Mathematical Modeling with Generalized Function
PROFESSOR DR. ADEM KILIÇMAN
Mathematical Modeling with Generalized Function

PROFESSOR DR. ADEM KILIÇMAN
B. Sc. (Hons), M.Sc.(Hacettepe, Turkey), Ph.D.(Leicester, England)

25 November 2011

Dewan Phillip Kotler
Universiti Putra Malaysia

Universiti Putra Malaysia Press
Serdang • 2011
http://www.penerbit.upm.edu.my
Mathematical Modeling with Generalized Function

Adem Kılıçman

B.Sc. (Hons.), M.Sc. (Hacettepe, Turkey), Ph.D. (Leicester, England)
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>4</td>
</tr>
<tr>
<td>Resource Allocations</td>
<td>5</td>
</tr>
<tr>
<td>Assignment Problems</td>
<td>5</td>
</tr>
<tr>
<td>Transportation Problems</td>
<td>5</td>
</tr>
<tr>
<td>Integer programming Problem</td>
<td>6</td>
</tr>
<tr>
<td>Dynamic Programming</td>
<td>6</td>
</tr>
<tr>
<td>Iterative Algorithms</td>
<td>6</td>
</tr>
<tr>
<td>DIFFERENCE EQUATIONS</td>
<td>10</td>
</tr>
<tr>
<td>First order Difference Equation</td>
<td>10</td>
</tr>
<tr>
<td>Second Order Linear Difference Equations</td>
<td>11</td>
</tr>
<tr>
<td>DIFFERENTIAL EQUATIONS</td>
<td>12</td>
</tr>
<tr>
<td>Population Model</td>
<td>13</td>
</tr>
<tr>
<td>Harmonic Oscillator</td>
<td>14</td>
</tr>
<tr>
<td>Competing Species</td>
<td>14</td>
</tr>
<tr>
<td>Stochastic Environment</td>
<td>16</td>
</tr>
<tr>
<td>DISTRIBUTIONS</td>
<td>18</td>
</tr>
<tr>
<td>Need to Study Distributions</td>
<td>19</td>
</tr>
<tr>
<td>Historical Developments</td>
<td>20</td>
</tr>
</tbody>
</table>
In His Name, be He glorified! And there is nothing but its glorifies Him with praise.

In the Name of God, the Merciful, the Compassionate.

There is no god but God, He is One, He has no partner; His is the dominion and His is the praise; He alone grants life, and deals death, and He is living and dies not; all good is in His hand, He is powerful over all things, and with Him all things have their end.

Be certain of this, that the highest aim of creation and its most important result are belief in God. And the most exalted rank in humanity and its highest degree are the knowledge of God contained within belief in God. And the most radiant happiness and sweetest bounty for jinn and human beings are the love of God contained within the knowledge of God. And the purest joy for the human spirit and the sheerest delight for man’s heart are the rapture of the spirit contained within the love of God. Indeed, all true happiness, pure joy, sweet bounties, and untroubled pleasure lie in knowledge of God and love of God; they cannot exist without them.

From the Risale-i Nur Collection
ABSTRACT
In recent years there has been a growing interest in setting up the modeling and solving mathematical problems in order to explain numerous experimental findings which are relevant to industrial applications.

Distributions also known as generalized functions which generalize classical functions and allow us to extend the concept of derivative to all continuous functions. The theory of distributions have applications in various fields especially in science and engineering where many non-continuous phenomena that naturally lead to differential equations whose solutions are distributions, such as the delta distribution therefore distributions can help us to develop an operational calculus in order to investigate linear ordinary differential equations as well as partial differential equations with constant coefficients through their fundamental solutions.

Further, some regular operations which are valid for ordinary functions such as addition, multiplication by scalars are extended into distributions. Other operations can be defined only for certain restricted subclasses; these are called irregular operations.

They allow us to extend the concept of derivative to all continuous functions and beyond and are used to formulate generalized solutions of partial differential equations.
They are important in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution.

In this work we aim to show how differential equations arise in the mathematical modeling of certain problems in industry. The focus of the presentations will be on the use of mathematics to advance the understanding of specific problems that arise in industry. For this purpose we let $D$ be the space of infinitely differentiable functions with compact support and let $D'$ be the space of distributions defined on $D$ then we provide some particular examples how to use the generalized functions in Statistics and Economics. At the end of the study we relate the Tomography and The Radon Transform on using the generalized functions.

In mathematical analysis, distributions also known as generalized functions are objects which generalize functions and probability distributions.

We apply the distributions to the some mathematical problems. For this purpose we let $\rho$ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$. 

Define, the function $\delta_n$ by putting

$$\delta_n(x) = n\rho(nx) \quad \text{for } n = 1, 2, \ldots,$$

it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $D$ be the space of infinitely differentiable functions with compact support and let $D'$ be the space of distributions defined on $D$. Then if $f$ is an arbitrary distribution in $D'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

**Key Words and Phrases:** Differential equations, Delta-function, Generalized functions, Probability Functions, Tomography and Radon Transforms.
INTRODUCTION

Over the past decades mathematical programming has become a widely used tool to help managers with decision making. The problems that were impossible to solve in the past are now very easy and solved by standard computer programs. In fact the development of mathematical programming has generated much interest in mathematical modeling through out business of all types and all sizes.

In mathematics we deal with equations. An equation is a statement that two mathematical expressions are equal. The two expressions that make up an equation are called its sides. We have several types of the equations such as polynomial equations, exponential equations, logarithmic equations, trigonometric equations, difference Equations, differential equations, integral equations, etc.

We solve an equation by using the some properties of equality to transform the equation into an equivalent statement of the form until we get

\[ ax + b = c \Rightarrow x = ? \]
\[ e^{a(x)} = d \Rightarrow x = ? \]
\[ \log f(x) = m \Rightarrow x = ? \]
\[ \frac{df}{dx} = g(x) \Rightarrow f(x) = ? \]
\[ \int h(x)dx = v(x) \Rightarrow h(x) = ? \]

The best way to demonstrate the scope of the mathematical modeling is to show the wide variety of the problems in the real world to which that model can be applied. With the exception, the cases enable us to grasp
the range and the full dimension of the problems that can be formulated and solved by mathematical modeling. The following examples have the common characteristic:

**Resource Allocations:** Consider a company in Malaysia manufactures two type of products for distributions to retailers in all over Malaysia. Then the problem is to identify the

- Raw material
- Man hours
- Specific profit for each products that company turns out.
- The objective function is to be maximize the profit made by the products.

**Assignment Problems:** Consider a textile factory and has different 20 machines, and each machine has capability of performing various jobs. On the specific day the manager of the factory has to check the performance of the 15 different jobs. Each assignment can be accomplished by any of the 20 machines.

- The time it takes to perform the job $n$ on machine $m$ is $t_{nm}$
- Time varies from machine to machine
- If the $t_{nm}$ goes to infinity then job $n$ cannot be performed on the machine $m$
- To determine the best 15 machines.

**Transportation Problems:** After the production process is finished or completed the major problem is the distributions. So each side try to maximize the capacity and has to satisfy the demand for the product for a
definite time.

**Integer programming Problem:** It is a mathematical optimization problem which study feasibility in where some or all of the variables are restricted to be integers. In many settings the term refers to integer linear programming, which is also known as mixed integer programming.

**Dynamic Programming:** Dynamic programming is a method for solving the complex problems by breaking into simpler small or subproblems. It is applicable to problems exhibiting the properties of overlapping subproblems which are only slightly smaller and optimal substructure. Then combine the solutions of the subproblems to reach an overall solution. When applicable, the method takes far less time than naive methods.

**Iterative Algorithms:** The iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. A specific implementation of an iterative method, including the termination criteria, is an algorithm of the iterative method. An iterative method is called convergent if the corresponding sequence converges for given initial approximations. A mathematically rigorous convergence analysis of an iterative method is usually performed; however, heuristic-based iterative methods are also common.

However, a mathematical model is a description of an particular system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. Mathematical
models are used not only in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science); physicists, engineers, statisticians, operations research analysts and economists use mathematical models most extensively.

Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models, as far as logic is taken as a part of mathematics. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed. Many mathematical models can be classified in some of the following ways:

**Linear vs. nonlinear** Mathematical models are usually composed by variables, which are abstractions of quantities of interest in the described systems, and operators that act on these variables, which can be algebraic operators, functions, differential operators, etc. If all the operators in a mathematical model exhibit linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise.
The question of linearity and nonlinearity is dependent on context, and linear models may have nonlinear expressions in them. For example, in a statistical linear model, it is assumed that a relationship is linear in the parameters, but it may be nonlinear in the predictor variables. Similarly, a differential equation is said to be linear if it can be written with linear differential operators, but it can still have nonlinear expressions in it. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is regarded as a linear model. If one or more of the objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.

Nonlinearity, even in fairly simple systems, is often associated with phenomena such as chaos and irreversibility. Although there are exceptions, nonlinear systems and models tend to be more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one is trying to study aspects such as irreversibility, which are strongly tied to nonlinearity.

**Deterministic vs. probabilistic (stochastic)** A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables. Therefore, deterministic models perform the same way for a given set of initial conditions. Conversely, in a stochastic model, randomness is present, and variable states are not described by unique values, but rather by probability distributions.
**Static vs. dynamic** A static model does not account for the element of time, while a dynamic model does. Dynamic models typically are represented with difference equations or differential equations.

**Discrete vs. Continuous:** A discrete model does not take into account the function of time and usually uses time-advance methods, while a Continuous model does. Continuous models typically are represented with f(t) and the changes are reflected over continuous time intervals.

**Deductive, inductive, or floating:** A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation, but is merely the invocation of expected structure. Application of mathematics in social sciences outside of economics has been criticized for unfounded models, see [1]. Application of catastrophe theory in science has been characterized as a floating model, see [2].

The purpose of mathematical modeling is to describe the essential features of a phenomenon or a system in a manner which allows us to use of various mathematical methods for a deeper analysis. However, to formulate a mathematical model can be a challenging task. It requires both a solid understanding of the basic interactions governing the system under study and a good knowledge of mathematical methods. Thus the goal of the mathematical modeling is to find values for the some decision variables that decision makers will have choice to optimize his objective which may include, for example, Maximize profit, Maximize utilization of equipment, Minimize cost, Minimize used of raw material or resources and minimize
traveling time. All these problems are now standard and solutions are easy on using the many software packages that available for mathematical programming.

**DIFFERENCE EQUATIONS**

These equations occur in many mathematical model and as tools in numerical analysis. We can easily develop a theory and devise methods for solving linear difference equations.

**First order Difference Equation**

A recurrence relation can be defined by a difference equation of the form

\[ x_{n+1} = f(x_n) \]

where \( x_{n+1} \) is derived from \( x_n \) and \( n = 0, 1, 2, 3, \ldots \). If the first one starts with an initial value, say \( x_0 \) then the iteration of the difference equation leads to a sequence of the form

\[ \{x_i : i = 0 \rightarrow \infty \} = \{x_0, x_1, x_2, x_3, \ldots x_n\}. \]

Consider the following simple example,

The difference equation might be used to model the interest in a bank account compounded \( k \) times per year and the

\[ x_{n+1} = a \cdot x_n \]

where \( a > 1 \) and a constant.
Second Order Linear Difference Equations

Recurrence relations involving terms whose suffices differ by two are known as a second-order linear difference equations: The general form of these equations with constant coefficients is

\[ a \ x_{n+2} = b \ x_{n+1} + c \ x_n. \]

For example, A model for the population dynamics under immigration involves the equation

\[ P_{t+1} - P_t = a \ P_t + b \]

where

- the \( P_{t+1} - P_t \) is the difference,
- the constant \( a \) is the difference between birth rate and the death rate,
- and \( b \) is the rate at which people immigrate to the country.

Similarly, suppose that the national income of a country in year \( n \) is given by

\[ I_n = S_n + P_n + G_n \]

where \( S_n, P_n \) and \( G_n \) represent national spending by populous, private investment and the government spending. If the national income increase from one year to the next, then assume consumers will spend more for the following year. In this case suppose that consumer spend \( \frac{3}{4} \) of the previous years income. Then \( S_{n+1} = \frac{3}{4} I_n \). An increase in consumer spending should also lead to in creased investment to the following year. Assume that \( P_{n+1} = S_{n+1} - S_n \). Substitution for \( S_n \) gives \( P_{n+1} = \frac{3}{4} (I_n - I_{n-1}) \). Now if the government spending is kept constant then the economic model is

\[ I_{n+2} = \frac{3}{4} I_{n+1} - \frac{3}{4} I_n + G \]
where \( I_n \) is the national income in year \( n \) and \( G \) is the initial national income. Now if the national income one year later is \( \frac{5}{4}G \) then we can easily determine the following:

- a general solution to this model
- the national income after 10 years
- long term state of the economy.

In general, \( k \)-order linear difference equation is an equation of the form

\[
a_k(n) x_{n+k} + a_{k-1}(n) x_{n+k-1} + \ldots + a_1(n) x_{n+1} + a_0(n) x_n = b_n
\]

where \( n = 0, 1, 2, \ldots, a_i(n) \) and \( b_n \) are defined for all nonnegative integers \( n \).

### Differential Equations

A linear homogeneous ordinary differential equation has the form

\[
a_s(x) \frac{d^s y}{dx^s} + a_{s-1}(x) \frac{d^{s-1} y}{dx^{s-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0
\]

and if the general solution exits then it looks like

\[
Y(c_1, c_2, c_3, c_4, \ldots, c_n) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + \ldots + c_s y_s(x)
\]

where \( y_i(x) \) are independent set of particular solutions. The constants \( c_i \) can be adjusted so that the solution satisfies specified initial values

\[
y_0(x_0) = b_0, \quad y_1(x_0) = b_1, \quad y_2(x_0) = b_2, \quad \ldots, \quad y_{s-1}(x_0) = b_{s-1}
\]

at any point \( x_0 \) where the equation coefficient \( a_i(x) \) are continuous and \( a_s(x) \) is non zero. The general form of the linear ordinary differential equation is given by the equation

\[
a_s(x) \frac{d^s y}{dx^s} + a_{s-1}(x) \frac{d^{s-1} y}{dx^{s-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x) y = f(x)
\]
in short form we write

\[ L(y) = f(x) \]

then quickly one can observe that if

\[ L(y_1) = f \quad \text{and} \quad L(y_2) = 0 \Rightarrow L(y_1 + y_2) = f(x) \]

conversely, if

\[ L(y_1) = f \quad \text{and} \quad L(y_2) = f \Rightarrow L(y_1 - y_2) = 0. \]

Remark: Only the linear homogeneous equations possess the property that any linear combination of solutions is again a solution.

We start with the basic and simple one.

**Population Model** Let \( P(t) \) be the function of the population of a given species at the time \( t \) and let \( r(t, p) \) be the difference between its birth rate and its death rate. If the population is isolated (there is no immigration or emigration) then

\[ \frac{dp}{dt} \]

will be the rate of the population, equals to \( r \ p(t) \).

In the most simplistic model we assume that \( r \) is constant (does not change with either time or population). Then the differential equation governing the population growth is:

\[ \frac{dp}{dt} = a \ p(t), \quad a = \text{constant}. \]

This is a linear equation and is known as the population growth. Now if the population of the given species is \( p_0 \) at time \( t_0 \), then \( p(t) \) satisfies the
initial-value problem
\[ \frac{dp}{dt} = a p(t), \quad p(t_0) = p_0. \]

The solution of this initial-value problem is
\[ p(t) = p_0 e^{a(t-t_0)}. \]

**Harmonic Oscillator** In order to model a mass on a spring we use the differential equation
\[ y'' = \lambda y \]
which arises repeatedly in Engineering applications and the general solution can be expressed as
\[
Y(x) = \begin{cases} 
  c_1 \cosh \sqrt{\lambda}(x - x_0) + c_2 \sinh \sqrt{\lambda}(x - x_0) & \text{if } \lambda > 0 \\
  c_1 + c_2 x & \text{if } \lambda = 0 \\
  c_1 \cos \sqrt{-\lambda}(x - x_0) + c_2 \sin \sqrt{-\lambda}(x - x_0) & \text{if } \lambda < 0
\end{cases}.
\]

We note that whenever the function \( y = f(x) \) has a specific interpretation in one of the sciences, its derivative will also have a specific interpretation as a rate of change. Several concepts in economics that have to do with rates of change as well can be effectively described with the methods of calculus. All the properties of the functions which we measure show how one variable changes due to changes in another variable. Thus, differential calculus is concerned with how one quantity changes in relation to another quantity. Actually, the central concept of differential calculus is the derivative.

**Competing Species** Suppose there are two species in competition with one another in an environment where the common food supply is limited. For example, sea lions and the penguins, red and grey squirrels, ants and
termites are all species that is in this category. Then there are two types of outcome, coexistence and the mutual exclusion. Then find the possible phase solution for the following system:

\[
\dot{x} = x(a - bx - cy) \\
\dot{y} = x(d - ex - fy)
\]

where \(a, b, c, d, e\) and \(f\) are all positive constants with \(x(t)\) and \(y(t)\) both representation of two populations, see for example [55].

**Cooking a Roast** The process of cooking a Roast involves taking a piece of meat at an initial temperature \(T_{\text{cold}}\) and placing it in an oven at a constant temperature \(T_{\text{oven}}\) until a meat thermometer indicates that the temperature at a specific location say \((x, y, z) = (0, 0, 0)\) has reached the value \(T_{\text{done}}\) this request a cooking time \(t_{\text{done}}\).

Let, \(T(x, y, z, t)\) denote the temperature of the roast, if the thermal diffusivity of the meat is \(k\) then its temperature evolves according to the heat equation:

\[
\frac{\partial T}{\partial t} = k \nabla^2 = k \left\{ \frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y} + \frac{\partial^2 T}{\partial^2 z} \right\}
\]

if \(t = 0\) then the final temperature \(T_{\text{cold}}\) is

\[
T(x, y, z, 0) = T_{\text{cold}}
\]

the temperature on the skin or "boundary" \(B\) of the roast is maintained by the oven at

\[
T(x, y, z, t) = T_{\text{oven}}(x, y, z)
\]
on the boundary $B$ all $t > 0$. And the cooking time is specified by the condition
\[ T(0, 0, 0, t_{\text{done}}) = T_{\text{done}} \]
see the details in [47]. Now the question if largest roast is placed in the oven what equation does its temperature satisfy?

**Stochastic Environment** Any variable its value changes over time in an uncertain way is said to be stochastic variable and this process is called stochastic process. Might be discrete or continuous.

Stochastic methods have become increasingly important in the analysis of a broad range of phenomena in natural sciences and economics. Many processes are described by differential equations where some of the parameters and/or the initial data are not known with complete certainty due to lack of information, uncertainty in the measurements, or incomplete knowledge of the mechanisms themselves. To compensate for this lack of information one introduces stochastic noise in the equations, either in the parameters or in the initial data which results in stochastic differential equations. Since the stochastic environment allows for some randomness in some of the differential equations we can get the more realistic model for the most of the situation. How to make the stochastic differential equations?

Consider the ordinary differential equation
\[ \frac{df}{dt} = g(x, t) \]
then we suppose that system has random part (component), added to the system
\[
\frac{df}{dt} = g(x, t) + h(x, t)\epsilon(x).
\]
Since we have some randomness the solution to the equation might be difficult. Then we write the system as follows
\[
df = g(x, t)dt + h(x, t)\epsilon(x)dt
\]
then the solution to the equation by performing the integration yields
\[
f(t) = f(0) + g(x, s)ds + h(x, s)\epsilon(s)ds.
\]
If we reconsider the population model
\[
\frac{dp}{dt} = ap(t), \quad a = \text{constant}.
\]
Now if we ask what will happen \(a(t)\) is a function and is not completely known, but subject to some random environmental effects, so that we have
\[
a(t) = r(t) + \text{unknown parts}
\]
in fact we call this unknown part as the noise where we do not know the exact behavior of the noise term, then how do we solve this problem?

Consider we want to make some investments: We can do some safe investment like fixed deposit or bond where the price per unit at the time \(t\) grows exponentially:
\[
\frac{dp}{dt} = ap
\]
where \(a\) is a constant function. We can also do some risky investment such as stock where the price of the unit at time \(t\) satisfies stochastic differential
equation of the type
\[
\frac{dp}{dt} = (b(t) + k \text{ noise}) \ p
\]
where \( b(t) > 0 \) and \( k \in \mathbb{R} \) are constants. Now we can easily choose how much amount of the money we want to place in the risky investment. If the utility function \( U \) and the terminal time \( T \) the problem is to find the optimal portfolio \( u_t \in [0, 1] \). That is to find the investment distribution \( u_t \) subject to \( 0 \leq t \leq T \) which maximize the expected utility of the corresponding terminal fortune \( X_T^{(u)} \).

\[
\max_{0 \leq u_t \leq 1} \left\{ E \left[ U \left( X_T^{(u)} \right) \right] \right\}.
\]

In short, stochastic differential equations or SDEs, allow for inherited randomness in physical or biological systems by adding a random noise term to classical differential equations.

**DISTRIBUTIONS**

In classical models, the physical world is modeled as a continuum, and the objects in study are thought as infinitely divisible and observable with arbitrarily good accuracy. In real life, physical phenomena are only observable to a maximum degree of precision dictated by the limitations of the instruments used or even by uncertainty principles inherent to the very nature of reality. Using the classical tools derived from Calculus, it is not only necessary to adopt this continuum model but often the quantities in study must satisfy regularity properties, they must show a certain degree of “smoothness”. In many situations these assumptions are impractical and several important problems are not treatable using this classic approach to modeling.
Physicists, staring with the work of Dirac, again solved this shortcoming of the classical theory by introducing new objects (now called distribution or generalized functions) based in their physical intuition. This more modern approach opened the door to treat all sort of models where the smoothness assumptions are more relaxed, allowing for discontinuities and other types of singularities. Distributions theory in its full scope is not only difficult but also requires a sophisticated mathematical background.

**Need to Study Distributions**

In classical models, the physical world is modeled as a continuum, and the objects in study are thought as infinitely divisible and observable with arbitrarily good accuracy. In real life, physical phenomena are only observable to a maximum degree of precision dictated by the limitations of the instruments used or even by uncertainty principles inherent to the very nature of reality. Using the classical tools derived from Calculus, it is not only necessary to adopt this continuum model but often the quantities in study must satisfy regularity properties, they must show a certain degree of "smoothness". In many situations these assumptions are impractical and several important problems are not treatable using this classic approach to modeling.

Physicists, staring with the work of Dirac, again solved this shortcoming of the classical theory by introducing new objects (now called distribution or generalized functions) based in their physical intuition. This more modern approach opened the door to treat all sort of models where the smoothness assumptions are more relaxed, allowing for discontinuities and other types of singularities. Distributions theory in its full scope is not only difficult
but also requires a sophisticated mathematical background.

Before we study distribution there are several questions that we have to ask in order to motivate ourself to study distributions theory: For example, in the classical analysis, it is well known that every differentiable function is continuous. In general, converse is not true. With generalized functions one can overcome of this problem further discontinuous function is differentiable in the distributional sense, that is, we can differentiate nearly any function as many times as we like, regardless of discontinuities.

If the \( \lim_{n \to \infty} \int_\mathbb{R} f_n \cdot \phi \) exists for all very nice test functions \( \phi \) then the \( \lim_{n \to \infty} f_n \) exists as a generalized function.

The history of the distribution is also relatively new, only in the 1930s Hadamard, Sobolev, and others made systematic use of non-classical generalized functions.

In 1952 Laurent Schwartz won a Fields Medal for systematic treatment of these ideas. Since then, generalized functions have found many applications in various fields of science and engineering. The well known example of this formalism is by considering the delta function that we will discuss later.

**Historical Developments**

Although now known as the Dirac delta function, the delta function \( \delta(x) \) can be said to have been first introduced by Kirchhoff in [43]. He defined
\[ \delta(x) = \lim_{\mu \to \infty} \pi^{-1/2} \mu \exp(-\mu^2 x^2). \]

It is easily seen that \( \delta(x) = 0 \) for \( x \neq 0 \) and \( \delta(0) = \infty \). Defining \( \int_{-\infty}^{x} \delta(t) \, dt \) by
\[ \int_{-\infty}^{x} \delta(t) \, dt = \lim_{\mu \to \infty} \pi^{-1/2} \mu \int_{-\infty}^{x} \exp(-\mu^2 t^2) \, dt, \]
it follows that
\[ \int_{-\infty}^{x} \delta(t) \, dt = \begin{cases} 
0, & x < 0, \\
1, & x > 0 
\end{cases} \tag{1} \]
and thus \( \delta \) is not a function in the mathematical sense, since its infinite value takes us out of the usual domain of definition of functions so Kirchhoff referred to \( \delta \) as the unit impulse function and mathematicians call it a distribution, a limit of a sequence of functions that really only has meaning in integral expressions such as
\[ \int_{-\infty}^{\infty} f(x) \delta(x - a) \, dx = f(a) \tag{2} \]
Let us evaluate (2) for the special case if \( f(x) = 1 \) then we get
\[ \int_{-\infty}^{\infty} \delta(x - a) \, dx = 1 \]
The delta function was next used by Heaviside, see [20]. Heaviside’s function \( H \) is the locally summable function defined to be equal to 0 for \( x < 0 \) and equal to 1 for \( x > 0 \). Heaviside appreciated that the derivative of \( H \) was in some sense equal to \( \delta \).

When Dirac considered the delta function, see [6], he treated it as though it were a function that was zero everywhere except at the origin where it
was infinite in such a way that
\[ \int_{-\infty}^{\infty} \delta(x) dx = 1. \]

Dirac used \( \delta \) to represent a unit point charge at the origin and the derivative \( \delta' \) of \( \delta \) as to represent a dipole of unit electric moment at the origin since
\[ \int_{-\infty}^{\infty} x \delta'(x) dx = \lim_{\mu \to \infty} \pi^{-1/2} \mu \int_{-\infty}^{\infty} x \left[ \exp(-\mu^2 x^2) \right]' dx = -1. \]

Higher derivatives of \( \delta \) can be used to represent more complicated multiple-layers and have been used in the physical and engineering sciences for some time, see [23]. Therefore we note that for physicists the delta function is well designed to represent, for example, the charge density of a point particle: there is some total charge on the particle, but since the particle is point-like, the charge density is zero except at the single location of the particle.

Dirac gave no rigorous theory for the delta function and its derivatives but used intuitively obvious results such as
\[ \int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a), \]
\[ \int_{-\infty}^{\infty} f(x) \delta'(x - a) dx = -f'(a), \]
\[ \int_{-\infty}^{\infty} \delta(a - x) \delta(x - b) dx = \delta(a - b). \]

It was left to Sobolev, see [51] and he defined the main operations on distributions such as the derivation and the product by infinitely differentiable functions, and used them for a study of partial differential equations. However, it was not until 1950 when Schwartz published his theory of distributions that a really comprehensive and mathematically account was
However, it was not until 1950 when Schwartz published his theory of distributions that a really comprehensive and mathematically account was given. After the Schwartz work, the theory of distributions has led to the essential progress in several mathematical disciplines and such as the theory of partial differential equations and mathematical physics.

In general, Schwartz’s well-known result is on the property of distribution theory, multiplication of distributions is not possible, cannot be easily given a meaning to product of arbitrary distributions. This fact is an obstacle to the usage of distributions in the theory of nonlinear equations and the theory of generalized coefficients equations and in particular, one arrives at the impossibility to give a meaning to such objects as

$$H(x)\delta(x), \quad x^{-1}\delta(x), \quad \delta(x)\delta(x), \quad \delta'\delta, \quad \delta^{(n)}\delta^{(s)}$$

where $\delta$ is delta function and $H$ is the Heaviside function, are of special interest to physicists and widely used, for example, in quantum theory. The problem proved to be very natural and to have a lot of applications. Therefore it has attracted attention at once after the creation of the distributions theory.

Since theory of distributions is a linear theory. We can extend some operations to $D'$ which are valid for ordinary functions such operations are called regular operations such as addition, multiplication by scalars. Other operations can be defined only for particular class of distributions or for certain restricted subclasses of distributions; these are called irregular operations
such as: product, convolution product, composition, Fourier Transform and differential equations.

**Definition 1.** Let $f$ be a continuous, real (or complex) valued function defined on the real line. Then $\text{supp } f$, the *support* of $f$, is the closure of the set on which $f(x) \neq 0$.

If $f$ is a differentiable function, its derivative $f'$ is also another function; thus the new function $f'$ may have a derivative of its own and it is denoted by $(f')' = f''$. This new function $f''$ is called the *second* derivative of $f$ or derivative of the first derivative. The process can be continued as long as successive derivatives are differentiable. We note that, in general, if $f$ is differentiable then $f'$ need not necessarily to be continuous. That is to say the function might have derivative but the derivative not necessarily be continuous however we have the following definition.

**Definition 2.** Let $f$ be defined on an open subset $S \subseteq \mathbb{R}$. Then if $f'$ is continuous on $S \subseteq \mathbb{R}$, we say that $f \in C^1(S)$ that is, $C^1(S)$ is the set of all functions which have continuous first order derivatives in $S$.

**Example 3.** Let $f$ be defined by $f(s) = e^s$ then $f'$ is continuous on $\mathbb{R}$, then $f \in C^1(\mathbb{R})$.

**Definition 4.** If the $f''$ exists and continuous at each point of $S$ then $f$ is the member of the $C^2(S)$, or we say $f \in C^2(S)$. Thus it is obvious that

$$C^2(S) \subset C^1(S).$$

The set $C^\infty(S)$ contains the functions such that having continuous derivatives of all orders in $S$. We call this class infinitely continuously differentiable function class. Of course we can also show that for any natural
number $m$, we have the following implication

$$C^\infty(S) \subset \ldots \subset C^{m+1}(S) \subset C^m(S) \subset \ldots \subset C^2(S) \subset C^1(S)$$

The function having continuous derivatives of all orders is also known as smooth function.

Now if we let $\phi$ be an infinitely differentiable real valued function with compact support. Then $\phi$ is said to be a test function. The set of all test functions with the usual definition of sum and product by a scalar is a vector space and is denoted by $D$. If $\phi$ is given by

$$\phi(x) = \begin{cases} 
e^\left(\frac{1}{x-a} - \frac{1}{x-b}\right), & a < x < b \\
0, & x \leq a \quad x \geq b, \end{cases}$$

then $\phi \in D$ and $\text{supp} \phi = [a, b]$.

Note that if $\phi$ is in $D$, then $\phi^{(r)}$ is in $D$ for $r = 1, 2, \ldots$, we also note that if $\phi \in D$ and $g$ is any infinitely differentiable function, then $g\phi \in D$ but $\Phi(x) = \int_{-\infty}^{x} \phi(x) \, dx$ is not necessarily in $D$, since $\int_{-\infty}^{\infty} \phi(x) \, dx$ may not be equal to zero, in which case $\Phi$ will not have compact support. Thus, with the above if $\phi \in D$ then

$$\cos x \phi(x), \; \sin x \phi(x), \; \left(\sum_{i=0}^{r} a_i x^i\right) \phi(x), \; \ln |x - c| \phi(x)$$

where $c < a$, are all in $D$.

**Example 5.** Let $f$ be defined by

$$f(s) = \begin{cases} \ne^{-\frac{1}{2}} & \text{if } s \in (0, \infty) \\
0 & \text{if } s \in (-\infty, 0] \end{cases}$$
then $f$ is infinitely differentiable for all $s$. That is $f \in C^\infty(S)$ for any $S \subseteq \mathbb{R}$.

We note that

(i) The polynomials are infinitely continuously differentiable function thus polynomials are in $C^\infty(S)$.

(ii) The function $f(t) = \sin t$, $g(t) = \cos t$ and $h(t) = e^t$ are not polynomial however they are continuously differentiable thus they are in $C^\infty(S)$.

**Definition 6.** Let $\{\phi_n\}$ be a sequence of functions in $D$. Then the sequence $\{\phi_n\}$ is said to **converge to zero** if $\exists$ a bounded interval $[a, b]$, with $\text{supp}\, \phi_n \subseteq [a, b]$ for all $n$ and $\lim_{n \to \infty} \phi_n^{(r)}(x) = 0$ for all $x$ and $r = 0, 1, 2, \ldots$. Further, let $f$ be a linear functional on $D$. Then $f$ is said to be **continuous (bounded)** if $\lim_{n \to \infty} \langle f, \phi_n \rangle = 0$ whenever $\{\phi_n\}$ is a sequence in $D$ converging to zero. A continuous linear functional on $D$ is said to be a **distribution or generalized function**. The set of all distributions is a vector space and is denoted by $D'$.

We note that in the classical sense, we represent a function as a table of ordered pairs $(x, f(x))$. Of course, often this table has an uncountably infinite number of ordered pairs. We show this table as a curve representing the function in a plane. In generalized function theory, we also describe $f(x)$ by a table of numbers. These numbers are produced by the relation

$$F(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

(3)

where the function $\phi(x)$ comes from a given space of functions called the test function space thus generalized functions are defined as continuous linear functionals over a space of infinitely differentiable functions $D$ therefore
\( \langle f, \phi \rangle \) is an action on \( \phi \) rather than a pointwise value.

Now let \( f \) be a locally summable function. (i.e. \( \int_{a}^{b} f(x) \, dx \) exists for every bounded interval \([a, b]\).) Then \( f \) defines a linear functional on \( D \) if we put

\[
\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x) \phi(x) \, dx = \int_{a}^{b} f(x) \phi(x) \, dx
\]

if \( \text{supp}\phi \subseteq [a, b] \). Further, \( f \) is continuous and so a distribution, since if \( \{\phi_n\} \) converges to zero, \(|\phi_n(x)| < \epsilon \) for \( n > N \) and so

\[
|\langle f, \phi_n \rangle| \leq \epsilon \int_{a}^{b} |f(x)| \, dx
\]

Thus it is clear that \( D \subset D' \). In fact one can prove that \( D = D' \).

**Example 7.** A distribution \( \delta \) (the *Dirac delta-function*) can be defined by putting

\[
\langle \delta, \phi \rangle = \phi(0).
\]

However, \( \delta \) is not defined by a locally summable function since if

\[
\int_{-\infty}^{\infty} \delta(x) \phi(x) \, dx = \phi(0)
\]

for all \( \phi \in D \), then

\[
\int_{-\infty}^{\infty} \delta(x) \phi(x, a) \, dx = \phi(0, a) = e^{-1}
\]

for all \( a > 0 \). Letting \( a \to 0 \), LHS \( \to 0 \), giving a contradiction.

Although it seems impossible to give a suitable definition which will define the product of any two distributions, it is possible to define the product of a distribution \( f \) and an infinitely differential function \( g \) and this is given in the next definition.
Definition 8. Let \( f \) be a distribution and let \( g \) be an infinitely differentiable function. Then the product \( fg = gf \) is defined by

\[
\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle
\]

for all \( \phi \) in \( D \). Note that in this definition the product \( g\phi \) is in \( D \) and so the product \( fg = gf \) is defined as a distribution.

Example 9. Let \( g \) be an infinitely differentiable function. Then

\[
\langle \delta g, \phi \rangle = \langle g\delta, \phi \rangle = \langle \delta, g\phi \rangle = g(0)\phi(0) = g(0)\langle \delta, \phi \rangle
\]

for all \( \phi \) in \( D \). Thus

\[
\delta g = g\delta = g(0)\delta.
\]

Theorem 10. Let \( f \) be a distribution and let \( g \) be an infinitely differentiable function. Then

\[
f^{(r)}g = g f^{(r)} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [f g^{(i)}]^{(r-i)}.
\]

In particular, Leibnitz’s Theorem holds for the product \( fg = gf \), i.e.

\[
(fg)' = fg' + f'g.
\]

Example 11. Let \( g \) be an infinitely differentiable function. Then

\[
\delta' g = g\delta' = g(0)\delta' - g'(0)\delta.
\]

In particular, with \( g(x) = x \),

\[
\delta' x = x\delta' = -\delta.
\]

More general
\[
\delta^{(r)} g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [\delta g^{(i)}]^{(r-i)} \\
= \sum_{i=0}^{r} \binom{r}{i} (-1)^i g^{(i)}(0) \delta^{(r-i)}.
\]

In particular
\[
\delta^{(r)} x^s = x^s \delta^{(r)} = \begin{cases} 
\frac{r!}{(r-s)!} (-1)^s \delta^{(r-s)}, & 0 \leq s \leq r, \\
0, & s > r.
\end{cases}
\]

**Note:** The product of distributions is not necessarily associative. To see this, we have
\[
\langle x^{-1}x, \phi \rangle = \int_0^\infty x^{-1} [x\phi(x) + x\phi(-x)] \, dx \\
= \int_0^\infty [\phi(x) + \phi(-x)] \, dx \\
= \int_{-\infty}^\infty \phi(x) \, dx = \langle 1, \phi \rangle,
\]
for all \(\phi\) in \(D\) and so \(x^{-1}x = 1\). Thus
\[
(x^{-1}x)\delta = 1\delta = \delta
\]
on the other side
\[
x^{-1}(x\delta) = x^{-1}0 = 0.
\]

In physics products of distributions such as \(H\delta\) or \(\delta^2\) can be interpreted in many different ways. In the literature, several definitions have been proposed for \(\delta^2\) ranging from
\[
\delta^2 = 0, \quad c\delta, \quad cx^{-2}, \quad c\delta + \frac{1}{2\pi i}\delta', \quad c\delta + c'\delta'
\]
with arbitrary constants $c, c'$. This has opened up a new area of mathematical research, with many attempts to try and give a satisfactory definition for the product of two distributions.

Despite the non-associativity, still there are some distributions, for example the product of two distributions such as

$$\ln^p |x| \delta^{(r)}(x), \ x^{-p} \delta^{(r)}(x) \quad \text{or} \quad (x + i0)^{-p} \delta^{(r)}(x)$$

for $p = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$ which do not exist in the ordinary sense. In order to compute highly singular products, another generalization of product, the neutrix products was introduced by Fisher which is based on the concept of neutrix limits due to van der Corput, see [56]. The essential use of the neutrix limit is to extract the finite part from a divergent quantity as one has usually done to subtract the divergent terms via rather complicated procedures in the renormalization theory, see [48]. In fact we can consider the neutrices as the generalization of the Hadamard finite parts, see [18].

**Definition 12.** Let $\{f_n\}$ be a sequence of distributions. Then $\{f_n\}$ is said to converge to the distribution $f$ if

$$\lim_{n \to \infty} \langle f_n, \phi \rangle = \langle f, \phi \rangle, \quad \forall \phi \in D.$$  

More generally, if $\{f_n\}$ is a sequence of distributions converging to the distribution $f$. Then the sequence $\{f_n^{(r)}\}$ converges to the distribution $f^{(r)}$ for $r = 1, 2, \ldots$. 
Example 13. Let $f_n$ be the function defined by $f_n(x) = x^{-1}$ for $|x| \geq 1/n$ and $f_n(x) = 0$ for $|x| < 1/n$. Then $\lim_{n \to \infty} f_n = x^{-1}$. Similarly, if $f_\nu = \nu \pi^{-1}(\nu^2 + x^2)^{-1}$. Then $\lim_{\nu \to 0} f_\nu = \delta$.

Now suppose that $f$ is a continuous function having a continuous derivative $f'$. Then for arbitrary $\phi \in D$ with $\text{supp} \phi = [a, b]$, we have

$$\langle f', \phi \rangle = \int_a^b f'(x)\phi(x) \, dx$$

$$= \left[ f(x)\phi(x) \right]_a^b - \int_a^b f(x)\phi'(x) \, dx$$

$$= -\langle f, \phi' \rangle.$$

Thus the derivative $f'$ of $f$ is the distribution defined by

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle,$$

for all $\phi \in D$.

It is clear that $f'$ is in fact a distribution, $f^{(n)}$ is a distribution and

$$\langle f^{(n)}, \phi \rangle = (-1)^n\langle f, \phi^{(n)} \rangle$$

for $n = 1, 2, \ldots$.

Example 14. Let $x_+$ be the locally summable function defined by

$$x_+ = \begin{cases} 
  x, & x > 0, \\
  0, & x \leq 0,
\end{cases}$$

Then its derivative, denoted by $H$ (Heaviside’s function), is given by

$$\langle H, \phi \rangle = -\langle x_+, \phi' \rangle = -\int_0^\infty x\phi'(x) \, dx = \int_0^\infty \phi(x) \, dx$$

and so $H$ corresponds to the locally summable function defined by

$$H(x) = \begin{cases} 
  1, & x > 0, \\
  0, & x < 0.
\end{cases}$$
The derivative $H'$ of $H$ is given by

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = - \int_0^\infty \phi'(x) \, dx = \phi(0)$$

and so $H' = \delta$. We note that the function $H(x)$ is a very useful function in the study of the generalized function (distributions theory), especially in the discussion of the functions with jump discontinuities. For instance, let $F(x)$ be a function that is continuous everywhere except for the point $x = \xi$, at which point it has a jump discontinuity,

$$F(x) = \begin{cases} F_1(x), & x < \xi, \\ F_2(x), & x > \xi. \end{cases}$$

Then it can be written that

$$F(x) = F_1(x)H(\xi - x) + F_2(x)H(x - \xi).$$

This concept can be extended to enable one to write a function that has jump discontinuities at several points.

The $r$-th derivative $\delta^{(r)}$ of $\delta$ is given by

$$\langle \delta^{(r)}, \phi \rangle = (-1)^r \langle \delta, \phi^{(r)} \rangle = (-1)^r \phi^{(r)}(0).$$

More generally, let $x_+^\lambda (\lambda > -1)$ be the locally summable function defined by

$$x_+^\lambda = \begin{cases} x, & x > 0 \\ 0, & x < 0. \end{cases}$$

If $\lambda > 0$, its derivative is the locally summable function $\lambda x_+^{\lambda - 1}$ but if $-1 < \lambda < 0$, $x_+^{\lambda - 1}$ is not a locally summable function. If $-1 < \lambda < 0$, we will still
denote the derivative of $x_+^\lambda$ by $\lambda x_+^{\lambda-1}$ but it must be defined by

$$\langle (x_+^\lambda)', \phi \rangle = -\langle x_+^\lambda, \phi' \rangle = -\int_0^\infty x^\lambda d[\phi(x) - \phi(0)] = \lambda \int_0^\infty x^{\lambda-1}[\phi(x) - \phi(0)] \, dx.$$ 

Thus if $-2 < \lambda < -1$, we have defined $x_+^\lambda$ by

$$\langle x_+^\lambda, \phi \rangle = \int_0^\infty x^\lambda[\phi(x) - \phi(0)] \, dx.$$ 

The distribution $x_+^\lambda \ln^m x_+$ is defined by differentiating $x_+^\lambda$ $m$ times partially with respect to $\lambda$. Thus if $-r - 1 < \lambda < -r$ then

$$\langle x_+^\lambda \ln^m x_+, \phi \rangle = \int_0^\infty x^\lambda \ln^m x \left[ \phi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!} \phi^{(i)}(0) \right] \, dx$$

and if $-r - 1 < \lambda < -r + 1$ and $\lambda \neq -r$, then

$$\langle x_+^\lambda \ln^m x_+, \phi \rangle = \int_0^\infty x^\lambda \ln^m x \left[ \phi(x) - \sum_{i=0}^{r-2} \frac{x^i}{i!} \phi^{(i)}(0) \right. \right.$$  

$$\left. - \frac{x^{r-1}}{(r-1)!} \phi^{(r-1)}(0) H(1-x) \right] \, dx$$  

$$+ \frac{(-1)^m m! \phi^{(r-1)}(0)}{(r-1)! (\lambda + r)^{m+1}}.$$ 

**Example 15.** The locally summable function $\ln x_+$ is defined by

$$\ln x_+ = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0. \end{cases}$$
We define the distribution $x_+^{-1}$ to be the derivative of $\ln x_+$. Thus
\[
\langle x_+^{-1}, \phi \rangle = \langle (\ln x_+)', \phi \rangle = -\langle \ln x_+, \phi' \rangle
\]
\[
= - \int_0^1 \ln x d[\phi(x) - \phi(0)] - \int_1^\infty \ln x d\phi(x)
\]
\[
= \int_0^1 x^{-1}[\phi(x) - \phi(0)] dx + \int_1^\infty x^{-1}\phi(x) dx
\]
\[
= \int_0^\infty x^{-1}[\phi(x) - \phi(0)H(1 - x)] dx.
\]
The distribution $x_+^{-1}\ln^m x_+$ is defined for $m = 1, 2, \ldots$ by
\[
(\ln^{m+1} x_+)' = (m + 1)x_+^{-1}\ln^m x_+.
\]
Then the distribution $x_+^{-r}\ln^m x_+$ is then defined inductively. Suppose that $x_+^{-r}\ln^{m-1} x_+$ has been defined for $r = 1, 2, \ldots$ and some $m$. This is certainly true when $m = 1$. The distribution $x_+^{-1}\ln^m x_+$ has been defined for $m = 1, 2, \ldots$, so suppose that $x_+^{-r+1}\ln^m x_+$ has been defined for some $r$. Then $x_+^{-r}\ln^m x_+$ is defined by
\[
(x_+^{-r+1}\ln^m x_+)' = -(r - 1)x_+^{-r}\ln^m x_+ + mx_+^{-r}\ln^{m-1} x_+.
\]
The distribution $|x|^\lambda \ln^m |x|$ is defined by
\[
|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ + x_-^\lambda \ln^m x_-,
\]
the distribution $\text{sgn } x.|x|^\lambda \ln^m |x|$ is defined by
\[
\text{sgn } x.|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ - x_-^\lambda \ln^m x_-
\]
and the distribution $x^r$ is defined by
\[
x^r = x_+^r + (-1)^r x_-^r,
\]
for $r = 0, \pm 1, \pm 2, \ldots$. 
Example 16. The function \( \ln(x + i0) \) is defined by

\[
\ln(x + i0) = \lim_{y \to i0^+} \ln(x + iy)
\]

then it follows that

\[
\ln(x + i0) = \ln|x| + i\pi H(-x).
\]

Similarly, the distribution \((x + i0)^\lambda\) is defined by

\[
(x + i0)^\lambda = \lim_{y \to 0^+} (x + iy)^\lambda
\]

It follows that

\[
(x + i0)^\lambda = x_+^\lambda + e^{i\lambda \pi} x_-^\lambda, \quad \lambda \neq -1, -2, \ldots,
\]

\[
(x + i0)^r = x^r, \quad r = 0, 1, 2, \ldots,
\]

\[
(x + i0)^{-r} = \lim_{\lambda \to -r} (x + i0)^\lambda = x^{-r} + \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(x)
\]

\[r = 1, 2, \ldots.\] Similarly,

\[
(x - i0)^\lambda = \lim_{y \to 0^-} (x + iy)^\lambda
\]

and then

\[
(x - i0)^\lambda = x_+^\lambda + e^{-i\lambda \pi} x_-^\lambda, \quad \lambda \neq -1, -2, \ldots,
\]

\[
(x - i0)^r = x^r, \quad r = 0, 1, 2, \ldots,
\]

\[
(x - i0)^{-r} = \lim_{\lambda \to -r} (x - i0)^\lambda = x^{-r} - \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(x),
\]

\[r = 1, 2, \ldots.\]

More generally we can define the distribution

\[
(x + i0)^\lambda \ln^m(x + i0)
\]
is defined by
\[(x + i0)\lambda \ln^m(x + i0) = \frac{\partial^m}{\partial \lambda^m}(x + i0)^\lambda\]
\[= x_+^\lambda \ln^m x_+ + \sum_{k=0}^{m} \binom{m}{k} (i\pi)^{m-k} e^{i\lambda \pi} x_+^\lambda \ln^k x_-.
\]
In particular, we have
\[(x + i0)^\lambda \ln(x + i0) = x_+^\lambda \ln x_+ + e^{i\lambda \pi} x_-^\lambda \ln x_- + i\pi e^{i\lambda \pi} x_-^\lambda\]
\[(x + i0)^\lambda \ln^2(x + i0) = x_+^\lambda \ln^2 x_+ + e^{i\lambda \pi} x_-^\lambda \ln^2 x_- + 2i\pi e^{i\lambda \pi} x_-^\lambda\]
for \(\lambda \neq -1, -2, \ldots,
\[(x + i0)^r \ln(x + i0) = x^r \ln |x| + (-1)^r i\pi x_-^r\]
\[(x + i0)^r \ln^2(x + i0) = x^r \ln^2 |x| + 2(-1)^r i\pi x_-^r \ln x_- - (-1)^r \pi^2 x_-^r\]
for \(r = 0, 1, 2, \ldots\) and
\[(x + i0)^{-r} \ln(x + i0) = \lim_{\lambda \to -r} (x + i0)^\lambda \ln(x + i0)\]
\[= x^{-r} \ln |x| + (-1)^r i\pi F(x_-, -r) - \frac{(-1)^r \pi^2}{2(r - 1)!} \delta^{(r-1)}(x)\]
\[(x + i0)^{-r} \ln^2(x + i0) = \lim_{\lambda \to -r} (x + i0)^\lambda \ln^2(x + i0)\]
\[= x^{-r} \ln^2 |x| + 2(-1)^r i\pi F(x_-, -r) \ln x_- + (-1)^r \pi^2 F(x_-, -r) + \frac{(-1)^r \pi^3}{3(r - 1)!} \delta^{(r-1)}(x)\]
for \(r = -1, -2, \ldots\) In general, it can be proved that any distribution \(f\) defined on the bounded interval \((a, b)\) is the \(r\)-th derivative of a continuous
function $F$ on the interval $(a,b)$ for some $r$, see [19].

Similar to the previous examples, consider the Gamma function $\Gamma$ then this function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

and it follows that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. $\Gamma(x)$ is then defined by

$$\Gamma(x) = x^{-1}\Gamma(x+1)$$

for $-1 < x < 0$. Further we can express this function as follows

$$\Gamma(x) = x^{-1} + f(x) = x^{-1} + \sum_{i=1}^\infty \frac{\Gamma^{(i)}(1)}{i!} x^{i-1}$$

where $x^{-1}$ is interpreted in the distributional sense. The distribution $\Gamma$ is of course an ordinary summable function for $x > 0$, see [31].

The related distribution $\Gamma(x_+)$ by equation

$$\Gamma(x_+) = x_+^{-1} + f(x_+) = x_+^{-1} + \sum_{i=1}^\infty \frac{\Gamma^{(i)}(1)}{i!} x_+^{i-1}$$ (4)

and the distribution $\Gamma(x_-)$ by equation

$$\Gamma(x_-) = x_-^{-1} + f(x_-) = x_-^{-1} + \sum_{i=1}^\infty \frac{\Gamma^{(i)}(1)}{i!} x_-^{i-1}$$ (5)

where $x_+^{-1}$ and $x_-^{-1}$ are interpreted in the distributional sense, see [26]. It follows that

$$\Gamma(x) = \Gamma(x_+) - \Gamma(x_-)$$ (6)
Differentiating equation (4) \( s \) times we have

\[
\Gamma^{(s)}(x_+) = (-1)^s s! x_+^{-s-1} + f^{(s)}(x_+)
\]

\[
= (-1)^s s! x_+^{-s-1} + \sum_{i=0}^{\infty} \frac{\Gamma(s+i+1)(1)}{(s+i+1)!} x_+^i
\]

(7)

and differentiating equation (5) \( s \) times we have

\[
\Gamma^{(s)}(x_-) = s! x_-^{-s-1} + f^{(s)}(x_-)
\]

\[
= (-1)^s s! x_-^{-s-1} + \sum_{i=0}^{\infty} \frac{\Gamma(s+i+1)(1)}{(s+i+1)!} x_-^i
\]

(8)

Similarly, now let us consider the Beta function \( B(x, y) \). This function can be defined for \( x, y > 0 \) by

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt
\]

and it follows that

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]

for \( x, y, x+y \neq 0, -1, -2, \ldots \). In particular we have

\[
B(x, n) = B(n, x)
\]

\[
= (n-1)! [x(x+1)\ldots(x+n-1)]^{-1}
\]

\[
= \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (x+i)^{-1}
\]

for \( n = 1, 2, \ldots \), where the \((x+i)^{-1}\) for \( i = 0, 1, \ldots, n-1 \) are interpreted in the distributional sense, see [30].
Further, we define the distribution $B_+(x,n)$ by the equation

$$B_+(x,n) = x + \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i (x+i)^{-1} H(x) \quad (9)$$

for $n = 1, 2, \ldots$ and we define the distribution $B_-(x,n)$ by the equation

$$B_-(x,n) = x - \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i (x+i)^{-1} H(-x) \quad (10)$$

for $n = 1, 2, \ldots$, where $H$ denotes Heaviside’s function. It follows that

$$B(x,n) = B_+(x,n) - B_-(x,n).$$

Differentiating equation (9) $r$ times we have

$$B_+^{(r)}(x,n) = (-1)^r r! x^{-r-1} + \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{r+i}(x+i)^{-r-1} H(x) + \sum_{i=1}^{n-1} \sum_{j=1}^{r} \binom{n-1}{i} (-1)^{i+j} (j-1)! i^{-j} \delta^{(r-j)}(x) \quad (11)$$

and differentiating equation (10) $r$ times we have

$$B_-^{(r)}(x,n) = r! x^{-r-1} - \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{r+i}(x+i)^{-r-1} H(-x) + \sum_{i=1}^{n-1} \sum_{j=1}^{r} \binom{n-1}{i} (-1)^{i+j} (j-1)! i^{-j} \delta^{(r-j)}(x) \quad (12)$$

for $r = 1, 2, \ldots$, see details [49].

By using the distributional approach, it was proved in [11] that

$$\Gamma(0) = \int_0^\infty e^{-t} \ln t \, dt = \Gamma'(1) = -\gamma \quad (13)$$
where \( \gamma \) denotes Euler’s constant, and
\[
\Gamma(-m) + m^{-1} \Gamma(-m + 1) = \frac{(-1)^m}{mm!} \tag{14}
\]
for \( m = 1, 2, \ldots \). More generally, we have
\[
\Gamma^{(r)}(0) = \frac{1}{r + 1} \Gamma^{(r+1)}(1) \tag{15}
\]
for \( r = 1, 2, \ldots \) and
\[
\Gamma(-m) = \int_1^{\infty} t^{-m-1} e^{-t} dt + \int_0^1 t^{-m-1} \left[ e^{-t} - \sum_{i=0}^{m} \frac{(-1)^i}{i!} \right] dt
- \sum_{i=0}^{m-1} \frac{(-1)^i}{i!(m-i)} \tag{16}
\]
for \( m = 0, 1, 2, \ldots \).

Similarly, the incomplete Gamma function \( \gamma(\alpha, x) \) is defined for \( \alpha > 0 \) and \( x \geq 0 \) by
\[
\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u} du \tag{17}
\]
see [11], the integral diverging for \( \alpha \in [0, \infty) \). Now if we let \( \alpha > 0 \), then on integration by parts, we obtain that
\[
\gamma(\alpha + 1, x) = \alpha \gamma(\alpha, x) - x^\alpha e^{-x} \tag{18}
\]
and so we can use equation (17) to extend the the definition of \( \gamma(\alpha, x) \) to negative, non-integer values of \( \alpha \). In particular, it follows that if \(-1 < \alpha < 0 \) and \( x > 0 \), then
\[
\gamma(\alpha, x) = \alpha^{-1} \gamma(\alpha + 1, x) + \alpha^{-1} x^\alpha e^{-x}
= -\alpha^{-1} \int_0^x u^\alpha d(e^{-u} - 1) + \alpha^{-1} x^\alpha e^{-x}
\]
and on integration by parts, we see that
\[
\gamma(\alpha, x) = \int_0^x u^{\alpha-1}(e^{-u} - 1) \, du + \alpha^{-1} x^\alpha.
\]
More generally, it can be easily proved by induction that if \(-r < \alpha < -r+1\) and \(x > 0\), then
\[
\gamma(\alpha, x) = \int_0^x u^{\alpha-1}\left[e^{-u} - \sum_{i=0}^{r-1} \frac{(-u)^i}{i!}\right] \, du + \sum_{i=0}^{r-1} \frac{(-1)^i x^{\alpha+i}}{(\alpha+i)!}.
\]
(19)

As we can see in the above examples even locally integrable functions (in the Lebesgue sense) though discontinuous are infinitely differentiable as generalized functions.

**CONVOLUTION PRODUCTS**

**Definition 17.** Let \(f\) and \(g\) be functions. Then the convolution \(f \ast g\) is defined by
\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) \, dt
\]
for all points \(x\) for which the integral exists.

**Theorem 18.** If the convolution \(f \ast g\) exists, then \(g \ast f\) exists and
\[
(f \ast g) = g \ast f.
\]
Further, if \((f \ast g)'\) and \(f \ast g'\) (or \(f' \ast g\)) exist, then
\[
(f \ast g)' = f \ast g' \quad \text{(or \(f' \ast g\))}.
\]
(21)

**Example 19.** If \(\lambda, \mu > -1\), then \(x_+^{\lambda} \ast x_+^{\mu} = B(\lambda + 1, \mu + 1)x_+^{\lambda+\mu+1}\). Equivalently, \(f_+^{\lambda} \ast f_+^{\mu} = f_+^{\lambda+\mu+1}\). In particular,
\[
x_+^{\lambda} \ast H(x) = \frac{x_+^{\lambda+1}}{\lambda + 1} = \int_{-\infty}^{x} x_+^\lambda \, dx.
\]
Further, if $\lambda, \mu > -1 > \lambda + \mu$, then $x^\lambda \ast x^\mu = B(\lambda + 1, -\lambda - \mu - 1)x^\lambda x^{\mu + 1} + B(\mu + 1, -\lambda - \mu - 1)x^{\lambda + \mu + 1}$.

Now let $f, g$ be locally summable functions and suppose that $\text{supp } f \subseteq [a, b]$. Then if $G$ is a primitive of $g$ and $[c, d]$ is any interval,

$$\int_c^d g(x-t) \, dx = G(d-t) - G(c-t).$$

This implies that the function $\int_c^d g(x-t) \, dx$ is bounded on the interval $[a, b]$ and so $f(t) \int_c^d g(x-t) \, dx$ is a locally summable function. This proves that

$$(f \ast g)(x) = \int_{-\infty}^\infty f(t)g(x-t) \, dt$$
$$= \int_a^b f(t)g(x-t) \, dt$$

exists and further

$$\int_c^d (f \ast g)(x) \, dx = \int_c^d \left\{ \int_a^b f(t)g(x-t) \, dt \right\} dx$$
$$= \int_a^b f(t) \left\{ \int_c^d g(x-t) \, dx \right\} dt,$$

proving that $f \ast g$ is a locally summable function if $f$ has compact support. Similarly, $f \ast g$ is a locally summable function if $g$ has compact support and in either case $f \ast g = g \ast f$.

Finally suppose that $(f \ast g)(x) \neq 0$. Then there exists a point $t_0$ such that $f(t_0)g(x-t_0) \neq 0$ which implies that $t_0 \in \text{supp } f$ and $x-t_0 \in \text{supp } g$. Thus $x \in \text{supp } f + \text{supp } g$ or

$$\text{supp}(f \ast g) \subseteq \text{supp } f + \text{supp } g.$$
We now consider the problem of defining the convolution \( f * g \) of two distributions \( f \) and \( g \). First of all suppose that \( f, g \) are locally summable functions and that \( f * g \) exists. Then for arbitrary \( \phi \in D \) we can write

\[
\langle f * g, \phi \rangle = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t)g(x-t)dt \right\}\phi(x)\,dx
\]

We put

\[
\psi(t) = \int_{-\infty}^{\infty} g(x-t)\phi(x)\,dx
\]

\[
= \int_{-\infty}^{\infty} g(x)\phi(x+t)\,dx = \langle g(x), \phi(x+t) \rangle,
\]

where \( \phi(x+t) \in D \) as a function of \( x \), and \( \psi(t) = \langle g(x), \phi(x+t) \rangle \) in fact exists for every distribution \( g \) and for all \( t \) since \( \phi \) has compact support. It is easy to prove that \( \psi(t) \) is a continuous function for every distribution \( g \in D \). To see this, let \( \{t_n\} \) be an arbitrary sequence converging to \( t_0 \). Then \( \phi(x+t_n) \) converges uniformly to \( \phi(x+t_0) \) together with all its derivatives and each \( \phi(x+t_n) \) has its support contained in some fixed bounded interval. Since \( g \) is a continuous linear functional,

\[
\psi(t_n) = \langle g(x), \phi(x+t_n) \rangle \to \langle g(x), \phi(x+t_0) \rangle = \psi(t_0),
\]

proving the continuity of \( \psi \). Further,

\[
\frac{\psi(t_n) - \psi(t_0)}{t_n - t_0} = \left\langle g(x), \frac{\phi(x+t_n) - \psi(x+t_0)}{t_n - t_0} \right\rangle
\]

\[
\to \left\langle g(x), \phi'(x+t_0) \right\rangle,
\]

proving that \( \psi \) is differentiable with derivative

\[
\psi'(t) = \langle g(x), \phi'(x+t) \rangle.
\]
**Definition 20.** Let $f, g$ be distributions in $D'$ and suppose that either $f$ or $g$ has bounded support or that $f$ and $g$ are bounded on the same side. Then $f \ast g$, the convolution of $f$ and $g$, is defined by

$$\langle f \ast g, \phi \rangle = \langle f(t), \langle g(x), \phi(x + t) \rangle \rangle,$$

for all $\phi \in D$.

**Example 21.** Let $f$ be an arbitrary distribution. Then

$$f \ast \delta^{(r)} = f^{(r)}, \quad r = 0, 1, 2, \ldots$$

and $x^\lambda_+ \ast x^\mu_+ = B(\lambda + 1, \mu + 1)x^{\lambda + \mu + 1}_+$ for $\lambda, \mu, \lambda + \mu + 1 \neq -1, -2, \ldots$

**Multiplication of Distributions**

If $f$ is a distribution in $D'$ and $g$ is an infinitely differentiable function then the product $fg = gf$ is defined by

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all $\phi \in D$ and satisfies the rule

$$f^{(r)}g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[ Fg^{(i)} \right]^{(r-i)}$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}$$

for $r = 1, 2, \ldots$. The first extension of the product of a distribution and an infinitely differentiable function is the following definition, see for example [9].

**Definition 22.** Let $f$ and $g$ be distributions in $D'$ for which on the interval $(a, b)$, $f$ is the $k$-th derivative of a locally summable function $F$ in $L^p(a, b)$
and $g^{(k)}$ is a locally summable function in $L^q(a,b)$ with $1/p + 1/q = 1$.

Then the product $fg = gf$ of $f$ and $g$ is defined on the interval $(a,b)$ by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$  

In the literature, many attempts have been made to define a product of distributions. König was the first to develop a systematic treatment of the subject in an abstract way and showed that there are actually many possible product theories if one gives up some requirements such as associativity, see König [44].

König theory was followed by Güttinger on noticing that certain products of distributions could be defined on subspaces of $D'$. For example, if

$$D_0 = \{ \phi \in D : \phi(0) = 0 \}$$

and $\phi \in D_0$, then $x^{-1}\phi(x) = \psi(x)$ is a continuous function. He then defined the product $\delta(x) \cdot x^{-1}$ on $D_0$ by the equation

$$\langle \delta(x) \cdot x^{-1}, \phi(x) \rangle = \psi(0) = \phi'(0) = -\langle \delta'(x), \phi(x) \rangle.$$  

It follows that $\delta(x) \cdot x^{-1} = -\delta'(x)$ on $D_0$, see [17]. Extending the linear functional to the whole of $D'$ by the Hahn–Banach Theorem, he obtained the equation

$$\delta(x) \cdot x^{-1} = -\delta'(x) + c_0\delta(x),$$

where $c_0$ is an arbitrary constant. More generally,

$$\delta^{(r)}(x) \cdot x^{-1} = (-1)^{r+1}\delta^{(r+1)}(x) + \sum_{i=0}^{r} c_i \delta^{(i)}(x), \quad (22)$$
where \( c_0, c_1, \ldots, c_r \) are arbitrary constants. Formal differentiation of Equation (22) gives

\[
\delta^{(r+1)}(x) \cdot x^{-1} - \delta^{(r)}(x) \cdot x^{-2} = (-1)^{r+1} \delta^{(r+2)}(x) + \sum_{i=0}^{r} c_i \delta^{(i+1)}(x)
\]

and so \( \delta^{(r)}(x) \cdot x^{-2} \) is defined by

\[
\delta^{(r)}(x) \cdot x^{-2} = \delta^{(r+1)} \cdot x^{-1} + (-1)^r \delta^{(r+2)}(x) - \sum_{i=0}^{r} c_i \delta^{(i+1)}(x)
\]

\[
= 2(-1)^r \delta^{(r+2)}(x) + c_0 \delta(x) + \sum_{i=1}^{r+1} (c_i - c_{i-1}) \delta^{(i)}(x). \tag{23}
\]

Note that the \( c_i - c_{i-1} \) are not considered to be further independent constants and expression for the product \( \delta^{(r)}(x) \cdot x^{-3} \) can now be found by formally differentiating equation (23).

In a similar way, Güttinger obtained the product

\[
\delta^{(r)}(x) \cdot H(x) = - \sum_{i=0}^{r} b_i \delta^{(r-i)}(x), \tag{24}
\]

where \( b_0, b_1, \ldots, b_r \) are again arbitrary constants. Formal differentiation of equation (24) gives

\[
\delta^{(r+1)}(x) \cdot H(x) + \delta^{(r)}(x) \cdot \delta(x) = - \sum_{i=0}^{r} b_i \delta^{(r-i+1)}(x)
\]

and so \( \delta^{(r)}(x) \cdot \delta(x) \) is defined by

\[
\delta^{(r)}(x) \cdot \delta(x) = -\delta^{(r+1)}(x) \cdot H(x) - \sum_{i=0}^{r} b_i \delta^{(r-i+1)}(x) = b_{r+1} \delta(x).
\]

The constants in equations (23) and (24) are completely independent and they would also be completely independent of any new constants introduced.
to define further products, unless of course they are obtained by formal differentation.

However the products which result from this approach are generally neither commutative nor associative, and there is an inherent arbitrariness as the presence of the arbitrary constants in the examples given above makes clear. For these reasons Konig and Güttinger approach has generally found less favor than the sequential treatments developed by Mikusiński in [46]. These have been generally guided by the idea of developing a product which remains consistent with the Schwartz product and which further extends its domain of definition, see Hoskins [23].

However, the products of some singular distributions very important to applications, but does not exist in the sense of definition 22. Then there are some further extension of this definition in order to apply the product to the wide range of distributions. In order to include the singular distributions the definition of product was extended in two directions as follows:

One way is the Fourier Transform method by using the convolution method one can define the product of distributions which is known as the Fourier Transform method. For given two distributions \( f, g \in D' \) assume that their Fourier transforms \( F(f), F(g) \) exists. Then the product of two distributions \( f \) and \( g \) is defined by following equation

\[
f.g = F^{-1} (F(f) \ast F(g)) ,
\]

(25)

see for example Bremermann [4].
The second method is the regularization and passage to the limit. Sometimes it is known as the methods of the sequential completion for the product of distributions that is also compatible with the ordinary product. This method was first used by Mikusiński and Sikorski in [46] for a wide range of irregular distributions. To deal with the sequential completion approach we need the following concept of delta sequences.

**Delta Sequences and Convergence**

**Definition 23.** A sequence $\delta_n : \mathbb{R} \rightarrow \mathbb{R}$ is called a delta sequence of ordinary functions which converges to the singular distribution $\delta(x)$ and satisfy the following conditions:

(i) $\delta_n(x) \geq 0$ for all $x \in \mathbb{R}$,

(ii) $\delta_n$ is continuous and integrable over $\mathbb{R}$ with $\int_{-\infty}^{\infty} \delta_n(x) \, dx = 1$,

(iii) Given any $\epsilon > 0$,

$$\lim_{n \to \infty} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \delta_n(x) \, dx = 0.$$  

For example, $\delta_n(xt) = \frac{n}{\pi(n^2t^2 + 1)}$ then

$$\int_{a}^{b} \delta_n(t) \, dt = \int_{a}^{b} \frac{n}{\pi(n^2t^2 + 1)} \, dt$$

$$= \frac{1}{\pi} [\arctan(nb) - \arctan(an)]$$

now if we let $n \to \infty$ than it follows that $\delta_n$ is a delta sequence. In general, if $\phi$ is a continuous, nonnegative, $\phi(x) = 0$ for all $|x| \geq 1$ and $\int_{-1}^{1} \phi(x) \, dx = 1$, if we set $\delta_n(x) = n\phi(nx)$. Then one can show that $\delta_n$ is a delta sequence.
Thus the above examples show that there are several ways to construct a delta sequence. For our next definition we let $\rho(x)$ be a fixed infinitely differentiable function in $D$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$.

The function $\delta_n$ is then defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the delta function $\delta$. If now $f$ is an arbitrary distribution in $D'$, the function $f_n$ is defined by

$$f_n(x) = (f \ast \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f$.

Let $f, g \in D'$ then the convolutions $f_n = f \ast \delta_n$ and $g_n = g \ast \delta_n$ always exists as infinitely smooth functions for each $n$ if $(\delta_n)$ are delta sequences. By using the above regular sequence idea we can develop an alternative generalization of definition 22 and based on the product $f \phi = \phi f \in D'$ for a distribution $f \in D'$ and infinitely differentiable function $\phi \in C^\infty$ which was introduced by Schwartz in [50].

Therefore in the literature there are several product of distributions by using delta sequences. In summary, if we let $f, g \in D'$ be two distributions then product of two distributions can be defined either of the following
\begin{align*}
  f(g_n) &= \lim_{n \to \infty} f(g \ast \delta_n) \quad (26) \\
  (f_n)g &= \lim_{n \to \infty} (f \ast \delta_n)g \quad (27) \\
  (f_n)(g_n) &= \lim_{n \to \infty} (f \ast \delta_n)(g \ast \epsilon_n) \quad (28) \\
  (f \cdot g)_n &= \lim_{n \to \infty} (f \ast \delta_n)(g \ast \delta_n) \quad (29)
\end{align*}

provided that the limits exist in the space $D'$ for arbitrary delta sequences $(\delta_n)$ and $(\epsilon_n)$. Equation (26) is due to Mikusinski and Sikorski [46], equations (27) and (28) to Hirata and Ogata [21] required both simultaneously, (29) is due to Fisher [9]. It should be noted that if the respective limits exist in the above definitions then they are independent of the choice of the sequence defining the $\delta$ function. By using above definitions one can propose several results for $\delta^2(x)$ such as
\[ \delta^2(x) = 0, \quad c_1 \delta(x), \quad c_1 \delta(x) + \frac{1}{2\pi i} \delta'(x), \quad c_1 \delta(x) + c_2 \delta'(x) \]
with arbitrary constants $c_1$ and $c_2$. Later Tysk in [54] gave a comparison between equations (28) and (29).

However one can combine these two non-symmetric equations and generate another new commutative product as follows:
\begin{align*}
  \langle f \cdot g, \phi \rangle &= \lim_{n \to \infty} \frac{1}{2} \langle f(g_n) + (f_n)g, \phi \rangle \\
  &= \lim_{n \to \infty} \langle f(g \ast \delta_n) + (f \ast \delta_n)g, \phi \rangle \quad (30)
\end{align*}
for all $\phi \in D$, see Kılıçman [35].
APPLICATIONS OF DISTRIBUTIONS
With admission of the delta function (or distribution) we can also have a solution for the following equation

\[ x^n \cdot f(x) = g(x), \]

where \( g(x) \) assumed to be ordinary function. In fact this idea can also be extended further as follows. Consider that \( p(x) \) a polynomial having the zeros at \( x = a_1, x = a_2, \ldots x = a_n \) that is

\[ p(x) = (x - a_1) (x - a_2) (x - a_3) \ldots (x - a_n) = \prod_{i=1}^{n} (x - a_i). \]

Then the equation \( p(x) \cdot f(x) = g(x) \) has the distributional solutions

\[ f(x) = \frac{g(x)}{p(x)} + \sum_{i=0}^{n-1} c_i \delta^{(i)}(x - a_i), \quad (31) \]

for any constants \( c_1, c_2, c_3, \ldots, c_n \) and \( a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_n \), see Kılıçman [41].

To solve a differential equation there are several methods and each method requires different techniques and there are no general method that will solve all the differential equations.

To solve a differential equation there are several methods and each method requires different techniques and there are no general method that will solve all the differential equations. We list the common methods by using the some sophisticated software such as Scientific Work Place or MAPLE:

(a) Exact Solutions Method, In this method return exact solutions to a differential equation. This method is a more general method
that it can work for some nonlinear differential equations as well. Each of these options recognizes some functions that the other may not.

(b) Integral Transform Methods, Laplace transforms, Fourier transform, Mellin transform, sin and cos transforms that solve either homogeneous or non homogeneous systems in which the coefficients are all constants. Initial conditions appear explicitly in the solution.

(c) Numerical Solutions, Some Appropriate systems can be solved numerically. These numeric solutions are functions that can be evaluated at points or plotted. The method implemented by Maple for numerical solutions is a Fehlberg fourth-fifth order Runge-Kutta method.

(d) Series Solutions, For many applications, a few terms of a Taylor series solution are sufficient. We can also control the number of terms that appear in the solution by changing series order.

**Example 24.** Consider the following differential equations

\[ \frac{dy}{dx} = x \sin \frac{1}{x} \]

and the exact solution is given by:

\[ y(x) = \frac{1}{2} \left( \sin \frac{1}{x} \right) x^2 + \frac{1}{2} \left( \cos \frac{1}{x} \right) x + \frac{1}{2} \text{Si} \left( \frac{1}{x} \right) + C_1. \]

**Example 25.** Similar to the previous example, consider to find the general solution of differential equation

\[ x^2 \frac{dy}{dx} + xy = \sin x \]
then exact solution is given by
\[ y(x) = \frac{1}{x} (\text{Si}(x) + C_1). \]

**Distributional Solutions**

Consider the initial-value problem
\[
\frac{d^2 y}{dx^2} + y = \sum_{k=0}^{\infty} \delta(x - k\pi), \quad y(0) = y'(0) = 0
\]
then we give the solution as
\[
y(x) = \sum_{k=0}^{\infty} (-1)^k (H(x - k\pi)) \sin(x) \\
+ C_1 \sin(x) + C_2 \cos(x).
\]

However there are no serial solution for these differential equations, see Kilicman and Hassan [40].

In fact when we try to solve the differential equation
\[ P(D) y = f(x) \]
we might have either of the following cases, see the details by Kanwal [25].

(i) The solution \( y \) is a smooth function such that the operation can be performed in the classical sense and the resulting equation is an identity. Then \( y \) is a classical solution.

(ii) The solution \( y \) is not smooth enough, so that the operation cannot be performed but satisfies as a distributions.

(iii) The solution \( y \) is a singular distribution then the solution is a distributional solution.
**Example 26.** Let $f$ be the given distribution and if we can find a fundamental solution $g$, then we are able to solve the equation

$$ P(D) g = f $$

when $f \ast g$ is defined. Now consider to find the general solutions of the following ordinary differential equations:

$$ g'' + g = \delta, \quad f'' + f = \delta' $$

then on using the delta sequences we have

$$ y'''_n + y_n = \delta_n \rightarrow y'' + y = \delta \quad \text{and} \quad y''_n + y_n = (\delta_n)' \rightarrow y'' + y = \delta' $$

where $y''_n = y'' \ast \delta_n$ and $y_n = y \ast \delta_n$ as $n$ tends to $\infty$. Then we can take

$$ \delta_n(t) = n - nH \left( t - \frac{1}{n} \right) = \begin{cases} n & 0 < t < \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases} $$

By using the Laplace Transform one can easily show that

$$ y_n(t) = n - n \cos t - \left( n - n \cos \left( t - \frac{1}{n} \right) \right) H \left( t - \frac{1}{n} \right) $$

$$ = \begin{cases} n - n \cos t & 0 < t < \frac{1}{n} \\ n \cos \left( t - \frac{1}{n} \right) - n \cos t & \frac{1}{n} < t \end{cases} $$

for fix $t > 0$, then for large enough $n$ we have

$$ \lim_{n \to \infty} y_n(t) = \sin t $$

and also for $t = 0$,

$$ \lim_{n \to \infty} y_n(0) = 0 = \sin 0 $$

therefore

$$ \lim_{n \to \infty} y_n(t) = \sin t $$
as a symbolic solution. We can also have same result by using the Laplace Transform directly, see Nagle and Saff [47].

Example 27. But if we have differential equations in the form of

$$xf' = 1$$

then this equation has distributional solution

$$f = c_1 \ln |x| + c_1 H(x) + c_3$$

of course this not classical solution since $f$ is not differentiable at zero, see Kılıçman [41]. Now if we replace 1 by $\delta$ then we try to find the fundamental solution for

$$xf' = \delta \implies f' = x^{-1} \delta$$

or more general form of

$$x^s f' = \delta^{(r)} \implies f' = x^{-s} \delta^{(r)}, \quad \text{for } r, s = 0, 1, 2, 3, 4, \ldots$$

Now we can ask the following question: What is the interpretation of this?

There is no general method that can solve all the differential equations. Each might require different methods. Now consider to find the fundamental solution for

$$xy' = \delta(x) \implies y' = x^{-1} \delta(x)$$

or more general form of

$$x^s y^{(n)} = \delta^{(r)}(x) \implies y^{(n)} = x^{-s} \delta^{(r)}(x),$$

for $n, r, s = 0, 1, 2, 3, 4, \ldots$. 
The equation \( y'' = x^{-1} \delta \) has no classical solution on \((-1, 1)\). However on using the distributional approach then we have

\[
y(x) = (f \ast g)(x)
\]

\[
= (x^{-1} \delta(x)) \ast (\text{Heaviside}(x)x + C_1x + C_2)
\]

\[
y(x) = \int \left( \int \left( \frac{\delta(x)}{x} \right) dx + xC_1 dx \right) + C_2
\]

as a distributional solution since

\[
y(x) = \text{Heaviside}(x)x + C_1x + C_2
\]

is a solution for elementary equation

\[
y''(x) = \delta(x).
\]

In general, if we have the equation as \( y' = fg \) where \( f, g \in D' \) then we have the following three interpretations to solve it, \( y' \ast \delta_n = (f \ast \delta)g \), \( y' \ast \delta_n = f(g \ast \delta_n) \) and \( y' \ast \delta_n = (f \ast \delta_n)(g \ast \delta_n) \) by using the neutrix limit

\[
\lim_{n \to \infty} y' \ast \delta_n = \lim_{n \to \infty}(f \ast \delta_n)g \quad \text{(34)}
\]

\[
\lim_{n \to \infty} y' \ast \delta_n = \lim_{n \to \infty} f(g \ast \delta_n) \quad \text{(35)}
\]

\[
\lim_{n \to \infty} y' \ast \delta_n = \lim_{n \to \infty}(f \ast \delta_n)(g \ast \delta_n). \quad \text{(36)}
\]

So we can easily see that solving the differential equation is reduced to existence of the distributional products. The same procedure applies in the case of more general differential equations.

For example, suppose that we want to find the distribution \( g \) satisfying

\[
P(D)g = f,
\]

(37)
where $P(D)$ is the generalized differential operator given by

$$P(D) = a_0(x) \frac{d^s}{dx^s} + a_1(x) \frac{d^{s-1}}{dx^{s-1}} + \ldots + a_s(x)$$

Note if $f$ is a regular distribution generated by a locally integrable function but not continuous or if it is a singular distribution then equation (37) has no meaning in the classical sense. The solution in this case is called a weak or distributional solution.

While it is possible to add distributions, it is not possible to multiply distributions easily, especially when they have coinciding singular support. Despite this, it is possible to take the derivative of a distribution, to get another distribution. Consequently, they may satisfy a linear partial differential equation, in which case the distribution is called a weak solution. For example, given any locally integrable function $f$ it makes sense to ask for solutions $u$ of Poisson’s equation

$$\nabla^2 u = f$$

by only requiring the equation to hold in the sense of distributions, that is, both sides are the same distribution. For example, the problem for the Greens function is as follows. We scale cylindrical coordinates $(r, \theta, z)$ so that the boundary conditions are imposed on $r = 1$. The Greens function satisfies

$$\nabla^2 G = -4\pi \delta(x)$$
and the boundary condition \( \frac{\partial G}{\partial r} = 0 \) on \( r = 1 \). We know that \( u(x,t) = H(x - ct) \) solves the wave equation. This area still need some more research we only list Friedman [15] for the introductory level and more recently Farassat [7].

We can extend the single Laplace transform of delta function to double Laplace transform as follows:

\[
L_x L_t [\delta(t-a)\delta(x-b)] = \int_0^\infty e^{-px} \int_0^\infty e^{-st} \delta(t-a)\delta(x-b) dtdx
= e^{-sa-pb}
\]

and also double Laplace transform of the partial derivative with respect to \( x \) and \( t \) as

\[
L_x L_t \left[ \frac{\partial}{\partial t} \delta(t-a) \frac{\partial}{\partial x} \delta(x-b) \right] = \left[ \frac{\partial^2}{\partial x \partial t} (e^{-st-px}) \right]_{t=a,x=b}
= pse^{-sa-pb}.
\]

In general multiple Laplace transform of delta function in \( n \) dimensional given by

\[
L_{t_1} \cdots L_{t_n} [\delta(t_1-a_1)\delta(t_2-a_2)\cdots\delta(t_n-a_n)] = e^{-s_1a_1-s_2a_2-\cdots-s_na_n}
\]

where \( L_{t_n} \) means multiple Laplace transform in \( n \) dimensional. Kanwal, (2004) defined the classical derivative of a function

\[
f(t) = \begin{cases} 
g_2(t), & t > a 
g_1(t), & t < a
\end{cases}
\]

\[
f(t) = g_1(t)H(a-t) + g_2(t)H(t-a)
\]
where \( a > 0 \) and \( g_1, g_2 \) are continuously differentiable function by

\[
f'(t) = g'_1(t)H(a - t) + g'_2(t)H(t - a)
\]

for all \( t \neq a \) see [3]. We try to extend Kanwal’s result from single variable to two variables as

\[
f(x, t) = \begin{cases} 
  g_2(x, t), & x > a, \ t > b \\
  g_1(x, t), & x < a, \ t < b 
\end{cases} \tag{38}
\]

The above function can be written in the form

\[
f(x, t) = g_1(x, t)H(a - x)H(b - t) + g_2(x, t)H(x - a)H(t - b) \tag{39}
\]

where \( a > 0 \) and \( b > 0 \) and \( g_1, g_2 \) are continuously differentiable function the classical partial derivative respect to \( t, x \) given by

\[
f_t = \frac{\partial g_1(x, t)}{\partial t}H(a - x)H(b - t) + \frac{\partial g_2(x, t)}{\partial t}H(x - a)H(t - b). \tag{40}
\]

If we take the derivative with respect to \( x \) in equation (40) we obtain

\[
f_{tx} = \frac{\partial^2 g_1(x, t)}{\partial t \partial x}H(a - x)H(b - t) + \frac{\partial^2 g_2(x, t)}{\partial t \partial x}H(x - a)H(t - b) \tag{41}
\]

now if we take second partial derivative with respect to \( x \) we get

\[
f_{xx} = \frac{\partial^2 g_1(x, t)}{\partial x^2}H(a - x)H(b - t) + \frac{\partial^2 g_2(x, t)}{\partial x^2}H(x - a)H(t - b) \tag{42}
\]

similarly, we take second partial derivative with respect to \( t \)

\[
f_{tt} = \frac{\partial^2 g_1(x, t)}{\partial t^2}H(a - x)H(b - t) + \frac{\partial^2 g_2(x, t)}{\partial t^2}H(x - a)H(t - b) \tag{43}
\]
for all \( x \neq a \) and \( t \neq b \). The generalized partial derivative of equation (39) with respect to \( x \) follows

\[
\begin{align*}
\mathcal{F}_x(x,t) &= \frac{\partial g_1(x,t)}{\partial x} H(a-x)H(b-t) - g_1(x,t)\delta(a-x)H(b-t) \\
&+ \frac{\partial g_2(x,t)}{\partial x} H(x-a)H(t-b) + g_2(x,t)\delta(x-a)H(t-b)
\end{align*}
\]

(44)

and the generalized partial derivative of equation (44) with respect to \( t \) given by

\[
\begin{align*}
\mathcal{F}_{xt}(x,t) &= \frac{\partial^2 g_1(x,t)}{\partial x \partial t} H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial x \partial t} H(x-a)H(t-b) \\
&+ \frac{\partial g_1(x,t)}{\partial t} H(a-x)\delta(t-b) - \frac{\partial g_1(x,t)}{\partial x} H(a-x)\delta(b-t) \\
&+ \frac{\partial g_2(x,t)}{\partial t} \delta(x-a)H(t-b) - \frac{\partial g_1(x,t)}{\partial t} \delta(a-x)H(b-t) \\
&+ g_2(x,t)\delta(x-a)\delta(t-b) + g_1(x,t)\delta(a-x)\delta(b-t).
\end{align*}
\]

(45)

Similar to the previous equation the generalized second partial derivative with respect to \( x \) follows

\[
\begin{align*}
\mathcal{F}_{xx}(x,t) &= \frac{\partial^2 g_1(x,t)}{\partial x^2} H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial x^2} H(x-a)H(t-b) \\
&- 2\frac{\partial g_1(x,t)}{\partial x} \delta(a-x)H(b-t) + 2\frac{\partial g_2(x,t)}{\partial x} \delta(x-a)H(t-b) \\
&+ g_1(x,t)\frac{\partial \delta(a-x)}{\partial x} H(b-t) + g_2(x,t)\frac{\partial \delta(x-a)}{\partial x} H(t-b)
\end{align*}
\]

(46)
and generalized second partial derivative with respect to $t$ given by

$$\mathcal{F}_{tt}(x, t) = \frac{\partial^2 g_1(x, t)}{\partial t^2} H(a - x)H(b - t) + \frac{\partial^2 g_2(x, t)}{\partial t^2} H(x - a)H(t - b)$$

$$-2 \frac{\partial g_1(x, t)}{\partial t} H(a - x)\delta(b - t) + 2 \frac{\partial g_2(x, t)}{\partial t} H(x - a)\delta(t - b)$$

$$+ g_1(x, t) H(a - x) \frac{\partial \delta(b - t)}{\partial t} + g_2(x, t) H(x - a) \frac{\partial \delta(t - b)}{\partial t}.$$  

(47)

Now we use double Laplace transform for equation (42)

$$f_{xx} = \frac{\partial^2 g_1(x, t)}{\partial x^2} H(a - x)H(b - t) + \frac{\partial^2 g_2(x, t)}{\partial x^2} H(x - a)H(t - b)$$

form definition of equation (38) and we take Laplace transform with respect to $x$ equation (42) becomes

$$L_x[f_{xx}] = H(b - t) \int_0^a e^{-px} \frac{\partial^2 g_1(x, t)}{\partial x^2} dx + H(t - b) \int_a^\infty e^{-px} \frac{\partial^2 g_2(x, t)}{\partial x^2} dx$$

(48)

if we integrate by part the first and second terms of equation (48), then we obtain

$$L_x[f_{xx}] = H(b - t) \left[ e^{-pa} \frac{\partial g_1(a, t)}{\partial x} - \frac{\partial g_1(0, t)}{\partial x} + pe^{-pa} g_1(a, t) \right]$$

$$+ H(b - t) \left[ -pg_1(0, t) + p^2 \int_0^a e^{-px} g_1(x, t) dx \right]$$

$$+ H(t - b) \left[ -e^{-pa} \frac{\partial g_2(a, t)}{\partial x} - pe^{-pa} g_2(a, t) \right]$$

$$+ H(t - b) p^2 \int_a^\infty e^{-px} g_2(x, t) dx.$$  

(49)
By taking Laplace transform with respect to \( t \) for equation (49), then we obtain double Laplace Transform for equation (42) as

\[
L_tL_x[f_{xx}] = p e^{-pa} \left[ \int_0^b e^{-st} g_1(a,t) dt - \int_b^\infty e^{-st} g_2(a,t) dt \right] + e^{-pa} \left[ \int_0^b e^{-st} \frac{\partial g_1(a,t)}{\partial x} dt - \int_b^\infty e^{-st} \frac{\partial g_2(a,t)}{\partial x} dt \right] - \int_0^b e^{-st} \frac{\partial g_1(0,t)}{\partial x} dt - p \int_0^b e^{-st} g_1(0,t) dt + p^2 F(p,s)
\]

where we assume that the integral exists. In particular if we substitute \( a = 0, b = 0 \) and \( x, t > 0 \) in equation (50), it is easy to see that the equation (50) gives double Laplace Transform of second order partial derivative with respect to \( x \) in classical sense as

\[
L_tL_x[f_{xx}] = p^2 F(p,s) - \frac{\partial g_2(0,s)}{\partial x} - pg_2(0,s)
\]

by the same way we take double Laplace transform with respect to \( x, t \), for equation (43) and we obtain

\[
L_xL_t[f_{tt}] = s e^{-sb} \left[ \int_0^a e^{-px} g_1(x,b) dx - \int_a^\infty e^{-px} g_2(x,b) dx \right] + e^{-sb} \left[ \int_0^a e^{-px} \frac{\partial g_1(x,b)}{\partial t} dx - \int_a^\infty e^{-px} \frac{\partial g_2(x,b)}{\partial t} dx \right] - \int_0^a e^{-px} \frac{\partial g_1(x,0)}{\partial t} dx - s \int_0^a e^{-px} g_1(x,0) dt + s^2 F(p,s)
\]

provided that the integrals exist. In particular if we substitute \( a = 0, b = 0 \) and \( x, t > 0 \) in equation (52) give double Laplace Transform of second order
partial derivative with respect to $t$ in classical sense as

$$L_x L_t [f_{tt}] = s^2 F(p, s) - sg_2(p, 0) - \frac{\partial g_2(p, 0)}{\partial t}$$

double Laplace transform of a mixed partial derivative of equation (41) by similar way we obtain double Laplace transform for mixed partial derivatives as follows

$$L_t L_x [f_{tx}] = e^{-pa}e^{-sb}[g_1(a, b) + g_2(a, b)] + g_1(0, 0) - e^{-pa}g_1(a, 0)$$

$$-e^{-sb}g_1(0, b) + se^{-pa} \left[ \int_0^b e^{-st}g_1(a, t)dt - \int_b^\infty e^{-st}g_2(a, t)dt \right]$$

$$+pe^{-sb} \left[ \int_0^a e^{-px}g_1(x, b)dx - \int_a^\infty e^{-px}g_2(x, b)dx \right]$$

$$-s \int_0^b e^{-st}g_1(0, t)dt - p \int_0^a e^{-px}g_1(x, 0)dx + psF(p, s)$$

(53)

In particular if we substitute $a = 0, b = 0$ and $x, t > 0$ equation (53) becomes

$$L_t L_x [f_{tx}] = psF(p, s) + g_2(0, 0) - s \int_0^\infty e^{-st}g_2(0, t)dt$$

$$-p \int_0^\infty e^{-px}g_2(x, 0)dx$$

(54)

equation (54) can be written in the form

$$L_t L_x [f_{tx}] = psF(p, s) - sF(0, s) - pF(p, 0) + g_2(0, 0)$$

(55)

equation (55) give double Laplace Transform in classical sense for mixed partial derivative with respect to $x, t$.  

Distribution Defined by Divergent Integrals

In this section we try to extend the idea of one dimension pseudo-function to two dimensional. Now if we examine the function in the form

\[ f(x, y) = \begin{cases} 
  x^{-n} y^{-n}, & x, y > 0 \\
  0, & x, y < 0 
\end{cases} \]

\[ = \frac{H(x, y)}{x^n y^n} \quad \text{(56)} \]

where \( n \) is positive integer and \( H(x, y) = \begin{cases} 
  1, & x, y > 0 \\
  0, & x, y < 0 
\end{cases} \) then we can write in the form of tensor product as \( H(x, y) = H(x) \otimes H(y) \). We first consider the simple case \( n = 1 \) there for study the integral

\[
\langle f(x,y), \phi(x,y) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H(x) \otimes H(y)}{xy} \phi(x,y) \, dx \, dy \\
= \int_{0}^{\infty} \frac{1}{y} \left[ \int_{0}^{\infty} \frac{1}{x} \phi(x,y) \, dx \right] \, dy. \quad \text{(57)}
\]

Now if we consider Taylor series as

\[ \phi(x, y) = \phi(0,0) + y\phi_y(0,0) + x\phi_x(0,0) + xy\psi(x,y) \quad \text{(58)} \]

where \( \psi(x, y) \) defined by

\[
\psi(x, y) = \frac{1}{2} xy^{-1} \phi_{xx}(0,0) + \phi_{xy}(0,0) + \frac{1}{2} y x^{-1} \phi_{yy}(0,0) + \\
+ \ldots + \frac{x^{n-1-k} y^{k-1}}{(n-k)!k!} \frac{\partial^n \phi(t,t)}{\partial x^{n-k} \partial y^k} \quad \text{for} \quad 0 < t < 1 \quad \text{(59)}
\]

is continuous function for \( x, y > 0 \), now further consider that the \( \text{supp} \phi(x,y) \subset [0,a] \times [0,b] \) and \( a,b > 0 \). Let us going back to the integral inside bracket in equation (57) have singularity at \( x = 0 \), for \( \varepsilon > 0 \) can be write it in the
form of improper integral

\[
\int_0^\infty \frac{1}{x} \phi(x, y) \, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^a \frac{1}{x} \phi(x, y) \, dx \\
= \lim_{\varepsilon \to 0} [\phi(0, 0) \ln a - \phi(0, 0) \ln \varepsilon + y\phi_y(0, 0) \ln a] \\
+ \lim_{\varepsilon \to 0} [-y\phi_y(0, 0) \ln \varepsilon + a\phi_x(0, 0) - \varepsilon\phi_x(0, 0)] \\
+ \lim_{\varepsilon \to 0} \int_\varepsilon^a y\psi(x, y) \, dx. \tag{60}
\]

Then it follows

\[
\int_0^\infty \frac{1}{x} \phi(x, y) \, dx = \lim_{\varepsilon \to 0} [\phi(0, 0) \ln a - \phi(0, 0) \ln \varepsilon + y\phi_y(0, 0) \ln a] \\
+ \lim_{\varepsilon \to 0} [-y\phi_y(0, 0) \ln \varepsilon + a\phi_x(0, 0) - \varepsilon\phi_x(0, 0)] \\
+ \lim_{\varepsilon \to 0} \int_\varepsilon^a y\psi(x, y) \, dx. \tag{61}
\]

We substitute (61) into (57) and apply the similar technique that we used in above, for \( \beta > 0 \), and calculating the integrals and taking the limit yields the Hadamard finite part of the divergent of equation (57) in the form of

\[
\operatorname{pf} \left( \frac{H(x) \otimes H(y)}{xy} \right) = \frac{\partial^2}{\partial x \partial y} \ln(x) \ln(y).
\]

In the next we study the pseudo-function, see Kanwal (2004) in case \( n = 2 \) as

\[
f(x, y) = \begin{cases} 
0, & x, y < 0 \\
-x^{-2}y^{-2}, & x, y > 0 
\end{cases} 
= \frac{H(x, y)}{x^2y^2}. \tag{62}
\]
Let us now examine the above function

\[
\langle f(x, y), \phi(x, y) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H(x, y)}{x^2 y^2} \phi(x, y) dx dy
\]

By similar way we obtain Hadamard finite part for two dimensional of above equation as follows

\[
FP \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H(x, y)}{x^2 y^2} \phi(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xy}(x, y) \left( \frac{H(x, y)}{xy} \right) dx dy
+ \phi_{xy}(0, 0)
\]

Finally yields the required the relation

\[
pf \left( \frac{H(x, y)}{x^2 y^2} \right) = \frac{\partial^2}{\partial x \partial y} \left[ pf \left( \frac{H(x, y)}{xy} \right) \right] + \delta_{xy}(x, y).
\] (64)

We can continue the above analysis to generalized equation (64) as

\[
pf \left( \frac{H(x, y)}{(xy)^{m+1}} \right) = \frac{\partial^2}{\partial x \partial y} pf \left( \frac{H(x, y)}{m^2 (xy)^m} \right) + \left( \frac{1}{m!} \right)^2 \frac{\partial^{2m}}{\partial x^m \partial y^m} \delta(x, y), \ m \geq 1.
\] (65)

We can generalize the distributional derivative from one dimension pseudo-function see R. F. Hoskins(1979) and Kanwal (2004) and to two dimensional cases.

**Question:** Now consider the equation

\[ P(D) u = f(x, y) \]
and multiply the differential operator by a function then what will happen to the classification. Since convolution compatible with differentiation then we can ask the question what will happen the new classification problem of the

$$(Q(x, t) \ast * P(D)) u = F(x, t).$$

For example,

$$(Q(x, t) \ast * P(\text{Elliptic} )) u = F(x, t)$$

when it will be elliptic and on what conditions. Similarly,

$$(Q(x, t) \ast * P(\text{Hyperbolic} )) u = F(x, t)$$

when it will be Hyperbolic and on what conditions.

We note that in the literature there is no systematic way to generate a partial differential equation with variable coefficients from the PDE with constant coefficients, however the most of the partial differential equations with variable coefficients depend on nature of particular problems, see Kılıçman and Eltayeb [8] and Kılıçman [13].

In particular, consider the differential equation in the form of

$$y''' - y'' + 4y' - 4y = 2 \cos(2t) - \sin(2t)$$

$$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 1.$$  \hfill (66)

Then, by taking the Sumudu transform, we obtain:

$$Y(u) = \frac{u^3 (2u + 1)}{(4u^2 + 1)(1 - u + 4u^2 - 4u^3)} + \frac{(u^2 + 3u + 1)}{(1 - u + 4u^2 - 4u^3)}. \hfill (67)$$
Replacing the complex variable $u$ by $\frac{1}{s}$, Eq. (67) turns to:

$$Y\left(\frac{1}{s}\right) = \frac{s(s + 2)}{(s^2 + 4)(s^2 + 4)(s - 1)} + \frac{s(s^2 + 3s + 1)}{(s^2 + 4)(s - 1)}.$$  \hspace{1cm} (68)

Now in order to obtain the inverse Sumudu transform for Eq.(68), we use

$$S^{-1}(Y(s)) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} Y\left(\frac{1}{s}\right) \frac{ds}{s} = \sum \text{residues} \left[ e^{st} Y\left(\frac{1}{s}\right) \frac{1}{s} \right].$$

Thus, the solution of Eq. (66) is given by:

$$y(t) = \frac{13}{8} \sin(2t) - \frac{1}{4} t \cos(2t) + e^t.$$

Now, if we consider to multiply the left hand side of Eq. (66) by the non constant coefficient $t^2$, then Eq. (66) becomes

$$t^2 \left( y''' - y'' + 4y' - 4y \right) = 2 \cos(2t) - \sin(2t)$$

$$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 1.$$  \hspace{1cm} (69)

By applying a similar method, we obtain the solution of Eq. (69) in the form:

$$y_1(t) = \cos(2t) - t \sin(2t) + \frac{3}{2} \sin(2t).$$

Now in order to see the effect of the convolutions we can see the difference as $||y - y_1||$ on $[0, 1]$, see the detail [42].

**Question:** How to generate a PDE with variable coefficients from the PDE with constants coefficients. For example, if we consider the wave equation
in the following example

\[ u_{tt} - u_{xx} = G(x, t) \quad (x, t) \in \mathbb{R}_+^2 \]
\[ u(x, 0) = f_1(x), \quad u_t(x, 0) = g_1(x) \]
\[ u(0, t) = f_2(t), \quad u_x(0, t) = g_2(t). \]  

(70)

Now, if we consider to multiply the left hand side equation of the above equation by non-constant coefficient \( Q(x, t) \) by using the double convolution with respect to \( x \) and \( t \) respectively, then the equation becomes

\[ Q(x, t) \ast \ast (u_{tt} - u_{xx}) = G(x, t) \quad (t, x) \in \mathbb{R}_+^2 \]  

(71)
\[ u(x, 0) = f_1(x), \quad u_t(x, 0) = g_1(x) \]
\[ u(0, t) = f_2(t), \quad u_x(0, t) = g_2(t). \]  

(72)

Thus the relationship between the solutions partial differential equations with constant coefficients and non constant coefficients was studied in [13].

**Application to Probability Theory**

In order to present the application of generalized function to the theory of probability and Random processes we assume the basic concepts are well known for the probability space. For a random variable \( X \) we define its probability distribution function \( F(x) \) as

\[ F(x) = P\{X < x\} = P\{X^{-1}(-\infty, x)\}, \quad x \in \mathbb{R} \]

and the function \( F \) has the following properties

- \( F \) is monotone
- \( F \) is continuous from left and
- \( \lim_{x \to \infty} F(x) = 1 \) and \( \lim_{x \to -\infty} F(x) = 0 \).
Now if the function $F(x)$ being a locally integrable function and defines a generalized function

$$\langle F(x), \phi(x) \rangle = \int_{-\infty}^{\infty} F(x)\phi(x)dx$$

where $\phi$ is infinitely differentiable function. Accordingly,

$$\langle F'(x), \phi(x) \rangle = -\langle F(x), \phi'(x) \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x)dx$$

$$= -|\phi(x)F(x)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(x) \ d(F(x))$$

$$= \langle f(x), \phi(x) \rangle$$

where $f(x) = \frac{dF}{dx}$ is called the probability density function. The density function $f(x)$ has the following properties:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$.
- $f(-\infty) = 0$, $f(\infty) = 1$ we find that

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

**Tossing a Coin** Let us consider the tossing of a coin. We assign the value $x = 0$ if we obtain heads and the value $x = 1$ if we get tails. In order to evaluate the probability distribution $F(x)$ we have the following:

(i) If head $x = 0$, if tails we have $x = 1$. Then the probability we have only two cases, either $\{x = 0\}$ or $\{x = 1\}$ so that

$$x < 0 \quad \text{yields} \quad F(x) = 0, \quad x \leq 0.$$

(ii) $0 < x \leq 1 \Rightarrow F(x) = \frac{1}{2}$ and
(iii) If $x > 1$, $F(x) = 1$ thus we find

$$F(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
\frac{1}{2} & \text{if } 0 < x \leq 1 \\
1 & \text{if } x > 1 
\end{cases}$$

$$= \frac{1}{2} \left[H(x) + H(x - 1)\right]$$

where $H$ is the Heaviside function. Then the probability density function is

$$f(x) = \frac{1}{2} \left[\delta(x) + \delta(x - 1)\right].$$

In the case of Random variable $X$ takes the values $a_1, a_2, a_3, \ldots, a_n$ with the probabilities $p_1, p_2, p_3, \ldots p_n$ respective such that

$$\sum_{k=0}^{n} p_k = 1$$

The generalized function

$$f(x) = \sum_{k=1}^{n} p_k \delta(x - a_k)$$

is the probability density function.

- If $X$ has the Binomial distribution then

$$f(x) = \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} \delta(x - k), \ 0 \leq p \leq 1, \ q = 1 - p$$

then the Probability function

$$F(x) = \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} H(x - k).$$
If $X$ has the Poisson distribution then the
\[
f(x) = e^{-\lambda} \sum_{k=1}^{\infty} \left( \frac{\lambda^k}{k!} \right) \delta(x - k), \quad \lambda > 0
\]
then the
\[
F(x) = e^{-\lambda} \sum_{k=1}^{\infty} \left( \frac{\lambda^k}{k!} \right) H(x - k).
\]

The Characterization of Random Variables

- **Expectation Value of $X$**
  \[
  E(x) = \int_{\mathbb{G}} X(u) dP(u) = \int_{-\infty}^{\infty} x dF(x) = \langle x, f \rangle.
  \]

- **The variance of $X$**
  \[
  D(x) = \int_{\mathbb{G}} (X(u) - E(x))^2 dP(u)
  = \int_{-\infty}^{\infty} (x - E(x))^2 dF(x)
  = \langle (x - E(x))^2, f(x) \rangle.
  \]

- **The $m$-moment of $X$**
  \[
  E(x^m) = \int_{\mathbb{G}} (X(u))^m dP(u)
  = \int_{-\infty}^{\infty} x^m dF(u) = \langle x^m, f(x) \rangle.
  \]

Since the $m$-moment function of $f(x)$ is defined by
\[
\langle f(x), x^m \rangle = \int_{-\infty}^{\infty} f(x) x^m dx.
\]

Then we consider a test function $\phi(x)$ and its Taylor series is
\[
\phi(x) = \sum_{m=0}^{\infty} \phi^{(m)}(0) \frac{x^m}{m!}.
\]
then it follows by putting that $\phi^{(m)}(0) = (-1)^m(\delta^{(m)}(x), \phi(x))$ then easily we see that

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m E(x^m)}{m!} \delta^{(m)}(x).$$

Now if we apply this to the asymptotic analysis we have

$$f(\lambda x) \sim \sum_{m=0}^{\infty} \frac{(-1)^m E(x^m)}{m! \lambda^{m+1}} \delta^{(m)}(x).$$

If $f(x) = e^{-x}H(x)$ then the moments are

$$E(x^m) = \int_{0}^{\infty} e^{-x}x^m = \Gamma(m + 1) = m!$$

the moment expansion is

$$H(x)e^{-x} = \sum_{m=0}^{\infty} (-1)^m \delta^{(m)}(x)$$

then the asymptotic expansion is

$$H(x)e^{-\lambda x} \sim \sum_{m=0}^{\infty} \frac{(-1)^m \delta^{(m)}(x)}{\lambda^{m+1}}, \lambda \to \infty$$

now if we set $\lambda = \frac{1}{\epsilon}$ then we have

$$H(x)e^{-\frac{x}{\epsilon}} = \sum_{m=0}^{\infty} (-1)^m \delta^{(m)}(x)e^{m+1}, \epsilon \to 0$$

in fact this is the case in the boundary layer problem.

The Characteristic function of a Random variable

Since the probability distribution $f(x)$ is a generalized function we can find its Fourier Transform. Thus

$$E(e^{iux}) = \int_{-\infty}^{\infty} e^{iux} f(x) \, dx = F(f) = \chi(u)$$
and the
\[ f(x) = F^{-1}(\chi(u)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu} \chi(u) \, du. \]

Now let us take \( \chi(u) = e^{\lambda u} \) then
\[
\begin{align*}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu} \chi(u) \, du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu} e^{\lambda u} \, du \\
&= \int_{-\infty}^{\infty} e^{-iu(x-\lambda)} \, du = \delta(x - \lambda). \end{align*}
\]

**Application to Economics** Most models of the dynamical behavior of an economic system usually assume that the variables of the system are the function of the time. In fact this not general but reasonable assumption, such as the price of the certain commodity or the prevailing interest rate.

The size of the capital stock can be observed at almost all times, but it might suffer jumps when additions are made in a very short period. A Dirac delta function placed at the instant of the jump is the best description of the investment. The basic dynamic model for the investment decision of the firm postulates that the investment schedule \( I(t) \) for \( t \geq t_0 \) is chosen at the time \( t = t_0 \) in such a way as to maximize the present value of the future stream of profits.

\[
\phi = \phi(I) = \int_{t_0}^{\infty} V(t-t_0) \left\{ R(t, K(t)) - r(I(t)) \right\} dt
\]

where

- \( K(t) \) is the capital stock at the time \( t \), then

\[
K'(t) = \frac{dK}{dt} = I(t)
\]
• $R(t, K)$ is the expected quasi-rent to be obtained from capital stock of size $K$ at time $t$,
• $r(I)$ is the cost of adjustment; and
• $V(t)$ is the discount factor.

Then the cost of the adjustment is a non linear functional of the investment $I(t)$ given by

$$C(V, I) = \int_{t_0}^{\infty} V(t - t_0)r(I(t))dt$$

for a given discount factor $V(t)$.

**The Radon Transform and Tomography**

Recently, impact of computer technology has informed us that there is a great need of further developments of distribution theory in Applied Sciences.

It turns out that all the conclusions about distributions in $D$ can be extended to the distributions on multidimensional spaces. One encounter the two-and three dimensional impulse symbols $\delta(x, y)$ and $\delta(x, y, z)$ as a natural expansion of $\delta$. For example we can interpret

- the $\delta(x, y)$ describes the pressure distribution over the $(x, y)$-plane when a concentrated unit force is applied at the origin.
- the $\delta(x, y, z)$ describes the charge density in a volume containing a unit charge at the point $(0, 0, 0)$. 
Then we have

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) dx \, dy = 1 \]
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y, z) dx \, dy \, dz = 1 \]
\[ \delta(x, y, z) = \delta(x, y) \delta(z) = \delta(x) \delta(y) \delta(z). \]

The basic problem of tomography is given a set of 1-D projections and the angles at which these projections were taken, then the problem is how to reconstruct the 2-D image from which these projections were taken.

In recent years the Radon transform have received much attention which is able to transform two dimensional images with lines into a domain of possible line parameters, where each line in the image will give a peak positioned at the corresponding line parameters.

This have lead to many line detection applications within image processing, computer vision, and seismic. There are several definitions of the Radon transform in the literature, but they are related, and a very popular form expresses lines in the form \( R = x \cos(\theta) + y \sin(\theta) \), where \( \theta \) is the angle and \( R \) the smallest distance to the origin of the coordinate system.

The Radon transform for a set of parameters \((R, \theta)\) is the line integral through the image \( f(x, y) \), where the line is positioned corresponding to the value of \((R, \theta)\). The delta \( \delta \) is the Dirac delta function which is infinite for argument 0 and zero for all other arguments (it integrates to one).
If the density distribution is \( f(x, y) \) which is not symmetrical but depends on two coordinates, the scans may still be taken but they will depend on the direction of the scanning \( \theta \), Calling the abscissa for each scan \( R \) we define the Radon Transform

\[
Q(\theta, R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(R - x \cos \theta - y \sin \theta) dx \, dy.
\]

The factor \( \delta(R - x \cos \theta - y \sin \theta) \) is zero everywhere except where its argument is zero, which is along the straight line \( x \cos \theta + y \sin \theta = R \). In fact this is equivalent with

\[
Q(\theta, R) = \int_{-\infty}^{\infty} f(R \cos \theta - t \sin \theta, R \sin \theta + t \cos \theta) \, dt.
\]

The collection of these \( Q(\theta, R) \) at all \( \theta \) is called the Radon Transform of image \( f(x, y) \).
Bibliography

Fisher, B and Kılıçman, A. (1994), *On the product of the function* $x^r \ln(x + i0)$ *and the distribution* $(x + i0)^{-4}$, Integral Transform and Special Functions, 2(4), 243–252.


Hadamard, J. (1923), Lectures on Cauchy’s Problem in Linear Partial Differential Equations.


[29] Kılıçman, A., Fisher, B. and Serpil Pehlivan (1998), *The neutrix convolution product of* $x^r_+ \ln x_+$ *and* $x^s_- \ln x_-$, Integral Transform and Special Functions, 7(3 – 4), 237–246.


[34] Kılıçman, A. (2001), *On the non-commutative neutrix product* $\Gamma^{(r)}(x_+) \circ x^r_+ \ln x_+$, Pertanika Journal of Science and Technology, 9(2), 157–167.


BIOGRAHPY

Adem Kılıçman was born in Hassa, Hatay, Turkey on 3rd February 1966. He obtained his early education in Tiyek Koyu Ilkokulu. He then completed his secondary school at Hassa Ortaokulu and continued his upper secondary in Hassa Lisesi.

Kılıçman graduated with Bachelor of Science (Hons.) in Mathematics at Hacettepe University, Ankara, Turkey in 1989 and stared to work in Ondokuz Mayıs University, in the Black Sea region. He then continued his Master in the Hacettepe University and graduated in 1991. Kılıçman was sent to do his PhD in University Leicester and obtained his PhD in 1995 from University of Leicester, England.

Currently, Adem Kılıçman is a Professor in the Department of Mathematics at University Putra Malaysia. His research interest includes Functional Analysis, Topology as well as Differential Equations. He has published several research papers in international journals and actively involved in several activities in Department of Mathematics, Institute of Mathematical Research(INSPEM) and Institute of Advanced Technology(ITMA). Adem Kılıçman is also the authors of three books Applied Mathematics for Business and Economics, An Introduction to Real Analysis and Distributions Theory and Neutrix Calculus.

Adem Kılıçman is a Member of Several Associations, for example, he is a Life Member of Malaysian Mathematical Sciences Society(PERSAMA), Member of American Mathematical Society(AMS), Member of Society for
Industrial Applications of Mathematics (SIAM), member of New York Academy of Sciences. He is an Editor-In-Chief in two international Journals and also member on the Editorial Board of International Journals and Bulletins which some in the ISI listing and Kılıçman is a Co-Editors in some proceedings of International Conferences as well as National Conferences. His research contributions are recognized and his name is listed in Who’s Who in Science and Engineering (MARQUIS)(USA), The Contemporary Who’s Who(USA), Great Minds of The 21st Century(UK), 2000 Outstanding Intellectuals of the 21st Century by International Biographical Center, Cambridge, UK.

Further, some of Kılıçman’s research were listed in the Top 8/25 Hottest articles, such as in Topology and its Applications (Elsevier), April - June 2007, and in the Top 5/25 Hottest articles, Applied Mathematics Letters (Elsevier), October - December 2008. Apart from being a reviewer in the several international Journals, Kılıçman is also a Reviewer for American Mathematical Society, Reviewer for Computing Reviews(USA) as well as Reviewer for Zentralblatt Mathematik(Germany) since 2002.
ACKNOWLEDGEMENT

And with Him all things have their end.

That is, human beings are sent to this world, which is the realm of trial and examination, with the important duties of trading and acting as officials. After they have concluded their trading, accomplished their duties, and completed their service, they will return and meet once more with their Generous Master and Glorious Creator Who sent them forth in the first place. Leaving this transient realm, they will be honoured and elevated to the presence of grandeur in the realm of permanence. That is to say, being delivered from the turbulence of causes and from the obscure veils of intermediaries, they will meet with their Merciful Sustainer without veil at the seat of His eternal majesty. Everyone will find his Creator, True Object of Worship, Sustainer, Lord, and Owner and will know Him directly. Thus, this phrase proclaims the following joyful news, which is greater than all the rest:

“O mankind! Do you know where you are going and to where you are being impelled? As is stated at the end of the Thirty-Second Word, a thousand years of happy life in this world cannot be compared to one hour of life in Paradise. And a thousand years of life in Paradise cannot be compared to one hour’s vision of the sheer loveliness of the Beauteous One of Glory. And you are going to the sphere of His mercy, and to His presence.

That is, everything will return to the realm of permanence from the transient realm, and will go to the seat of post-eternal sovereignty of the Sem-piternal Ever-Enduring One. They will go from the multiplicity of causes
to the sphere of power of the All-Glorious One of Unity, and will be transferred from this world to the Hereafter. Your place of recourse is His Court, therefore, and your place of refuge, His mercy.

Since this world is transitory, and since life is short, and since the truly essential duties are many, and since eternal life will be gained here, and since the world is not without an owner, and since this guest-house of the world has a most Wise and Generous director, and since neither good nor bad will remain without recompense, and since according to the verse,

*On no soul does God place a burden greater than it can bear*

_Qur’an 2: 286*

there is no obligation that cannot be borne, and since a safe way is preferable to a harmful way, and since worldly friends and ranks last only till the door of the grave, then surely the most fortunate is he who does not forget the hereafter for this world, and does not sacrifice the hereafter for this world, and does not destroy the life of the hereafter for worldly life, and does not waste his life on trivial things, but considers himself to be a guest and acts in accordance with the commands of the guest-house’s Owner, then opens the door of the grave in confidence and enters upon eternal happiness...

Thus first of all I am very grateful to Allah (s.w.t) that provided the life with faith and all praise to be upon Muhammad(s.a.w) who is a guidance for all the universe.
I sincerely also acknowledge that most of our research were partially supported by the University Putra Malaysia as well as Ministry of Science, Technology and Innovation (MOSTI) and Higher Education Ministry of Malaysia (MOHE).

I also thank all my Postgraduate students as well as students who made final year research project under our supervision which some of were completed as publication in the prestigious journals and to all my co-researchers all over the world.

My special thanks goes to my family members in particular my late father Huseyin Kılıçman and my mother Safiye Kılıçman from them I learned the true meaning of the word courage and determination.

Finally, I want to express my special and sincere thanks to my wife Dr. Arini Nuran Idris for her sacrifice and patience and my lovely children Muhammad Fateh and Muhammad Huseyin and Muhammad Hanif Idris.

O God! Grant blessings and peace to our master Muhammad (s.a.w) to the number of the particles of the universe, and to all his Family and Companions. And all praise be to God, the Sustainer of All the Worlds. May Allah grant all of us happiness in this world and in the hereafter (Amen).
LIST OF INAUGURAL LECTURES

1. Prof. Dr. Sulaiman M. Yassin
   *The Challenge to Communication Research in Extension*
   22 July 1989

2. Prof. Ir. Abang Abdullah Abang Ali
   *Indigenous Materials and Technology for Low Cost Housing*
   30 August 1990

3. Prof. Dr. Abdul Rahman Abdul Razak
   *Plant Parasitic Nematodes, Lesser Known Pests of Agricultural Crops*
   30 January 1993

4. Prof. Dr. Mohamed Suleiman
   *Numerical Solution of Ordinary Differential Equations: A Historical Perspective*
   11 December 1993

5. Prof. Dr. Mohd. Ariff Hussein
   *Changing Roles of Agricultural Economics*
   5 March 1994

6. Prof. Dr. Mohd. Ismail Ahmad
   *Marketing Management: Prospects and Challenges for Agriculture*
   6 April 1994

7. Prof. Dr. Mohamed Mahyuddin Mohd. Dahan
   *The Changing Demand for Livestock Products*
   20 April 1994

8. Prof. Dr. Ruth Kiew
   *Plant Taxonomy, Biodiversity and Conservation*
   11 May 1994

9. Prof. Ir. Dr. Mohd. Zohadie Bardaie
   *Engineering Technological Developments Propelling Agriculture into the 21st Century*
   28 May 1994
10. Prof. Dr. Shamsuddin Jusop
   *Rock, Mineral and Soil*
   18 June 1994

11. Prof. Dr. Abdul Salam Abdullah
   *Natural Toxicants Affecting Animal Health and Production*
   29 June 1994

12. Prof. Dr. Mohd. Yusof Hussein
   *Pest Control: A Challenge in Applied Ecology*
   9 July 1994

13. Prof. Dr. Kapt. Mohd. Ibrahim Haji Mohamed
   *Managing Challenges in Fisheries Development through Science and Technology*
   23 July 1994

14. Prof. Dr. Hj. Amat Juhari Moain
   *Sejarah Keagungan Bahasa Melayu*
   6 Ogos 1994

15. Prof. Dr. Law Ah Theem
   *Oil Pollution in the Malaysian Seas*
   24 September 1994

16. Prof. Dr. Md. Nordin Hj. Lajis
   *Fine Chemicals from Biological Resources: The Wealth from Nature*
   21 January 1995

17. Prof. Dr. Sheikh Omar Abdul Rahman
   *Health, Disease and Death in Creatures Great and Small*
   25 February 1995

18. Prof. Dr. Mohamed Shariff Mohamed Din
   *Fish Health: An Odyssey through the Asia - Pacific Region*
   25 March 1995
19. Prof. Dr. Tengku Azmi Tengku Ibrahim
\textit{Chromosome Distribution and Production Performance of Water Buffaloes}
6 May 1995

20. Prof. Dr. Abdul Hamid Mahmood
\textit{Bahasa Melayu sebagai Bahasa Ilmu- Cabaran dan Harapan}
10 Jun 1995

21. Prof. Dr. Rahim Md. Sail
\textit{Extension Education for Industrialising Malaysia: Trends, Priorities and Emerging Issues}
22 July 1995

22. Prof. Dr. Nik Muhammad Nik Abd. Majid
\textit{The Diminishing Tropical Rain Forest: Causes, Symptoms and Cure}
19 August 1995

23. Prof. Dr. Ang Kok Jee
\textit{The Evolution of an Environmentally Friendly Hatchery Technology for Udang Galah, the King of Freshwater Prawns and a Glimpse into the Future of Aquaculture in the 21st Century}
14 October 1995

24. Prof. Dr. Sharifuddin Haji Abdul Hamid
\textit{Management of Highly Weathered Acid Soils for Sustainable Crop Production}
28 October 1995

25. Prof. Dr. Yu Swee Yean
\textit{Fish Processing and Preservation: Recent Advances and Future Directions}
9 December 1995

26. Prof. Dr. Rosli Mohamad
\textit{Pesticide Usage: Concern and Options}
10 February 1996
27. Prof. Dr. Mohamed Ismail Abdul Karim  
*Microbial Fermentation and Utilization of Agricultural Bioresources and Wastes in Malaysia*  
2 March 1996

28. Prof. Dr. Wan Sulaiman Wan Harun  
*Soil Physics: From Glass Beads to Precision Agriculture*  
16 March 1996

29. Prof. Dr. Abdul Aziz Abdul Rahman  
*Sustained Growth and Sustainable Development: Is there a Trade-Off 1 or Malaysia*  
13 April 1996

30. Prof. Dr. Chew Tek Ann  
*Sharecropping in Perfectly Competitive Markets: A Contradiction in Terms*  
27 April 1996

31. Prof. Dr. Mohd. Yusuf Sulaiman  
*Back to the Future with the Sun*  
18 May 1996

32. Prof. Dr. Abu Bakar Salleh  
*Enzyme Technology: The Basis for Biotechnological Development*  
8 June 1996

33. Prof. Dr. Kamel Ariffin Mohd. Atan  
*The Fascinating Numbers*  
29 June 1996

34. Prof. Dr. Ho Yin Wan  
*Fungi: Friends or Foes*  
27 July 1996

35. Prof. Dr. Tan Soon Guan  
*Genetic Diversity of Some Southeast Asian Animals: Of Buffaloes and Goats and Fishes Too*  
10 August 1996
36. Prof. Dr. Nazaruddin Mohd. Jali
   *Will Rural Sociology Remain Relevant in the 21st Century?*
   21 September 1996

37. Prof. Dr. Abdul Rani Bahaman
   *Leptospirosis - A Model for Epidemiology, Diagnosis and Control of Infectious Diseases*
   16 November 1996

38. Prof. Dr. Marziah Mahmood
   *Plant Biotechnology - Strategies for Commercialization*
   21 December 1996

39. Prof. Dr. Ishak Hj. Omar
   *Market Relationships in the Malaysian Fish Trade: Theory and Application*
   22 March 1997

40. Prof. Dr. Suhaila Mohamad
   *Food and Its Healing Power*
   12 April 1997

41. Prof. Dr. Malay Raj Mukerjee
   *A Distributed Collaborative Environment for Distance Learning Applications*
   17 June 1998

42. Prof. Dr. Wong Kai Choo
   *Advancing the Fruit Industry in Malaysia: A Need to Shift Research Emphasis*
   15 May 1999

43. Prof. Dr. Aini Ideris
   *Avian Respiratory and Immunosuppressive Diseases - A Fatal Attraction*
   10 July 1999
44. Prof. Dr. Sariah Meon
_Biological Control of Plant Pathogens: Harnessing the Richness of Microbial Diversity_
14 August 1999

45. Prof. Dr. Azizah Hashim
_The Endomycorrhiza: A Futile Investment?_
23 Oktober 1999

46. Prof. Dr. Noraini Abdul Samad
_Molecular Plant Virology: The Way Forward_
2 February 2000

47. Prof. Dr. Muhamad Awang
_Do We Have Enough Clean Air to Breathe?_
7 April 2000

48. Prof. Dr. Lee Chnoong Kheng
_Green Environment, Clean Power_
24 June 2000

49. Prof. Dr. Mohd. Ghazali Mohayidin
_Managing Change in the Agriculture Sector: The Need for Innovative Educational Initiatives_
12 January 2002

50. Prof. Dr. Fatimah Mohd. Arshad
_Analisis Pemasaran Pertanian di Malaysia: Keperluan Agenda Pembaharuan_
26 Januari 2002

51. Prof. Dr. Nik Mustapha R. Abdullah
_Fisheries Co-Management: An Institutional Innovation Towards Sustainable Fisheries Industry_
28 February 2002

52. Prof. Dr. Gulam Rusul Rahmat Ali
_Food Safety: Perspectives and Challenges_
23 March 2002
53. Prof. Dr. Zaharah A. Rahman  
*Nutrient Management Strategies for Sustainable Crop Production in Acid Soils: The Role of Research Using Isotopes*  
13 April 2002

54. Prof. Dr. Maisom Abdullah  
*Productivity Driven Growth: Problems & Possibilities*  
27 April 2002

55. Prof. Dr. Wan Omar Abdullah  
*Immunodiagnosis and Vaccination for Brugian Filariasis: Direct Rewards from Research Investments*  
6 June 2002

56. Prof. Dr. Syed Tajuddin Syed Hassan  
*Agro-ento Bioinformation: Towards the Edge of Reality*  
22 June 2002

57. Prof. Dr. Dahlan Ismail  
*Sustainability of Tropical Animal-Agricultural Production Systems: Integration of Dynamic Complex Systems*  
27 June 2002

58. Prof. Dr. Ahmad Zubaidi Baharumshah  
*The Economics of Exchange Rates in the East Asian Countries*  
26 October 2002

59. Prof. Dr. Shaik Md. Noor Alam S.M. Hussain  
*Contractual Justice in Asean: A Comparative View of Coercion*  
31 October 2002

60. Prof. Dr. Wan Md. Zin Wan Yunus  
*Chemical Modification of Polymers: Current and Future Routes for Synthesizing New Polymeric Compounds*  
9 November 2002

61. Prof. Dr. Annuar Md. Nassir  
*Is the KLSE Efficient? Efficient Market Hypothesis vs Behavioural Finance*  
23 November 2002
62. Prof. Ir. Dr. Radin Umar Radin Sohadi
   *Road Safety Interventions in Malaysia: How Effective Are They?*
   21 February 2003

63. Prof. Dr. Shamsher Mohamad
   *The New Shares Market: Regulatory Intervention, Forecast Errors and Challenges*
   26 April 2003

64. Prof. Dr. Han Chun Kwong
   *Blueprint for Transformation or Business as Usual? A Structurational Perspective of the Knowledge-Based Economy in Malaysia*
   31 May 2003

65. Prof. Dr. Mawardi Rahmani
   *Chemical Diversity of Malaysian Flora: Potential Source of Rich Therapeutic Chemicals*
   26 July 2003

66. Prof. Dr. Fatimah Md. Yusoff
   *An Ecological Approach: A Viable Option for Aquaculture Industry in Malaysia*
   9 August 2003

67. Prof. Dr. Mohamed Ali Rajion
   *The Essential Fatty Acids-Revisited*
   23 August 2003

68. Prof. Dr. Azhar Md. Zain
   *Psychotheraphy for Rural Malays - Does it Work?*
   13 September 2003

69. Prof. Dr. Mohd. Zamri Saad
   *Respiratory Tract Infection: Establishment and Control*
   27 September 2003
70. Prof. Dr. Jinap Selamat  
*Cocoa-Wonders for Chocolate Lovers*  
14 February 2004

71. Prof. Dr. Abdul Halim Shaari  
*High Temperature Superconductivity: Puzzle & Promises*  
13 March 2004

72. Prof. Dr. Yaakob Che Man  
*Oils and Fats Analysis - Recent Advances and Future Prospects*  
27 March 2004

73. Prof. Dr. Kaida Khalid  
*Microwave Aquametry: A Growing Technology*  
24 April 2004

74. Prof. Dr. Hasanah Mohd. Ghazali  
*Tapping the Power of Enzymes- Greening the Food Industry*  
11 May 2004

75. Prof. Dr. Yusof Ibrahim  
*The Spider Mite Saga: Quest for Biorational Management Strategies*  
22 May 2004

76. Prof. Datin Dr. Sharifah Md. Nor  
*The Education of At-Risk Children: The Challenges Ahead*  
26 June 2004

77. Prof. Dr. Ir. Wan Ishak Wan Ismail  
*Agricultural Robot: A New Technology Development for Agro-Based Industry*  
14 August 2004

78. Prof. Dr. Ahmad Said Sajap  
*Insect Diseases: Resources for Biopesticide Development*  
28 August 2004
79. Prof. Dr. Aminah Ahmad  
*The Interface of Work and Family Roles: A Quest for Balanced Lives*  
11 March 2005

80. Prof. Dr. Abdul Razak Alimon  
*Challenges in Feeding Livestock: From Wastes to Feed*  
23 April 2005

81. Prof. Dr. Haji Azimi Hj. Hamzah  
*Helping Malaysian Youth Move Forward: Unleashing the Prime Enablers*  
29 April 2005

82. Prof. Dr. Rasedee Abdullah  
*In Search of An Early Indicator of Kidney Disease*  
27 May 2005

83. Prof. Dr. Zulkifli Hj. Shamsuddin  
*Smart Partnership: Plant-Rhizobacteria Associations*  
17 June 2005

84. Prof. Dr. Mohd Khanif Yusop  
*From the Soil to the Table*  
1 July 2005

85. Prof. Dr. Annuar Kassim  
*Materials Science and Technology: Past, Present and the Future*  
8 July 2005

86. Prof. Dr. Othman Mohamed  
*Enhancing Career Development Counselling and the Beauty of Career Games*  
12 August 2005

87. Prof. Ir. Dr. Mohd Amin Mohd Soom  
*Engineering Agricultural Water Management Towards Precision Framing*  
26 August 2005
88. Prof. Dr. Mohd Arif Syed
   *Bioremediation-A Hope Yet for the Environment?*
   9 September 2005

89. Prof. Dr. Abdul Hamid Abdul Rashid
   *The Wonder of Our Neuromotor System and the Technological Challenges They Pose*
   23 December 2005

90. Prof. Dr. Norhani Abdullah
   *Rumen Microbes and Some of Their Biotechnological Applications*
   27 January 2006

91. Prof. Dr. Abdul Aziz Saharee
   *Haemorrhagic Septicaemia in Cattle and Buffaloes: Are We Ready for Freedom?*
   24 February 2006

92. Prof. Dr. Kamariah Abu Bakar
   *Activating Teachers’ Knowledge and Lifelong Journey in Their Professional Development*
   3 March 2006

93. Prof. Dr. Borhanuddin Mohd. Ali
   *Internet Unwired*
   24 March 2006

94. Prof. Dr. Sundararajan Thilagar
   *Development and Innovation in the Fracture Management of Animals*
   31 March 2006

95. Prof. Dr. Zainal Aznam Md. Jelan
   *Strategic Feeding for a Sustainable Ruminant Farming*
   19 May 2006

96. Prof. Dr. Mahiran Basri
   *Green Organic Chemistry: Enzyme at Work*
   14 July 2006
97. Prof. Dr. Malik Hj. Abu Hassan
   Towards Large Scale Unconstrained Optimization
   20 April 2007

98. Prof. Dr. Khalid Abdul Rahim
   Trade and Sustainable Development: Lessons from Malaysia’s Experience
   22 June 2007

99. Prof. Dr. Mad Nasir Shamsudin
   Econometric Modelling for Agricultural Policy Analysis and Forecasting: Between Theory and Reality
   13 July 2007

100. Prof. Dr. Zainal Abidin Mohamed
    Managing Change - The Fads and The Realities: A Look at Process Reengineering, Knowledge Management and Blue Ocean Strategy
    9 November 2007

101. Prof. Ir. Dr. Mohamed Daud
    Expert Systems for Environmental Impacts and Ecotourism Assessments
    23 November 2007

102. Prof. Dr. Saleha Abdul Aziz
    Pathogens and Residues: How Safe is Our Meat?
    30 November 2007

103. Prof. Dr. Jayum A. Jawan
    Hubungan Sesama Manusia
    7 Disember 2007

104. Prof. Dr. Zakariah Abdul Rashid
    Planning for Equal Income Distribution in Malaysia: A General Equilibrium Approach
    28 December 2007
105. Prof. Datin Paduka Dr. Khatijah Yusoff
   *Newcastle Disease virus: A Journey from Poultry to Cancer*
   11 January 2008

106. Prof. Dr. Dzulkefly Kuang Abdullah
   *Palm Oil: Still the Best Choice*
   1 February 2008

107. Prof. Dr. Elias Saion
   *Probing the Microscopic Worlds by Lonizing Radiation*
   22 February 2008

108. Prof. Dr. Mohd Ali Hassan
   *Waste-to-Wealth Through Biotechnology: For Profit, People and Planet*
   28 March 2008

109. Prof. Dr. Mohd Maarof H. A. Moksin
   *Metrology at Nanoscale: Thermal Wave Probe Made It Simple*
   11 April 2008

110. Prof. Dr. Dzolkhifli Omar
   *The Future of Pesticides Technology in Agriculture: Maximum Target Kill with Minimum Collateral Damage*
   25 April 2008

111. Prof. Dr. Mohd. Yazid Abd. Manap
   *Probiotics: Your Friendly Gut Bacteria*
   9 May 2008

112. Prof. Dr. Hamami Sahri
   *Sustainable Supply of Wood and Fibre: Does Malaysia have Enough?*
   23 May 2008

113. Prof. Dato’ Dr. Makhdzir Mardan
   *Connecting the Bee Dots*
   20 June 2008
114. Prof. Dr. Maimunah Ismail  
*Gender & Career: Realities and Challenges*  
25 July 2008

115. Prof. Dr. Nor Aripin Shamaan  
*Biochemistry of Xenobiotics: Towards a Healthy Lifestyle and Safe Environment*  
1 August 2008

116. Prof. Dr. Mohd Yunus Abdullah  
*Penjagaan Kesihatan Primer di Malaysia: Cabaran Prospek dan Implikasi dalam Latihan dan Penyelidikan Perubatan serta Sains Kesihatan di Universiti Putra Malaysia*  
8 Ogos 2008

117. Prof. Dr. Musa Abu Hassan  
*Memanfaatkan Teknologi Maklumat & Komunikasi ICT untuk Semua*  
15 Ogos 2008

118. Prof. Dr. Md. Salleh Hj. Hassan  
*Role of Media in Development: Strategies, Issues & Challenges*  
22 August 2008

119. Prof. Dr. Jariah Masud  
*Gender in Everyday Life*  
10 October 2008

120 Prof. Dr. Mohd Shahwahid Haji Othman  
*Mainstreaming Environment: Incorporating Economic Valuation and Market-Based Instruments in Decision Making*  
24 October 2008

121. Prof. Dr. Son Radu  
*Big Questions Small Worlds: Following Diverse Vistas*  
31 Oktober 2008
122. Prof. Dr. Russly Abdul Rahman
   *Responding to Changing Lifestyles: Engineering the Convenience Foods*
   28 November 2008

123. Prof. Dr. Mustafa Kamal Mohd Shariff
   *Aesthetics in the Environment an Exploration of Environmental: Perception Through Landscape Preference*
   9 January 2009

124. Prof. Dr. Abu Daud Silong
   *Leadership Theories, Research & Practices: Farming Future Leadership Thinking*
   16 January 2009

125. Prof. Dr. Azni Idris
   *Waste Management, What is the Choice: Land Disposal or Biofuel?*
   23 January 2009

126. Prof. Dr. Jamilah Bakar
   *Freshwater Fish: The Overlooked Alternative*
   30 January 2009

127. Prof. Dr. Mohd. Zobir Hussein
   *The Chemistry of Nanomaterial and Nanobiomaterial*
   6 February 2009

128. Prof. Ir. Dr. Lee Teang Shui
   *Engineering Agricultural: Water Resources*
   20 February 2009

129. Prof. Dr. Ghizan Saleh
   *Crop Breeding: Exploiting Genes for Food and Feed*
   6 March 2009

130. Prof. Dr. Muzafar Shah Habibullah
   *Money Demand*
   27 March 2009
131. Prof. Dr. Karen Anne Crouse  
*In Search of Small Active Molecules*  
3 April 2009

132. Prof. Dr. Turiman Suandi  
*Volunteerism: Expanding the Frontiers of Youth Development*  
17 April 2009

133. Prof. Dr. Arbakariya Ariff  
*Industrializing Biotechnology: Roles of Fermentation and Bioprocess Technology*  
8 Mei 2009

134. Prof. Ir. Dr. Desa Ahmad  
*Mechanics of Tillage Implements*  
12 Jun 2009

135. Prof. Dr. W. Mahmood Mat Yunus  
*Photothermal and Photoacoustic: From Basic Research to Industrial Applications*  
10 Julai 2009

136. Prof. Dr. Taufiq Yap Yun Hin  
*Catalysis for a Sustainable World*  
7 August 2009

137. Prof. Dr. Raja Noor Zaliha Raja Abd. Rahman  
*Microbial Enzymes: From Earth to Space*  
9 Oktober 2009

138. Prof. Ir. Dr. Barkawi Sahari  
*Materials, Energy and CNGDI Vehicle Engineering*  
6 November 2009

139. Prof. Dr. Zulkifli Idrus  
*Poultry Welfare in Modern Agriculture: Opportunity or Threat?*  
13 November 2009
140. Prof. Dr. Mohamed Hanafi Musa
   Managing Phosphorus: Under Acid Soils Environment
   8 January 2010

141. Prof. Dr. Abdul Manan Mat Jais
   Haruan Channa striatus a Drug Discovery in an Agro-Industry Setting
   12 March 2010

142. Prof. Dr. Bujang bin Kim Huat
   Problematic Soils: In Search for Solution
   19 March 2010

143. Prof. Dr. Samsinar Md Sidin
   Family Purchase Decision Making: Current Issues & Future Challenges
   16 April 2010

144. Prof. Dr. Mohd Adzir Mahdi
   Lightspeed: Catch Me If You Can
   4 June 2010

145. Prof. Dr. Raha Hj. Abdul Rahim
   Designer Genes: Fashioning Mission Purposed Microbes
   18 June 2010

146. Prof. Dr. Hj. Hamidon Hj. Basri
   A Stroke of Hope, A New Beginning
   2 July 2010

147. Prof. Dr. Hj. Kamaruzaman Jusoff
   Going Hyperspectral: The "Unseen" Captured?
   16 July 2010

148. Prof. Dr. Mohd Sapuan Salit
   Concurrent Engineering for Composites
   30 July 2010
149. Prof. Dr. Shattri Mansor  
*Google the Earth: What's Next?*  
15 October 2010

150. Prof. Dr. Mohd Basyaruddin Abdul Rahman  
*Haute Couture: Molecules & Biocatalysts*  
29 October 2010

151. Prof. Dr. Mohd. Hair Bejo  
*Poultry Vaccines: An Innovation for Food Safety and Security*  
12 November 2010

152. Prof. Dr. Umi Kalsom Yusuf  
*Fern of Malaysian Rain Forest*  
3 December 2010

153. Prof. Dr. Ab. Rahim Bakar  
*Preparing Malaysian Youths for The World of Work: Roles of Technical and Vocational Education and Training (TVET)*  
14 January 2011

154. Prof. Dr. Seow Heng Fong  
*Are there "Magic Bullets" for Cancer Therapy?*  
11 February 2011

155. Prof. Dr. Mohd Azmi Mohd Lila  
*Biopharmaceuticals: Protection, Cure and the Real Winner*  
18 February 2011

156. Prof. Dr. Siti Shapor Siraj  
*Genetic Manipulation in Farmed Fish: Enhancing Aquaculture Production*  
25 March 2011

157. Prof. Dr. Ahmad Ismail  
*Coastal Biodiversity and Pollution: A Continuous Conflict*  
22 April 2011
158. Prof. Ir. Dr. Norman Mariun
   Energy Crisis 2050? Global Scenario and Way Forward for Malaysia
   10 June 2011

159. Prof. Dr. Mohd Razi Ismail
   Managing Plant Under Stress: A Challenge for Food Security
   15 July 2011

160. Prof. Dr. Patimah Ismail
   Does Genetic Polymorphisms Affect Health?
   23 September 2011

161. Prof. Dr. Sidek Ab. Aziz
   Wonders of Glass: Synthesis, Elasticity and Application
   7 October 2011

162. Prof. Dr. Azizah Osman
   Fruits: Nutritious, Colourful, Yet Fragile Gifts of Nature
   14 October 2011

163. Prof. Dr. Mohd. Fauzi Ramlan
   Climate Change: Crop Performance and Potential
   11 November 2011