$Y_{12} = [Y_{121}, Y_{122}, Y_{124}]'$

$Y_{23} = [Y_{231}, Y_{234}]'$

$Y_{31} = [Y_{311}, Y_{313}, Y_{314}]'$

$Y_{33} = [Y_{331}, Y_{332}, Y_{333}]'$

$Y = [Y'_{11}, Y'_{21}, Y'_{31}, Y'_{12}, Y'_{22}, Y'_{13}, Y'_{23}, Y'_{33}, Y'_{43}]'$

Y is a vector in the Euclidean space with dimension 30, R^{30}.

The cell means vector is written as:

$E(Y) = \sum_{j=1}^{3} \sum_{k=1}^{4} U_{jk} w_{jk}$ where w_{jk} is defined in Table 1

The set of w_{jk} vectors form a basis for the cell means space, C, having dimension 12. If we parameterized $U_{jk} = U + a_j + b_k + (ab)_{jk}$ subjected to the conditions

$$\sum_j a_j = \sum_k b_k = \sum_j (ab)_{jk} = \sum_k (ab)_{jk} = 0$$

then the cell means space, C, has a basis the set of vectors $\{ 1_{30}, a_1, a_2, b_1, b_2, b_3, (ab)_{11}, (ab)_{12}, (ab)_{13}, (ab)_{21}, (ab)_{22}, (ab)_{23} \}$ as defined in Table 2.

$Var(Y) = \sigma_E^2 I + \sigma_S^2 J$ where I is the n.. x n.. identity matrix and J is a matrix defined in Table 3.

We now define subspaces of R^{30} which facilitate the construction of test statistics. Let

$A = span \{ a_1, a_2 \}$,

$B = span \{ b_1, b_2, b_3 \}$,

$AB = span \{ (ab)_{11}, (ab)_{12}, (ab)_{13}, (ab)_{21}, (ab)_{22}, (ab)_{23} \}$,

'Within subject space', $W_S = span \{ s_{11}, s_{21}, s_{31}, s_{12}, s_{22}, s_{13}, s_{23}, s_{33}, s_{43} \}$ where s_{ij}'s are defined in Table 4, and

$T = W_S \boxplus B \boxplus AB$.

T is the smallest subspace containing both C and W_S. We defined the Error space, E, as the space orthogonal both to C and W_S i.e. $E = span \{ e_1, e_2 ,..., e_{12} \}$ where e's are defined in Table 5.

3. HYPOTHESES TESTING

The hypotheses of interest are:

H_A: $U_{j.} = U_{j'.}$

H_B: $U_{.k} = U_{.k'}$

H_{AB}: $U_{jk} - U_{j'k} = U_{jk'} - U_{j'k'}$

In general, when there are missing observations for subjects an exact test of H_A is not available.

Why not have an exact test for H_A?

The hypothesis for no main effect A in U_{jk} is

H_A: $U_{j.} = U_{j'.}$

$\iff a_j = 0 \iff K_A'U = 0 \iff G_A'E(Y) = 0 \iff E(Y) \subset W_A = 1_{30} \boxplus B \boxplus AB$ (3.1)

(K_A, U, and G_A are defined as in Table 6).

To assure a central F distribution when H_A is true, we need the numerator space for calculating Sum of squares for A, N_A, to be orthogonal to W_A.

If we want N_A, to be orthogonal to W_S also we would define $N_A = T \ominus [B \boxplus AB \boxplus W_S]$.

But $T = [B \boxplus AB \boxplus W_S]$, therefore, $N_A = \{ \ \}$ and we do not have test statistics. If we want $N_A \perp W_S$ (as in the case when all observations on a subject are present), note that

TABLE 2
After reparamaterization, alternative basis for C

1_{30}	a_1	a_2	b_1	b_2	b_3	$(ab)_{11}$	$(ab)_{12}$	$(ab)_{13}$	$(ab)_{21}$	$(ab)_{22}$	$(ab)_{23}$
1	1	0	1	0	0	1	0	0	0	0	0
1	1	0	0	1	0	0	1	0	0	0	0
1	1	0	−1	−1	−1	−1	−1	−1	0	0	0
1	1	0	1	0	0	1	0	0	0	0	0
1	1	0	0	1	0	0	1	0	0	0	0
1	1	0	0	0	1	0	0	1	0	0	0
1	1	0	−1	−1	−1	−1	−1	−1	0	0	0
1	1	0	1	0	0	1	0	0	0	0	0
1	1	0	0	0	1	0	0	1	0	0	0
1	1	0	−1	−1	−1	−1	−1	−1	0	0	0
1	0	1	1	0	0	0	0	0	1	0	0
1	0	1	0	1	0	0	0	0	0	1	0
1	0	1	−1	−1	−1	0	0	0	−1	−1	−1
1	0	1	1	0	0	0	0	0	1	0	0
1	0	1	0	1	0	0	0	0	0	1	0
1	0	1	0	0	1	0	0	0	0	0	1
1	0	1	−1	−1	−1	0	0	0	−1	−1	−1
1	−1	−1	1	0	0	−1	0	0	−1	0	0
1	−1	−1	0	1	0	0	−1	0	0	−1	0
1	−1	−1	0	0	1	0	0	−1	0	0	−1
1	−1	−1	−1	−1	−1	1	1	1	1	1	1
1	−1	−1	1	0	0	−1	0	0	−1	0	0
1	−1	−1	−1	−1	−1	1	1	1	1	1	1
1	−1	−1	1	0	0	−1	0	0	−1	0	0
1	−1	−1	0	1	0	0	−1	0	0	−1	0
1	−1	−1	0	0	1	0	0	−1	0	0	−1
1	−1	−1	1	0	0	−1	0	0	−1	0	0
1	−1	−1	0	1	0	0	−1	0	0	−1	0
1	−1	−1	0	0	1	0	0	−1	0	0	−1
1	−1	−1	−1	−1	−1	1	1	1	1	1	1

TABLE 3
J Matrix

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

TABLE 4

A basis for the within subject space, W_S

s_{11}	s_{21}	s_{31}	s_{12}	s_{22}	s_{13}	s_{23}	s_{33}	s_{43}
1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1

TABLE 5

A basis for the error space, E

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
2	2	1	0	0	0	0	0	0	0	0	0
–1	0	0	0	0	0	0	0	0	0	0	0
–1	-2	-1	0	0	0	0	0	0	0	0	0
–1	–1	-2	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	–1	0	0	0	0	0	0	0	0
–1	1	2	0	0	0	0	0	0	0	0	0
–1	–1	1	–1	0	0	0	0	0	0	0	0
–1	0	0	1	0	0	0	0	0	0	0	0
2	1	–1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	–1	0	0	0	0	0	0
0	0	0	0	–1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	–1	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	–1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	2	0	1	0	0
0	0	0	0	0	0	0	–2	1	0	0	0
0	0	0	0	0	0	–1	0	–1	–1	0	–1
0	0	0	0	0	0	1	0	0	0	0	–1
0	0	0	0	0	0	–1	0	0	0	0	1
0	0	0	0	0	0	–2	0	0	0	1	0
0	0	0	0	0	0	2	–1	0	0	–1	0
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	–1	0
0	0	0	0	0	0	–2	–1	0	–1	1	0
0	0	0	0	0	0	0	0	–1	0	0	0
0	0	0	0	0	0	2	0	1	1	0	0

TABLE 6

The matrices used in formulating H_A

$G_A =$

1/3	1/3
1/2	1/2
1/3	1/3
1/3	1/3
1/2	1/2
1/2	1/2
1/3	1/4
1/3	1/3
1/2	1/2
1/3	1/3
−1/2	0
−1/2	0
−1/2	0
−1/2	0
−1/2	0
−1	0
−1/2	0
0	−1/4
0	−1/3
0	−1/3
0	−1/3
0	−1/4
0	−1/3
0	−1/4
0	−1/3
0	−1/3
0	−1/4
0	−1/3
0	−1/3
0	−1/3

$K_A =$

1	1
1	1
1	1
1	1
−1	0
−1	0
−1	0
−1	0
0	−1
0	−1
0	−1
0	−1

$U =$

U_{11}
U_{12}
U_{13}
U_{14}
U_{21}
U_{22}
U_{23}
U_{24}
U_{31}
U_{32}
U_{33}
U_{34}

TABLE 7

The spanning vectors for $S = W_S \ominus (1_{30} \boxplus A)$

S_1	S_2	S_3	S_4	S_5	S_6
1/3	1	0	0	0	0
1/3	1	0	0	0	0
1/3	1	0	0	0	0
−1/4	0	0	0	0	0
−1/4	0	0	0	0	0
−1/4	0	0	0	0	0
−1/4	0	0	0	0	0
0	−1	0	0	0	0
0	−1	0	0	0	0
0	−1	0	0	0	0
0	0	1/3	0	0	0
0	0	1/3	0	0	0
0	0	1/3	0	0	0
0	0	−1/4	0	0	0
0	0	−1/4	0	0	0
0	0	−1/4	0	0	0
0	0	−1/4	0	0	0
0	0	0	1/4	1/4	1
0	0	0	1/4	1/4	1
0	0	0	1/4	1/4	1
0	0	0	1/4	1/4	1
0	0	0	−1/2	0	0
0	0	0	−1/2	0	0
0	0	0	0	−1/3	0
0	0	0	0	−1/3	0
0	0	0	0	−1/3	0
0	0	0	0	0	−1
0	0	0	0	0	−1
0	0	0	0	0	−1
0	0	0	0	0	−1

in general, there may not be any vectors in W_S that are orthogonal to W_A (although the last vector, s_6, in Table 7 is in W_S and orthogonal to W_A in our case). In addition, $J = 2P_{M_2} + 3P_{M_3} + 4P_{M_4}$ behaves differently on different vectors in W_S. Thus no exact test is available for testing H_A.

The hypothesis for no main effect B in U_{jk} is $H_B : U_{.k} = U_{.k'}$

$$\iff b_k = 0 \iff K_B' U = 0 \iff G_B' E(Y) = 0 \iff E(Y) \subseteq W_B = 1_{30} \boxplus A \boxplus AB \quad (3.2)$$

(K_B, and G_B are defined in Table 8).

Now $G_B \subset C \subset T$ and $G_B \perp W_B$ but $G_B \not\perp W_S$. Therefore, we take our numerator spaces as $N_B = T \ominus [W_S \boxplus AB]$. Then $N_B \perp W_B$ and

TABLE 8

The matrices used in formulating H_B

$G_B =$

1/3	1/3	1/3
1/2	0	0
0	0	−1/3
1/4	1/3	1/3
−1/2	0	0
0	−1/2	0
0	0	−1/3
1/3	1/3	1/3
0	−1/2	0
0	0	−1/3
1/2	1/2	1/2
−1/2	0	0
0	0	−1/2
1/2	1/2	1/2
−1/2	0	0
0	−1	0
0	0	−1/2
1/4	1/4	1/4
−1/3	0	0
0	−1/3	0
0	0	−1/3
1/4	1/4	1/4
0	0	−1/4
1/4	1/4	1/4
−1/3	0	0
0	−1/3	0
1/4	1/4	1/3
−1/3	0	0
0	−1/3	0
0	0	−1/3

$K_B =$

1	1	1
−1	0	0
0	−1	0
0	0	−1
1	1	1
−1	0	0
0	−1	0
0	0	−1
1	1	1
−1	0	0
0	−1	0
0	0	−1

$N_B \perp W_S$. The sum of squares for B is defined as $Y'P_{N_B}Y$ (the transpose of vector Y is to multiply the orthogonal projection onto the N_B space and then multiply again to the vector Y). To see how sum of squares of B is distributed we let $\{ b_j \}$ be the orthonormal basis for N_B. Then

$$Y'P_{N_B} Y = \sum_{j=1}^{b-1} (b_j'Y)^2 \qquad (3.3)$$

We note that $b_j'Y$ is distributed as a Normal random variable with mean $b_j'E(Y)$, variance $b_j' [\sigma_E^2 I + \sigma_S^2 J] b_j = \sigma_E^2$ since $Jb_j = 0$, and $Cov(b_j'Y, b_{j'}Y) = 0$ for $j \neq j'$. If we divide $b_j'Y$ by σ_E, then the result is a Normal random variable with mean $b_j'E(Y)$ and variance 1. Therefore, $Y'P_{N_B} Y/\sigma_E^2$ is $\chi^2(b\text{-}1,\lambda)$ (3.4)

It may appear that we are testing the hypothesis $N_B'E(Y) = 0$. However, we show below that $E(Y) \perp G_B$ if and only if $E(Y) \perp N_B$.

To show: $E(Y) \perp G_B \Longleftrightarrow E(Y) \perp N_B$

Note that $E(Y) \subset C$. Let $v \in C$. Then $v \perp G_B$ if and only if $v \perp N_B$.

Proof:

(only if) Let $v \perp G_B$, then $v \in W_B$. Since $N_B \perp W_B$ by defination, then $v \perp N_B$.

(if) Let $S = W_S \ominus (1_{30} \boxplus A)$. Then $S = span \{ s_1, s_2, ..., s_6 \}$ and S is linearly independent of C. Let $v \in T$ and $v \perp N_B$. Then $v \in AB \boxplus W_S = 1_{30} \boxplus A \boxplus AB \boxplus S$. Therefore, $v = kl_{30} + a + w + s$ where $kl_{30} \in 1_{30}$, $a \in A$, $w \in AB$, and $s \in S$. If $v \in C$, then $s = 0$ and $v \in W_B \Longrightarrow v \in G_B$.

The hypothesis for no AB interaction in U_{jk} is

$H_{AB} : U_{jk} - U_{j'k} = U_{jk'} - U_{j'K'}$ $\Longleftrightarrow (ab)_{jk} = 0 \Longleftrightarrow K_{AB}'U = 0 \Longleftrightarrow G_{AB}'E(Y) = 0 \Longleftrightarrow E(Y) \in W_{AB} = 1_{30} \boxplus A \boxplus B$ (3.5)

(K_{AB}, and G_{AB} are defined in Table 9).

As in the previous case, $G_{AB} \subset C \subset T$ and $G_{AB} \perp W_{AB}$ but $G_{AB} \perp W_S$. We will take $N_{AB} = T \ominus [W_S \boxplus B]$ as the numerator space. Now $N_{AB} \perp W_{AB}$ and $N_{AB} \perp W_S$.

The sum of square for AB is defined as $Y'P_{N_{AB}} Y$.

Let $\{ m_k \}$ be an orthonormal basis for N_{AB}. Then

$$Y'P_{N_{AB}} Y = \sum_{k=1}^{(a-1)(b-1)} (m_k'Y)^2 \qquad (3.6)$$

TABLE 9

The matrices used in formulating H_{AB}

$$G_{AB} = \begin{bmatrix}
1/3 & 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \\
-1/2 & 0 & 0 & -1/2 & 0 & 0 \\
0 & 0 & -1/3 & 0 & 0 & -1/3 \\[6pt]
1/3 & 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \\
-1/2 & 0 & 0 & -1/2 & 0 & 0 \\
0 & -1/2 & 0 & 0 & -1/2 & 0 \\
0 & 0 & -1/3 & 0 & 0 & -1/3 \\[6pt]
1/3 & 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \\
0 & -1/2 & 0 & 0 & -1/2 & 0 \\
0 & 0 & -1/3 & 0 & 0 & -1/3 \\[6pt]
-1/2 & -1/2 & -1/2 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 \\[6pt]
-1/2 & -1/2 & -1/2 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 \\[6pt]
0 & 0 & 0 & -1/4 & -1/4 & -1/4 \\
0 & 0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/3 \\[6pt]
0 & 0 & 0 & -1/4 & -1/4 & -1/4 \\
0 & 0 & 0 & 0 & 0 & 1/3 \\[6pt]
0 & 0 & 0 & -1/4 & -1/4 & -1/4 \\
0 & 0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 \\[6pt]
0 & 0 & 0 & -1/4 & -1/4 & -1/4 \\
0 & 0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/3
\end{bmatrix}
\qquad
K_{AB} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

With similar reasoning, $m_k'Y$ is $N\ (m_k'E(Y),\ \sigma_E^2)$ and Cov $(m_k'Y, m_{k'}'Y) = 0$ for $k \neq k'$. Therefore, $Y'P_{N_{AB}}Y/\ \sigma_E^2$ is $\chi^2(\ (a\text{-}1)(b\text{-}1),\ \lambda\ )$

(3.7)

The noncentrality parameter is due to the non-zero mean in the Normal random variable and we can define it as

$$\lambda = E(Y)'P_{N_X}E(Y)\ /\ 2\ \sigma_E^2 \quad \text{for} \quad X = B\ ,\ AB$$

Both hypotheses share the same sum of square for Error. Let $\{ e_i \}$ be an orthonormal basis for E.

Then $$Y'P_E\ Y = \sum_{i=1}^{t\text{-}ab\text{-}(n.\text{-}a)} (e_i'Y)^2 \qquad (3.8)$$

where t is the dimension of the observational space, $n. = n_1 + n_2 + n_3$, $e_i'Y$ is $N(0, \sigma_E^2)$. $Cov(e_i'Y, e_{i'}'Y) = 0$ for $i \neq i'$, $Cov(e_i'Y, b_j'Y) = 0$, and $Cov(e_i'Y, m_k'Y) = 0$.

Therefore, $Y'P_E Y$ is $\chi^2_{(\dim E)}$ (3.9)

The test statistics for H_x is

$$W = \frac{Y'P_X Y \ / \ df_X}{Y'P_E Y \ / \ df_E} \quad \text{where } X = N_B, N_{AB}.$$

W is distributed as a noncentral F with df_X and df_E as its degree of freedom and λ as its noncentrality parameter. When H_X is true, $\lambda = 0$ and thus W is distributed as central F. In this case we can find a critical value W such that

$Pr(\text{ reject } H_X | H_X \text{ true }) = \alpha$ and

$Pr(\text{ reject } H_X | H_X \text{ true }) > \alpha$.

Computation for sum of square:

Let us define some matrices as follows:

$S = [s_{11}, s_{21}, ..., s_{33}, s_{43}]$

$AB = [(ab)_{11}, (ab)_{13}, ..., (ab)_{22}, (ab)_{23}]$,

$B = [b_1, b_2, b_3]$,

$E = [e_1, e_2, ..., e_{12}]$,

$D = [S||AB||B]$ || is symbol for concatenation.

The || operator produces a new matrix by horizontally joining two matrices say A and B which must have the same number of rows.

$C = [S||AB]$, and $G = [S||B]$.

$$\begin{aligned} SSB &= Y'P_{N_B} Y \\ &= Y'P_{[T \ominus (AB \boxplus W_s)]} Y \\ &= Y'P_T Y - Y'P_{(AB \boxplus W_s)} Y \\ &= Y'D(D'D)^{-1}D'Y - Y'C(C'C)^{-1}C'Y \end{aligned}$$

$$\begin{aligned} SSAB &= Y'P_{N_{AB}} Y \\ &= Y'P_{[T \ominus (B \boxplus W_s)]} Y \\ &= Y'P_T Y - Y'P_{(B \boxplus W_s)} Y \\ &= Y'D(D'D)^{-1}D'Y - Y'G(G'G)^{-1}G'Y \end{aligned}$$

$$\begin{aligned} SSE &= Y'P_E Y \\ &= Y'E(E'E)^{-1}E'Y \\ \text{or } &= Y'(I - D(D'D)^{-1}D')Y \end{aligned}$$

Table for Analysis of Variance

Source	df	SS	MS	F
A		--- not available ---		
Error A		--- not available ---		
B	dim N_B	SSB	$MSB=SSB/df_B$	MSB/MSE
AB	dim N_{AB}	SSAB	$MSAB=SSAB/df_{AB}$	MSAB/MSE
Error	dim E	SSE	$MSE=SSE/df_E$	

(Note that the degree of freedom B, AB, Error is equal to the dimension of N_B, N_{AB}, E respectively.)

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