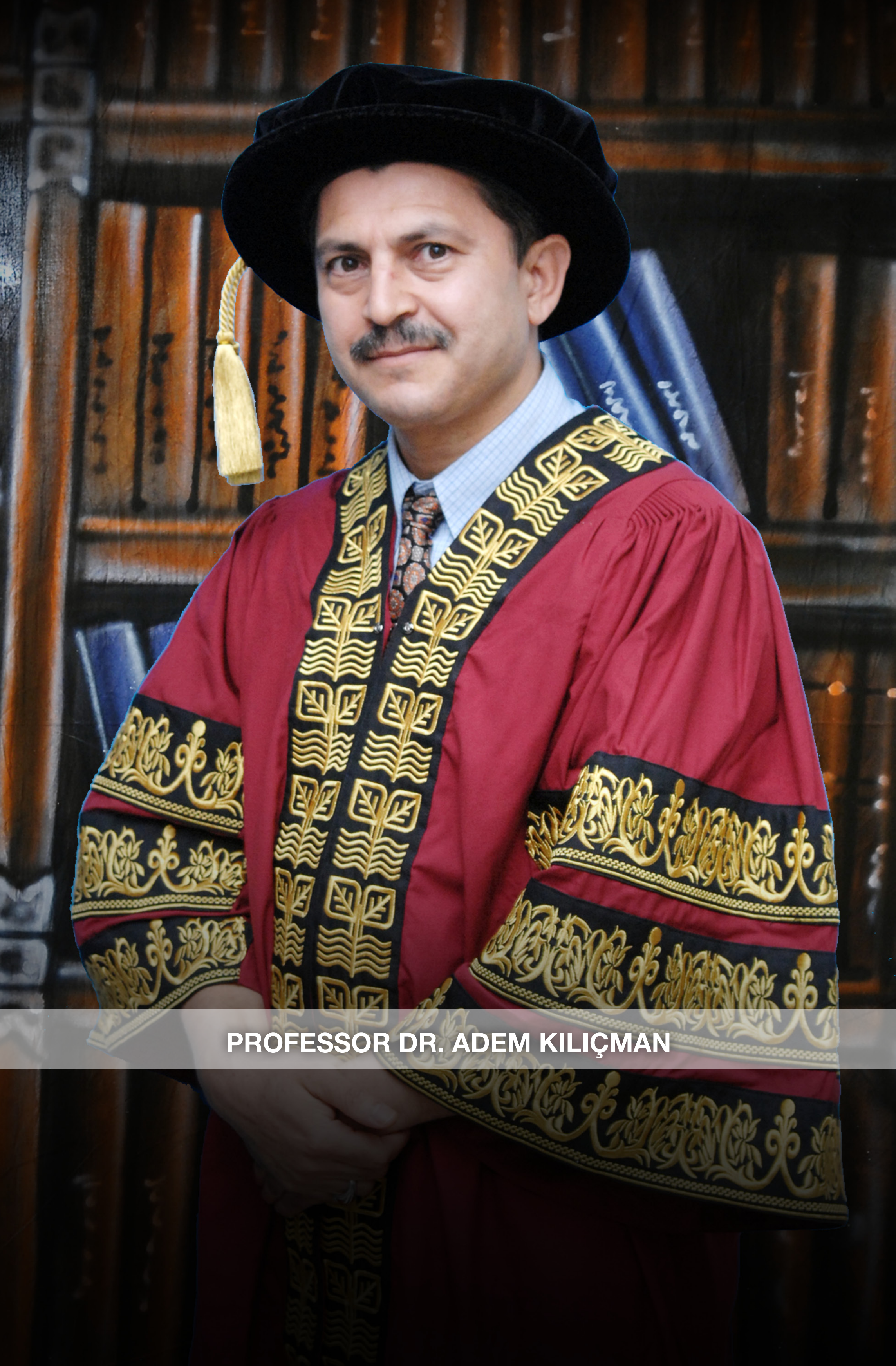
Mathematical Modeling with Generalized Function



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B.Sc. (Hons.), M.Sc.(Hacettepe, Turkey), Ph.D. (Leicester, England)

Contents

In His Name, be He glorified! And there is nothing but its glorifies Him with praise.

In the Name of God, the Merciful, the Compassionate.

There is no god but God, He is One, He has no partner; His is the dominion and His is the praise; He alone grants life, and deals death, and He is living and dies not; all good is in His hand, He is powerful over all things, and with Him all things have their end.

Be certain of this, that the highest aim of creation and its most important result are belief in God. And the most exalted rank in humanity and its highest degree are the knowledge of God contained within belief in God. And the most radiant happiness and sweetest bounty for jinn and human beings are the love of God contained within the knowledge of God. And the purest joy for the human spirit and the sheerest delight for man's heart are the rapture of the spirit contained within the love of God. Indeed, all true happiness, pure joy, sweet bounties, and untroubled pleasure lie in knowledge of God and love of God; they cannot exist without them.

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ABSTRACT

In recent years there has been a growing interest in setting up the modeling and solving mathematical problems in order to explain numerous experimental findings which are relevant to industrial applications.

Distributions also known as generalized functions which generalize classical functions and allow us to extend the concept of derivative to all continuous functions. The theory of distributions have applications in various fields especially in science and engineering where many non-continuous phenomena that naturally lead to differential equations whose solutions are distributions, such as the delta distribution therefore distributions can help us to develop an operational calculus in order to investigate linear ordinary differential equations as well as partial differential equations with constant coefficients through their fundamental solutions.

Further, some regular operations which are valid for ordinary functions such as addition, multiplication by scalars are extended into distributions. Other operations can be defined only for certain restricted subclasses; these are called irregular operations.

They allow us to extend the concept of derivative to all continuous functions and beyond and are used to formulate generalized solutions of partial differential equations.

They are important in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution.

In this work we aim to show how differential equations arise in the mathematical modeling of certain problems in industry. The focus of the presentations will be on the use of mathematics to advance the understanding of specific problems that arise in industry. For this purpose we let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D then we provide some particular examples how to use the generalized functions in Statistics and Economics. At the end of the study we relate the Tomography and The Radon Transform on using the generalized functions.

In mathematical analysis, distributions also known as generalized functions are objects which generalize functions and probability distributions.

We apply the distributions to the some mathematical problems. For this purpose we let ρ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Define, the function δ_n by putting

$$\delta_n(x) = n\rho(nx) \quad \text{for } n = 1, 2, \dots,$$

it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. Then if f is an arbitrary distribution in D', we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

Key Words and Phrases: Differential equations, Delta-function, Generalized functions, Probability Functions, Tomography and Radon Transforms.

INTRODUCTION

Over the past decades mathematical programming has become a widely used tool to help managers with decision making. The problems that was in the past years were impossible to solve are now very easy and solved by standard computer programs. In fact the development of mathematical programming has generated much interest in mathematical modeling trough out business of all types and all sizes.

In mathematics we deal with equations. An equation is a statement that two mathematical expressions are equal. The two expressions that make up an equation are called its sides. We have several types of the equations such as polynomial equations, exponential equations, logarithmic equations, trigonometric equations, difference Equations, differential equations, integral equations, etc.

We solve an equation by using the some properties of equality to transform the equation into an equivalent statement of the form until we get

$$\begin{aligned} ax+b &= c \Rightarrow x=? \\ e^{a(x)} &= d \Rightarrow x=? \\ \log f(x) &= m \Rightarrow x=? \\ \frac{df}{dx} &= g(x) \Rightarrow f(x)=? \\ \int h(x)dx &= v(x) \Rightarrow h(x)=? \end{aligned}$$

The best way to demonstrate the scope of the mathematical modeling is to show the wide variety of the problems in the real world to which that model can be applied. With the exception, the cases enable us to grasp the range and the full dimension of the problems that can be formulated and solved by mathematical modeling. The following examples have the common characteristic:

Resource Allocations: Consider a company in Malaysia manufactures two type of products for distributions to retailers in all over Malaysia. Then the problem is to identify the

- Raw material
- Man hours
- Specific profit for each products that company turns out.
- The objective function is to be maximize the profit made by the products.

Assignment Problems: Consider a textile factory and has different 20 machines, and each machine has capability of performing various jobs. On the specific day the manager of the factory has to check the performance of the 15 different jobs. Each assignment can be accomplished by any of the 20 machines.

- The time it takes to perform the job n on machine m is t_n
- Time varies from machine to machine
- If the t_{nm} goes to infinity then job n cannot be performed on the machine m
- To determine the best 15 machines.

Transportation Problems: After the production process is finished or completed the major problem is the distributions. So each side try to maximize the capacity and has to satisfy the demand for the product for a definite time.

Integer programming Problem: It is a mathematical optimization problem which study feasibility in where some or all of the variables are restricted to be integers. In many settings the term refers to integer linear programming, which is also known as mixed integer programming.

Dynamic Programming: Dynamic programming is a method for solving the complex problems by breaking into simpler small or subproblems. It is applicable to problems exhibiting the properties of overlapping subproblems which are only slightly smaller and optimal substructure. Then combine the solutions of the subproblems to reach an overall solution. When applicable, the method takes far less time than naive methods.

Iterative Algorithms: The iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. A specific implementation of an iterative method, including the termination criteria, is an algorithm of the iterative method. An iterative method is called convergent if the corresponding sequence converges for given initial approximations. A mathematically rigorous convergence analysis of an iterative method is usually performed; however, heuristic-based iterative methods are also common.

However, a mathematical model is a description of an particular system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. Mathematical models are used not only in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science); physicists, engineers, statisticians, operations research analysts and economists use mathematical models most extensively.

Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models, as far as logic is taken as a part of mathematics. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed. Many mathematical models can be classified in some of the following ways:

Linear vs. nonlinear Mathematical models are usually composed by variables, which are abstractions of quantities of interest in the described systems, and operators that act on these variables, which can be algebraic operators, functions, differential operators, etc. If all the operators in a mathematical model exhibit linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise.

The question of linearity and nonlinearity is dependent on context, and linear models may have nonlinear expressions in them. For example, in a statistical linear model, it is assumed that a relationship is linear in the parameters, but it may be nonlinear in the predictor variables. Similarly, a differential equation is said to be linear if it can be written with linear differential operators, but it can still have nonlinear expressions in it. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is regarded as a linear model. If one or more of the objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.

Nonlinearity, even in fairly simple systems, is often associated with phenomena such as chaos and irreversibility. Although there are exceptions, nonlinear systems and models tend to be more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one is trying to study aspects such as irreversibility, which are strongly tied to nonlinearity.

Deterministic vs. probabilistic (stochastic) A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables. Therefore, deterministic models perform the same way for a given set of initial conditions. Conversely, in a stochastic model, randomness is present, and variable states are not described by unique values, but rather by probability distributions.

Static vs. dynamic A static model does not account for the element of time, while a dynamic model does. Dynamic models typically are represented with difference equations or differential equations.

Discrete vs. Continuous: A discrete model does not take into account the function of time and usually uses time-advance methods, while a Continuous model does. Continuous models typically are represented with f(t) and the changes are reflected over continuous time intervals.

Deductive, inductive, or floating: A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation, but is merely the invocation of expected structure. Application of mathematics in social sciences outside of economics has been criticized for unfounded models, see [**1**]. Application of catastrophe theory in science has been characterized as a floating model, see [**2**].

The purpose of mathematical modeling is to describe the essential features of a phenomenon or a system in a manner which allows us to use of various mathematical methods for a deeper analysis. However, to formulate a mathematical model can be a challenging task. It requires both a solid understanding of the basic interactions governing the system under study and a good knowledge of mathematical methods. Thus the goal of the mathematical modeling is to find values for the some decision variables that decision makers will have choice to optimize his objective which may include, for example, Maximize profit, Maximize utilization of equipment, Minimize cost, Minimize used of raw material or resources and minimize traveling time. All these problems are now standard and solutions are easy on using the many software packages that available for mathematical programming.

DIFFERENCE EQUATIONS

These equations occur in many mathematical model and as tools in numerical analysis. We can easily develop a theory and devise methods for solving linear difference equations.

First order Difference Equation

A recurrence relation can be defined by a difference equation of the form

$$x_{n+1} = f(x_n)$$

where x_{n+1} is derived from x_n and $n = 0, 1, 2, 3, \ldots$. If the first one starts with an initial value, say x_0 then the iteration of the difference equation leads to a sequence of the form

$$\{x_i : i = 0 \to \infty\} = \{x_0, x_1, x_2, x_3, x_4, \ldots x_n\}.$$

Consider the following simple example,

The difference equation might be used to model the interest in a bank account compounded k times per year and the

$$x_{n+1} = a\ x_n$$

where $a > 1$ and a constant.

Second Order Linear Difference Equations Recurrence relations involving terms whose suffices differ by two are known as a second-order linear difference equations: The general form of these equations with constant coefficients is

$$a\ x_{n+2} = b\ x_{n+1} + c\ x_n.$$

For example, A model for the population dynamics under immigration involves the equation

$$P_{t+1} - P_t = a\ P_t + b$$

where

- the $P_{t+1} - P_t$ is the difference,
- the constant a is the difference between birth rate and the death rate,
- and b is the rate at which people immigrate to the country.

Similarly, suppose that the national income of a country in year n is given by

$$I_n = S_n + P_n + G_n$$

where S_n, P_n and G_n represent national spending by populous, private investment and the government spending. If the national income increase from one year to the next, then assume consumers will spend more for the following year. In this case suppose that consumer spend $\frac{3}{4}$ of the previous years income. Then $S_{n+1} = \frac{3}{4}I_n$. An increase in consumer spending should also lead to in creased investment to the following year. Assume that $P_{n+1} = S_{n+1} - S_n$. Substitution for S_n gives $P_{n+1} = \frac{3}{4}\left(I_n - I_{n-1}\right)$. Now if the government spending is kept constant then the economic model is

$$I_{n+2} = \frac{3}{4}I_{n+1} - \frac{3}{4}I_n + G$$

where I_n is the national income in year n and G is the initial national income. Now if the national income one year later is $\frac{5}{4}G$ then we can easily determine the following:

- a general solution to this model
- the national income after 10 years
- long term state of the economy.

In general, k- order linear difference equation is an equation of the form

$$a_k(n)\ x_{n+k} + a_{k-1}(n)\ x_{n+k-1} + \ldots + a_1(n)\ x_{n+1} + a_0(n)\ x_n = b_n$$

where $n = 0, 1, 2, \ldots$, $a_i(n)$ and b_n are defined for all nonnegative integers n.

DIFFERENTIAL EQUATIONS

A linear homogenous ordinary differential equation has the form

$$a_s(x)\frac{d^s y}{dx^s} + a_{s-1}(x)\frac{d^{s-1}y}{dx^{s-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and if the general solution exits then it looks like

$$Y(c_1, c_2, c_3, c_4, \ldots, c_n) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + \ldots + c_s y_s(x)$$

where $y_i(x)$ are independent set of particular solutions. The constants c_i can be adjusted so that the solution satisfies specified initial values

$$y_0(x_0) = b_0, \quad y_1(x_0) = b_1, \quad y_2(x_0) = b_2, \quad \ldots, \quad y_{s-1}(x_0) = b_{s-1}$$

at any point x_0 where the equation coefficient $a_i(x)$ are continuous and $a_s(x)$ is non zero. The general form of the linear ordinary differential equation is given by the equation

$$a_s(x)\frac{d^s y}{dx^s} + a_{s-1}(x)\frac{d^{s-1}y}{dx^{s-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

in short form we write

$$L(y) = f(x)$$

then quickly one can observe that if

$$L(y_1) = f \quad \text{and} \quad L(y_2) = 0 \Rightarrow L(y_1 + y_2) = f(x)$$

conversely, if

$$L(y_1) = f \quad \text{and} \quad L(y_2) = f \Rightarrow L(y_1 - y_2) = 0.$$

Remark: Only the linear homogenous equations posses the property that any linear combination of solution is again a solution.

We start with the basic and simple one.

Population Model Let $P(t)$ be the function of the population of a given species at the time t and let $r(t, p)$ be the difference between its birth rate and its death rate. If the population is isolated(there is no immigration or emigration) then

$$\frac{dp}{dt}$$

will be the rate of the population, equals to $r\, p(t)$.

In the most simplistic model we assume that r is constant (does not change with either time or population). Then the differential equation governing the population growth is :

$$\frac{dp}{dt} = a\, p(t), \qquad \text{a} \;=\; \text{constant}.$$

This is a linear equation and is known as the population growth. Now if the population of the given species is p_0 at time t_0, then $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = a\,p(t), \quad p(t_0) = p_0.$$

The solution of this initial-value problem is

$$p(t) = p_0 \; e^{a(t-t_0)}.$$

Harmonic Oscillator In order to model a mass on a spring we use the differential equation

$$y'' = \lambda y$$

which arises repeatedly in Engineering applications and the general solution can be expressed as

$$Y(x) = \begin{cases} c_1 \cosh \sqrt{\lambda}(x - x_0) + c_2 \sinh \sqrt{\lambda}(x - x_0) & \text{if } \lambda > 0 \\ c_1 + c_2 x & \text{if } \lambda = 0 \\ c_1 \cos \sqrt{-\lambda}(x - x_0) + c_2 \sin \sqrt{-\lambda}(x - x_0) & \text{if } \lambda < 0 \end{cases}.$$

We note that whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will also have a specific interpretation as a rate of change. Several concepts in economics that have to do with rates of change as well can be effectively described with the methods of calculus. All the properties of the functions which we measure show how one variable changes due to changes in another variable. Thus, differential calculus is concerned with how one quantity changes in relation to another quantity. Actually, the central concept of differential calculus is the derivative.

Competing Species Suppose there are two species in competition with one another in an environment where the common food supply is limited. For example, sea lions and the penguins, red and grey squirrels, ants and termites are all species that is in this category. Then there are two types of outcome, coexistence and the mutual exclusion. Then find the possible phase solution for the following system:

$$\begin{aligned} \dot{x} &= x(a - bx - cy) \\ \dot{y} &= x(d - ex - fy) \end{aligned}$$

where a, b, c, d, e and f are all positive constants with $x(t)$ and $y(t)$ both representation of two populations, see for example [**55**].

Cooking a Roast The process of cooking a Roast involves taking a piece of meat at an initial temperature T_{cold} and placing it in an oven at a constant temperature T_{oven} until a meat thermometer indicates that the temperature at a specific location say $(x, y, z) = (0, 0, 0)$ has reached the value T_{done} this request a cooking time t_{done}.

Let, $T(x, y, z, t)$ denote the temperature of the roast, if the thermal diffusivity of the meat is k then its temperature evolves according to the heat equation:

$$\frac{\partial T}{\partial t} = k\bigtriangledown^2 = k\left\{\frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y} + \frac{\partial^2 T}{\partial^2 z}\right\}$$

if $t = 0$ then the finial temperature T_{cold} is

$$T(x, y, z, 0) = T_{\text{cold}}$$

the temperature on the skin or " boundary" B of the roast is maintained by the oven at

$$T(x, y, z, t) = T_{\text{oven}}(x, y, z)$$

on the boundary B all $t > 0$. And the cooking time is specified by the condition

$$T(0, 0, 0, t_{\text{done}}) = T_{\text{done}}$$

see the details in [**47**]. Now the question if largest roast is placed in the oven what equation does its temperature satisfy?

Stochastic Environment Any variable its value changes over time in an uncertain way is said to be stochastic variable and this process is called stochastic process. Might be discrete or continuous.

Stochastic methods have become increasingly important in the analysis of a broad range of phenomena in natural sciences and economics. Many processes are described by differential equations where some of the parameters and/or the initial data are not known with complete certainty due to lack of information, uncertainty in the measurements, or incomplete knowledge of the mechanisms themselves. To compensate for this lack of information one introduces stochastic noise in the equations, either in the parameters or in the initial data which results in stochastic differential equations. Since the stochastic environment allows for some randomness in some of the differential equations we can get the more realistic model for the most of the situation. How to make the stochastic differential equations?

Consider the ordinary differential equation

$$\frac{df}{dt} = g(x, t)$$

then we suppose that system has random part (component), added to the system

$$\frac{df}{dt} = g(x,t) + h(x,t)\epsilon(x).$$

Since we have some randomness the solution to the equation might be difficult. Then we write the system as follows

$$df = g(x,t)dt + h(x,t)\epsilon(x)dt$$

then the solution to the equation by performing the integration yields

$$f(t) = f(0) + g(x,s)ds + h(x,s)\epsilon(s)ds.$$

If we reconsider the population model

$$\frac{dp}{dt} = a\,p(t), \qquad \text{a} \;=\; \text{constant.}$$

Now if we ask what will happen $a(t)$ is a function and is not completely known, but subject to some random environmental effects, so that we have

$$a(t) = r(t) + \text{ unknown parts}$$

in fact we call this unknown part as the noise where we do not know the exact behavior of the noise term, then how do we solve this problem?

Consider we want to make some investments: We can do some safe investment like fixed deposit or bond where the price per unit at the time t grows exponentially:

$$\frac{dp}{dt} = ap$$

where a is a constant function. We can also do some risky investment such as stock where the price of the unit at time t satisfies stochastic differential equation of the type

$$\frac{dp}{dt} = (b(t) + k \text{ noise})\ p$$

where $b(t) > 0$ and $k \in \mathbb{R}$ are constants. Now we can easily choose how much amount of the money we want to place in the risky investment. If the utility function U and the terminal time T the problem is to find the optimal portfolio $u_t \in [0, 1]$. That is to find the investment distribution u_t subject to $0 \leq t \leq T$ which maximize the expected utility of the corresponding terminal fortune $X_T^{(u)}$:

$$\max_{0 \leq u_t \leq 1} \left\{ E\left[U\left(X_T^{(u)} \right) \right] \right\}.$$

In short, stochastic differential equations or SDEs, allow for inherited randomness in physical or biological systems by adding a random noise term to classical differential equations.

DISTRIBUTIONS

In classical models, the physical world is modeled as a continuum, and the objects in study are thought as infinitely divisible and observable with arbitrarily good accuracy. In real life, physical phenomena are only observable to a maximum degree of precision dictated by the limitations of the instruments used or even by uncertainty principles inherent to the very nature of reality. Using the classical tools derived from Calculus, it is not only necessary to adopt this continuum model but often the quantities in study must satisfy regularity properties, they must show a certain degree of "smoothness". In many situations these assumptions are impractical and several important problems are not treatable using this classic approach to modeling.

Physicists, staring with the work of Dirac, again solved this shortcoming of the classical theory by introducing new objects (now called distribution or generalized functions) based in their physical intuition. This more modern approach opened the door to treat all sort of models where the smoothness assumptions are more relaxed, allowing for discontinuities and other types of singularities. Distributions theory in its full scope is not only difficult but also requires a sophisticated mathematical background.

Need to Study Distributions

In classical models, the physical world is modeled as a continuum, and the objects in study are thought as infinitely divisible and observable with arbitrarily good accuracy. In real life, physical phenomena are only observable to a maximum degree of precision dictated by the limitations of the instruments used or even by uncertainty principles inherent to the very nature of reality. Using the classical tools derived from Calculus, it is not only necessary to adopt this continuum model but often the quantities in study must satisfy regularity properties, they must show a certain degree of "smoothness". In many situations these assumptions are impractical and several important problems are not treatable using this classic approach to modeling.

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but also requires a sophisticated mathematical background.

Before we study distribution there are several questions that we have to ask in order to motivate ourself to study distributions theory: For example, in the classical analysis, it is well known that every differentiable function is continuous. In general, converse is not true. With generalized functions one can overcome of this problem further discontinuous function is differentiable in the distributional sense, that is, we can differentiate nearly any function as many times as we like, regardless of discontinuities.

If the $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \cdot \phi$ exists for all very nice test functions ϕ then the $\lim_{n\to\infty} f_n$ exists as a generalized function.

The history of the distribution is also relatively new, only in the 1930s Hadamard, Sobolev, and others made systematic use of non-classical generalized functions.

In 1952 Laurent Schwartz won a Fields Medal for systematic treatment of these ideas. Since then, generalized functions have found many applications in various fields of science and engineering. The well known example of this formalism is by considering the delta function that we will discuss later.

Historical Developments

Although now known as the Dirac delta function, the delta function $\delta(x)$ can be said to have been first introduced by Kirchhoff in [**43**]. He defined

$\delta(x)$ by

$$\delta(x) = \lim_{\mu\to\infty} \pi^{-1/2}\mu \exp(-\mu^2 x^2).$$

It is easily seen that $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$. Defining $\int_{-\infty}^{x} \delta(t)\,dt$ by

$$\int_{-\infty}^{x} \delta(t)\,dt = \lim_{\mu\to\infty} \pi^{-1/2}\mu \int_{-\infty}^{x} \exp(-\mu^2 t^2)\,dt,$$

it follows that

$$\int_{-\infty}^{x} \delta(t)\,dt = \begin{cases} 0, & x < 0, \\ 1, & x > 0 \end{cases} \tag{1}$$

and thus δ is not a function in the mathematical sense, since its infinite value takes us out of the usual domain of definition of functions so Kirchhoff referred to δ as the unit impulse function and mathematicians call it a distribution, a limit of a sequence of functions that really only has meaning in integral expressions such as

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx \quad = \quad f(a) \tag{2}$$

Let us evaluate (2) for the special case if $f(x) = 1$ then we get

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx \quad = \quad 1$$

The delta function was next used by Heaviside, see [**20**]. Heaviside's function H is the locally summable function defined to be equal to 0 for $x < 0$ and equal to 1 for $x > 0$. Heaviside appreciated that the derivative of H was in some sense equal to δ.

When Dirac considered the delta function, see [**6**], he treated it as though it were a function that was zero everywhere except at the origin where it was infinite in such a way that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Dirac used δ to represent a unit point charge at the origin and the derivative δ' of δ as to represent a dipole of unit electric moment at the origin since

$$\int_{-\infty}^{\infty} x\delta'(x)\, dx \quad = \quad \lim_{\mu\to\infty} \pi^{-1/2}\mu \int_{-\infty}^{\infty} x \left[\exp(-\mu^2 x^2)\right]' dx = -1.$$

Higher derivatives of δ can be used to represent more complicated multiple–layers and have been used in the physical and engineering sciences for some time, see [**23**]. Therefore we note that for physicists the delta function is well designed to represent, for example, the charge density of a point particle: there is some total charge on the particle, but since the particle is point-like, the charge density is zero except at the single location of the particle.

Dirac gave no rigorous theory for the delta function and its derivatives but used intuitively obvious results such as

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx &= f(a), \\
\int_{-\infty}^{\infty} f(x)\delta'(x-a)\, dx &= -f'(a), \\
\int_{-\infty}^{\infty} \delta(a-x)\delta(x-b)\, dx &= \delta(a-b).
\end{aligned}$$

It was left to Sobolev, see [**51**] and he defined the main operations on distributions such as the derivation and the product by infinitely differentiable functions, and used them for a study of partial differential equations. However, it was not until 1950 when Schwartz published his theory of distributions that a really comprehensive and mathematically account was given, see [**50**].

However, it was not until 1950 when Schwartz published his theory of distributions that a really comprehensive and mathematically account was given. After the Schwartz work, the theory of distributions has led to the essential progress in several mathematical disciplines and such as the theory of partial differential equations and mathematical physics.

In general, Schwartz's well-known result is on the property of distribution theory, multiplication of distributions is not possible, can not be easily given a meaning to product of arbitrary distributions. This fact is an obstacle to the usage of distributions in the theory of nonlinear equations and the theory of generalized coefficients equations and in particular, one arrives at the impossibility to give a meaning to such objects as

$$H(x)\delta(x), \quad x^{-1}\delta(x), \quad \delta(x)\delta(x), \quad \delta'\delta, \quad \delta^{(n)}\delta^{(s)}$$

where δ is delta function and H is the Heaviside function, are of special interest to physicists and widely used, for example, in quantum theory. The problem proved to be very natural and to have a lot of applications. Therefore it has attracted attention at once after the creation of the distributions theory.

Since theory of distributions is a linear theory. We can extend some operations to D' which are valid for ordinary functions such operations are called regular operations such as addition, multiplication by scalars. Other operations can be defined only for particular class of distributions or for certain restricted subclasses of distributions; these are called irregular operations such as: product, convolution product, composition, Fourier Transform and differential equations.

Definition 1. Let f be a continuous, real (or complex) valued function defined on the real line. Then $\operatorname{supp} f$, the *support* of f, is the closure of the set on which $f(x) \neq 0$.

If f is a differentiable function, its derivative f' is also an another function; thus the new function f' may have a derivative of its own and it is denoted by $\left(f'\right)' = f''$. This new function f'' is called the second derivative of f or derivative of the first derivative. The process can be continued as long as successive derivatives are differentiable. We note that, in general, if f is differentiable then f' need not necessarily to be continuous. That is to say the function might have derivative but the derivative not necessarily be continuous however we have the following definition.

Definition 2. Let f be defined on an open subset $S \subseteq \mathbb{R}$. Then if f' is continuous on $S \subseteq \mathbb{R}$, we say that $f \in C^1(S)$ that is, $C^1(S)$ is the set of all functions which have continuous first order derivatives in S.

Example 3. Let f be defined by $f(s) = e^s$ then f' is continuous on $\mathbb{R}$, then $f \in C^1(\mathbb{R})$.

Definition 4. If the f'' exists and continuous at each point of S then f is the member of the $C^2(S)$, or we say $f \in C^2(S)$. Thus it is obvious that

$$C^2(S) \subset C^1(S).$$

The set $C^\infty(S)$ contains the functions such that having continuous derivatives of all orders in S. We call this class infinitely continuously differentiable function class. Of course we can also show that for any natural number m, we have the following implication

$$C^{\infty}(S) \subset \ldots \subset C^{m+1}(S) \subset C^{m}(S) \subset \ldots \subset C^{2}(S) \subset C^{1}(S)$$

The function having continuous derivatives of all orders is also known as smooth function.

Now if we let ϕ be an infinitely differentiable real valued function with compact support. Then ϕ is said to be a *test function.* The set of all test functions with the usual definition of sum and product by a scalar is a vector space and is denoted by D. If ϕ is given by

$$\phi(x) = \begin{cases} e^{\left(\frac{1}{x-b} - \frac{1}{x-a}\right)}, & a < x < b \\ 0, & x \leq a \quad x \geq b, \end{cases}$$

then $\phi \in D$ and $\operatorname{supp} \phi = [a, b]$.

Note that if ϕ is in D, then $\phi^{(r)}$ is in D for $r = 1, 2, \ldots$, we also note that if $\phi \in D$ and g is any infinitely differentiable function, then $g\phi \in D$ but $\Phi(x) = \displaystyle\int_{-\infty}^{x} \phi(x)\, dx$ is not necessarily in D, since $\displaystyle\int_{-\infty}^{\infty} \phi(x)\, dx$ may not be equal to zero, in which case Φ will not have compact support. Thus, with the above if $\phi \in D$ then

$$\cos x\, \phi(x),\ \sin x\, \phi(x),\ \left(\sum_{i=0}^{r} a_i x^i\right) \phi(x),\ \ln|x - c|\, \phi(x)$$

where $c < a$, are all in D.

Example 5. Let f be defined by

$$f(s) = \begin{cases} e^{-\frac{1}{s}} & \text{if } s \in (0, \infty) \\ 0 & \text{if } s \in (-\infty, 0] \end{cases}$$

then f is infinitely differentiable for all s. That is $f \in C^{\infty}(S)$ for any $S \subseteq \mathbb{R}$.

We note that

(i) The polynomials are infinitely continuously differentiable function thus polynomials are in $C^{\infty}(S)$.

(ii) The function $f(t) = \sin t$, $g(t) = \cos t$ and $h(t) = e^t$ are not polynomial however they are continuously differentiable thus they are in $C^{\infty}(S)$.

Definition 6. Let $\{\phi_n\}$ be a sequence of functions in D. Then the sequence $\{\phi_n\}$ is said to *converge to zero* if $\exists$ a bounded interval $[a, b]$, with $\operatorname{supp} \phi_n \subseteq [a, b]$ for all n and $\lim_{n \to \infty} \phi_n^{(r)}(x) = 0$ for all x and $r = 0, 1, 2, \ldots$. Further, let f be a linear functional on D. Then f is said to be *continuous (bounded)* if $\lim_{n \to \infty} \langle f, \phi_n \rangle = 0$ whenever $\{\phi_n\}$ is a sequence in D converging to zero. A continuous linear functional on D is said to be a *distribution or generalized function*. The set of all distributions is a vector space and is denoted by D'.

We note that in the classical sense, we represent a function as a table of ordered pairs $(x, f(x))$. Of course, often this table has an uncountably infinite number of ordered pairs. We show this table as a curve representing the function in a plane. In generalized function theory, we also describe $f(x)$ by a table of numbers. These numbers are produced by the relation

$$F(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx \tag{3}$$

where the function $\phi(x)$ comes from a given space of functions called the test function space thus generalized functions are defined as continuous linear functionals over a space of infinitely differentiable functions D therefore

$\langle f, \phi \rangle$ is an action on ϕ rather than a pointwise value.

Now let f be a locally summable function. (i.e. $\int_a^b f(x)\,dx$ exists for every bounded interval $[a, b]$.) Then f defines a linear functional on D if we put

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx = \int_a^b f(x)\phi(x)\,dx$$

if $\operatorname{supp}\phi \subseteq [a, b]$. Further, f is continuous and so a distribution, since if $\{\phi_n\}$ converges to zero, $|\phi_n(x)| < \epsilon$ for $n > N$ and so

$$|\langle f, \phi_n \rangle| \leq \epsilon \int_a^b |f(x)|\,dx$$

Thus it is clear that $D \subset D'$. In fact one can prove that $\overline{D} = D'$.

Example 7. A distribution δ (the *Dirac delta-function*) can be defined by putting

$$\langle \delta, \phi \rangle = \phi(0).$$

However, δ is not defined by a locally summable function since if

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx = \phi(0)$$

for all $\phi \in D$, then

$$\int_{-\infty}^{\infty} \delta(x)\phi(x, a)\,dx = \phi(0, a) = e^{-1}$$

for all $a > 0$. Letting $a \to 0$, LHS $\to 0$, giving a contradiction.

Although it seems impossible to give a suitable definition which will define the product of any two distributions, it is possible to define the product of a distribution f and an infinitely differential function g and this is given in the next definition.

Definition 8. Let f be a distribution and let g be an infinitely differentiable function. Then the *product* $fg = gf$ is defined by

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ in D. Note that in this definition the product $g\phi$ is in D and so the product $fg = gf$ is defined as a distribution.

Example 9. Let g be an infinitely differentiable function. Then

$$\langle \delta g, \phi \rangle = \langle g\delta, \phi \rangle = \langle \delta, g\phi \rangle = g(0)\phi(0) = g(0)\langle \delta, \phi \rangle$$

for all ϕ in D. Thus

$$\delta g = g\delta = g(0)\delta.$$

Theorem 10. Let f be a distribution and let g be an infinitely differentiable function. Then

$$f^{(r)}g = gf^{(r)} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[fg^{(i)} \right]^{(r-i)}.$$

In particular, Leibnitz's Theorem holds for the product $fg = gf$, i.e.

$$(fg)' = fg' + f'g.$$

Example 11. Let g be an infinitely differentiable function. Then

$$\delta' g = g\delta' = g(0)\delta' - g'(0)\delta.$$

In particular, with $g(x) = x$,

$$\delta' x = x\delta' = -\delta.$$

More general

$$\begin{aligned}\delta^{(r)}g &= \sum_{i=0}^{r}\binom{r}{i}(-1)^i\left[\delta g^{(i)}\right]^{(r-i)}\\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^i g^{(i)}(0)\delta^{(r-i)}.\end{aligned}$$

In particular

$$\delta^{(r)}x^s = x^s\delta^{(r)} = \begin{cases}\frac{r!}{(r-s)!}(-1)^s\delta^{(r-s)}, & 0\le s\le r,\\ 0, & s>r.\end{cases}.$$

Note: The product of distributions is not necessarily associative. To see this, we have

$$\begin{aligned}\langle x^{-1}x,\phi\rangle &= \int_0^\infty x^{-1}[x\phi(x)+x\phi(-x)]\,dx\\ &= \int_0^\infty[\phi(x)+\phi(-x)]\,dx\\ &= \int_{-\infty}^\infty\phi(x)\,dx = \langle 1,\phi\rangle,\end{aligned}$$

for all ϕ in D and so $x^{-1}x=1$. Thus

$$(x^{-1}x)\delta = 1\delta = \delta$$

on the other side

$$x^{-1}(x\delta) = x^{-1}0 = 0.$$

In physics products of distributions such as $H\delta$ or δ^2 can be interpreted in many different ways. In the literature, several definitions have been proposed for δ^2 ranging from

$$\delta^2 = 0,\quad c\delta,\quad cx^{-2},\quad c\delta+\frac{1}{2\pi i}\delta',\quad c\delta+c'\delta'$$

with arbitrary constants c, c'. This has opened up a new area of mathematical research, with many attempts to try and give a satisfactory definition for the product of two distributions.

Despite the non-associativity, still there are some distributions, for example the product of two distributions such as

$$\ln^p |x| \; \delta^{(r)}(x), \quad x^{-p} \; \delta^{(r)}(x) \quad \text{or} \quad (x + i0)^{-p} \; \delta^{(r)}(x)$$

for $p = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$ which do not exist in the ordinary sense. In order to compute highly singular products, another generalization of product, the neutrix products was introduced by Fisher which is based on the concept of neutrix limits due to van der Corput, see [**56**]. The essential use of the neutrix limit is to extract the finite part from a divergent quantity as one has usually done to subtract the divergent terms via rather complicated procedures in the renormalization theory, see [**48**]. In fact we can consider the neutrices as the generalization of the Hadamard finite parts, see [**18**].

Definition 12. Let $\{f_n\}$ be a sequence of distributions. Then $\{f_n\}$ is said to converge to the distribution f if

$$\lim_{n \to \infty} \langle f_n, \phi \rangle = \langle f, \phi \rangle, \quad \forall \phi \in D.$$

More generally, if $\{f_n\}$ is a sequence of distributions converging to the distribution f. Then the sequence $\{f_n^{(r)}\}$ converges to the distribution $f^{(r)}$ for $r = 1, 2, \ldots$.

Example 13. Let f_n be the function defined by $f_n(x) = x^{-1}$ for $|x| \geq 1/n$ and $f_n(x) = 0$ for $|x| < 1/n$. Then $\lim_{n\to\infty} f_n = x^{-1}$. Similarly, if $f_\nu = \nu\pi^{-1}(\nu^2 + x^2)^{-1}$. Then $\lim_{\nu\to 0} f_\nu = \delta$.

Now suppose that f is a continuous function having a continuous derivative f'. Then for arbitrary $\phi \in D$ with $\operatorname{supp}\phi = [a, b]$, we have

$$\begin{aligned}\langle f', \phi\rangle &= \int_a^b f'(x)\phi(x)\,dx \\ &= \Big[f(x)\phi(x)\Big]_a^b - \int_a^b f(x)\phi'(x)\,dx \\ &= -\langle f, \phi'\rangle.\end{aligned}$$

Thus the *derivative* f' of f is the distribution defined by

$$\langle f', \phi\rangle = -\langle f, \phi'\rangle, \quad \text{for all } \phi \in D.$$

It is clear that f' is in fact a distribution, $f^{(n)}$ is a distribution and

$$\langle f^{(n)}, \phi\rangle = (-1)^n \langle f, \phi^{(n)}\rangle$$

for $n = 1, 2, \ldots$.

Example 14. Let x_+ be the locally summable function defined by

$$x_+ = \begin{cases} x, & x > 0, \\ 0. & x \leq 0, \end{cases}.$$

Then its derivative, denoted by H (Heaviside's function), is given by

$$\langle H, \phi\rangle = -\langle x_+, \phi'\rangle = -\int_0^\infty x\phi'(x)\,dx = \int_0^\infty \phi(x)\,dx$$

and so H corresponds to the locally summable function defined by

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}.$$

The derivative H' of H is given by

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)\, dx = \phi(0)$$

and so $H' = \delta$. We note that the function $H(x)$ is a very useful function in the study of the generalized function (distributions theory), especially in the discussion of the functions with jump discontinuities. For instance, let $F(x)$ be a function that is continuous every where except for the point $x = \xi$, at which point it has a jump discontinuity,

$$F(x) = \begin{cases} F_1(x), & x < \xi, \\ F_2(x), & x > \xi \end{cases} .$$

Then it can be written that

$$F(x) = F_1(x) H(\xi - x) + F_2(x) H(x - \xi).$$

This concept can be extended to enable one to write a function that has jump discontinuities at several points.

The r-th derivative $\delta^{(r)}$ of δ is given by

$$\langle \delta^{(r)}, \phi \rangle = (-1)^r \langle \delta, \phi^{(r)} \rangle = (-1)^r \phi^{(r)}(0).$$

More generally, let x_+^λ $(\lambda > -1)$ be the locally summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0 \\ 0, & x < 0. \end{cases}$$

If $\lambda > 0$, its derivative is the locally summable function $\lambda x_+^{\lambda-1}$ but if $-1 < \lambda < 0$, $x_+^{\lambda-1}$ is not a locally summable function. If $-1 < \lambda < 0$, we will still denote the derivative of x_+^λ by $\lambda x_+^{\lambda-1}$ but it must be defined by

$$\begin{aligned}\langle (x_+^\lambda)', \phi\rangle &= -\langle x_+^\lambda, \phi'\rangle \\ &= -\int_0^\infty x^\lambda \, d[\phi(x) - \phi(0)] \\ &= \lambda \int_0^\infty x^{\lambda-1}[\phi(x) - \phi(0)]\, dx.\end{aligned}$$

Thus if $-2 < \lambda < -1$, we have defined x_+^λ by

$$\langle x_+^\lambda, \phi\rangle = \int_0^\infty x^\lambda [\phi(x) - \phi(0)]\, dx.$$

The distribution $x_+^\lambda \ln^m x_+$ is defined by differentiating x_+^λ m times partially with respect to λ. Thus if $-r-1 < \lambda < -r$ then

$$\langle x_+^\lambda \ln^m x_+, \phi\rangle = \int_0^\infty x^\lambda \ln^m x \Big[\phi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!}\phi^{(i)}(0)\Big]\, dx$$

and if $-r-1 < \lambda < -r+1$ and $\lambda \neq -r$, then

$$\begin{aligned}\langle x_+^\lambda \ln^m x_+, \phi\rangle &= \int_0^\infty x^\lambda \ln^m x \Big[\phi(x) - \sum_{i=0}^{r-2} \frac{x^i}{i!}\phi^{(i)}(0) \\ &- \frac{x^{r-1}}{(r-1)!}\phi^{(r-1)}(0) H(1-x)\Big]\, dx \\ &+ \frac{(-1)^m m! \phi^{(r-1)}(0)}{(r-1)!(\lambda+r)^{m+1}}.\end{aligned}$$

Example 15. The locally summable function $\ln x_+$ is defined by

$$\ln x_+ = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0. \end{cases}.$$

We define the distribution x_+^{-1} to be the derivative of $\ln x_+$. Thus

$$\begin{aligned}\langle x_+^{-1}, \phi\rangle &= \langle (\ln x_+)', \phi\rangle = -\langle \ln x_+, \phi'\rangle \\ &= -\int_0^1 \ln x \, d[\phi(x) - \phi(0)] - \int_1^\infty \ln x \, d\phi(x) \\ &= \int_0^1 x^{-1}[\phi(x) - \phi(0)] \, dx + \int_1^\infty x^{-1}\phi(x) \, dx \\ &= \int_0^\infty x^{-1}[\phi(x) - \phi(0)H(1-x)] \, dx.\end{aligned}$$

The distribution $x_+^{-1} \ln^m x_+$ is defined for $m = 1, 2, \ldots$ by

$$(\ln^{m+1} x_+)' = (m+1)x_+^{-1} \ln^m x_+.$$

Then the distribution $x_+^{-r} \ln^m x_+$ is then defined inductively. Suppose that $x_+^{-r} \ln^{m-1} x_+$ has been defined for $r = 1, 2, \ldots$ and some m. This is certainly true when $m = 1$. The distribution $x_+^{-1} \ln^m x_+$ has been defined for $m = 1, 2, \ldots$, so suppose that $x_+^{-r+1} \ln^m x_+$ has been defined for some r. Then $x_+^{-r} \ln^m x_+$ is defined by

$$(x_+^{-r+1} \ln^m x_+)' = -(r-1)x_+^{-r} \ln^m x_+ + mx_+^{-r} \ln^{m-1} x_+.$$

The distribution $|x|^\lambda \ln^m |x|$ is defined by

$$|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ + x_-^\lambda \ln^m x_-,$$

the distribution $\operatorname{sgn} x.|x|^\lambda \ln^m |x|$ is defined by

$$\operatorname{sgn} x.|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ - x_-^\lambda \ln^m x_-$$

and the distribution x^r is defined by

$$x^r = x_+^r + (-1)^r x_-^r,$$

for $r = 0, \pm 1, \pm 2, \ldots$.

Example 16. The function $\ln(x+i0)$ is defined by

$$\ln(x+i0) = \lim_{y\to 0+} \ln(x+iy)$$

then it follows that

$$\ln(x+i0) = \ln|x| + i\pi H(-x).$$

Similarly, the distribution $(x+i0)^\lambda$ is defined by

$$(x+i0)^\lambda = \lim_{y\to 0+} (x+iy)^\lambda$$

It follows that

$$\begin{aligned}
(x+i0)^\lambda &= x_+^\lambda + e^{i\lambda\pi} x_-^\lambda, \quad \lambda \neq -1, -2, \dots, \\
(x+i0)^r &= x^r, \quad r = 0, 1, 2, \dots, \\
(x+i0)^{-r} &= \lim_{\lambda\to -r} (x+i0)^\lambda = x^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(x)
\end{aligned}$$

$r = 1, 2, \dots$. Similarly,

$$(x-i0)^\lambda = \lim_{y\to 0-} (x+iy)^\lambda$$

and then

$$\begin{aligned}
(x-i0)^\lambda &= x_+^\lambda + e^{-i\lambda\pi} x_-^\lambda, \quad \lambda \neq -1, -2, \dots, \\
(x-i0)^r &= x^r, \quad r = 0, 1, 2, \dots, \\
(x-i0)^{-r} &= \lim_{\lambda\to -r} (x-i0)^\lambda = x^{-r} - \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(x),
\end{aligned}$$

$r = 1, 2, \dots$.

More generally we can define the distribution

$$(x+i0)^\lambda \ln^m(x+i0)$$

is defined by

$$
\begin{aligned}
(x+i0)^{\lambda}\ln^m(x+i0) &= \frac{\partial^m}{\partial\lambda^m}(x+i0)^{\lambda} \\
&= x_+^{\lambda}\ln^m x_+ + \sum_{k=0}^{m}\binom{m}{k}(i\pi)^{m-k}e^{i\lambda\pi}x_-^{\lambda}\ln^k x_-.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
(x+i0)^{\lambda}\ln(x+i0) &= x_+^{\lambda}\ln x_+ + e^{i\lambda\pi}x_-^{\lambda}\ln x_- + i\pi e^{i\lambda\pi}x_-^{\lambda} \\
(x+i0)^{\lambda}\ln^2(x+i0) &= x_+^{\lambda}\ln^2 x_+ + e^{i\lambda\pi}x_-^{\lambda}\ln^2 x_- \\
&+ 2i\pi e^{i\lambda\pi}x_-^{\lambda}\ln x_- - \pi^2 e^{i\lambda\pi}x_-^{\lambda}
\end{aligned}
$$

for $\lambda \neq -1, -2, \ldots,$

$$
\begin{aligned}
(x+i0)^{r}\ln(x+i0) &= x^r\ln|x| + (-1)^r i\pi x_-^r \\
(x+i0)^{r}\ln^2(x+i0) &= x^r\ln^2|x| + 2(-1)^r i\pi x_-^r\ln x_- \\
&- (-1)^r\pi^2 x_-^r
\end{aligned}
$$

for $r = 0, 1, 2, \ldots$ and

$$
\begin{aligned}
(x+i0)^{-r}\ln(x+i0) &= \lim_{\lambda\to -r}(x+i0)^{\lambda}\ln(x+i0) \\
&= x^{-r}\ln|x| + (-1)^r i\pi F(x_-, -r) \\
&\qquad -\frac{(-1)^r\pi^2}{2(r-1)!}\delta^{(r-1)}(x) \\
(x+i0)^{-r}\ln^2(x+i0) &= \lim_{\lambda\to -r}(x+i0)^{\lambda}\ln^2(x+i0) \\
&= x^{-r}\ln^2|x| + 2(-1)^r i\pi F(x_-, -r)\ln x_- \\
&\quad +(-1)^r\pi^2 F(x_-, -r) + -\frac{(-1)^r i\pi^3}{3(r-1)!}\delta^{(r-1)}(x)
\end{aligned}
$$

for $r = -1, -2, \ldots$. In general, it can be proved that any distribution f defined on the bounded interval (a, b) is the r-th derivative of a continuous function F on the interval (a, b) for some r, see [**19**].

Similar to the previous examples, consider the Gamma function Γ then this function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

and it follows that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. $\Gamma(x)$ is then defined by

$$\Gamma(x) = x^{-1}\Gamma(x+1)$$

for $-1 < x < 0$. Further we can express this function as follows

$$\Gamma(x) = x^{-1} + f(x) = x^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!} x^{i-1}$$

where x^{-1} is interpreted in the distributional sense. The distribution Γ is of course an ordinary summable function for $x > 0$, see [**31**].

The related distribution $\Gamma(x_+)$ by equation

$$\Gamma(x_+) = x_+^{-1} + f(x_+) = x_+^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!} x_+^{i-1} \tag{4}$$

and the distribution $\Gamma(x_-)$ by equation

$$\Gamma(x_-) = x_-^{-1} + f(x_-) = x_-^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!} x_-^{i-1} \tag{5}$$

where x_+^{-1} and x_-^{-1} are interpreted in the distributional sense, see [**26**]. It follows that

$$\Gamma(x) = \Gamma(x_+) - \Gamma(x_-) \tag{6}$$

Differentiating equation (4) s times we have

$$\begin{aligned}\Gamma^{(s)}(x_+) &= (-1)^s s! x_+^{-s-1} + f^{(s)}(x_+) \\ &= (-1)^s s! x_+^{-s-1} + \sum_{i=0}^{\infty} \frac{\Gamma^{(s+i+1)}(1)}{(s+i+1)i!} x_+^i \end{aligned} \tag{7}$$

and differentiating equation (5) s times we have

$$\begin{aligned}\Gamma^{(s)}(x_-) &= s! x_-^{-s-1} + f^{(s)}(x_-) \\ &= (-1)^s s! x_-^{-s-1} + \sum_{i=0}^{\infty} \frac{\Gamma^{(s+i+1)}(1)}{(s+i+1)i!} x_-^i \end{aligned} \tag{8}$$

Similarly, now let us consider the Beta function $B(x, y)$. This function can be defined for $x, y > 0$ by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

and it follows that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y, x + y \neq 0, -1, -2, \ldots$. In particular we have

$$\begin{aligned} B(x, n) &= B(n, x) \\ &= (n-1)! \left[x(x+1)\ldots(x+n-1)\right]^{-1} \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (x+i)^{-1} \end{aligned}$$

for $n = 1, 2, \ldots$, where the $(x+i)^{-1}$ for $i = 0, 1, \ldots, n-1$ are interpreted in the distributional sense, see [**30**].

Further, we define the distribution $B_+(x,n)$ by the equation

$$B_+(x,n) = x_+^{-1} + \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i (x+i)^{-1} H(x) \tag{9}$$

for $n = 1, 2, \ldots$ and we define the distribution $B_-(x,n)$ by the equation

$$B_-(x,n) = x_-^{-1} - \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i (x+i)^{-1} H(-x) \tag{10}$$

for $n = 1, 2, \ldots,$ where H denotes Heaviside's function. It follows that

$$B(x,n) = B_+(x,n) - B_-(x,n).$$

Differentiating equation (9) r times we have

$$\begin{aligned} B_+^{(r)}(x,n) &= (-1)^r r! x_+^{-r-1} + r! \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{r+i} (x+i)^{-r-1} H(x) + \\ &\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{r} \binom{n-1}{i} (-1)^{i+j} (j-1)! i^{-j} \delta^{(r-j)}(x) \end{aligned} \tag{11}$$

and differentiating equation (10) r times we have

$$\begin{aligned} B_-^{(r)}(x,n) &= r! x_-^{-r-1} - r! \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{r+i} (x+i)^{-r-1} H(-x) + \\ &\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{r} \binom{n-1}{i} (-1)^{i+j} (j-1)! i^{-j} \delta^{(r-j)}(x) \end{aligned} \tag{12}$$

for $r = 1, 2, \ldots$, see details [**49**].

By using the distributional approach, it was proved in [**11**] that

$$\Gamma(0) = \int_0^\infty e^{-t} \ln t \, dt = \Gamma'(1) = -\gamma \tag{13}$$

where γ denotes Euler's constant, and

$$\Gamma(-m) + m^{-1}\Gamma(-m+1) = \frac{(-1)^m}{mm!} \tag{14}$$

for $m = 1, 2, \ldots$. More generally, we have

$$\Gamma^{(r)}(0) = \frac{1}{r+1}\Gamma^{(r+1)}(1) \tag{15}$$

for $r = 1, 2, \ldots$ and

$$\begin{aligned} \Gamma(-m) &= \int_1^\infty t^{-m-1}e^{-t}\,dt + \int_0^1 t^{-m-1}\left[e^{-t} - \sum_{i=0}^{m}\frac{(-t)^i}{i!}\right]dt \\ &\qquad - \sum_{i=0}^{m-1}\frac{(-1)^i}{i!(m-i)} \end{aligned} \tag{16}$$

for $m = 0, 1, 2, \ldots$.

Similarly, the incomplete Gamma function $\gamma(\alpha, x)$ is defined for $\alpha > 0$ and $x \geq 0$ by

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1}e^{-u}\,du \tag{17}$$

see [**11**], the integral diverging for $\alpha \in [0, \infty)$. Now if we let $\alpha > 0$, then on integration by parts, we obtain that

$$\gamma(\alpha+1, x) = \alpha\gamma(\alpha, x) - x^\alpha e^{-x} \tag{18}$$

and so we can use equation (17) to extend the the definition of $\gamma(\alpha, x)$ to negative, non-integer values of α. In particular, it follows that if $-1 < \alpha < 0$ and $x > 0$, then

$$\begin{aligned} \gamma(\alpha, x) &= \alpha^{-1}\gamma(\alpha+1, x) + \alpha^{-1}x^\alpha e^{-x} \\ &= -\alpha^{-1}\int_0^x u^\alpha\,d(e^{-u}-1) + \alpha^{-1}x^\alpha e^{-x} \end{aligned}$$

and on integration by parts, we see that

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1}(e^{-u} - 1)\, du + \alpha^{-1}x^{\alpha}.$$

More generally, it can be easily proved by induction that if $-r < \alpha < -r+1$ and $x > 0$, then

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1}\left[e^{-u} - \sum_{i=0}^{r-1}\frac{(-u)^i}{i!}\right] du + \sum_{i=0}^{r-1}\frac{(-1)^i x^{\alpha+i}}{(\alpha+i)i!}. \tag{19}$$

As we can see in the above examples even locally integrable functions(in the Lebesgue sense) though discontinuous are infinitely differentiable as generalized functions.

CONVOLUTION PRODUCTS

Definition 17. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt$$

for all points x for which the integral exists.

Theorem 18. If the convolution $f * g$ exists, then $g * f$ exists and

$$f * g = g * f. \tag{20}$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{21}$$

Example 19. If $\lambda, \mu > -1$, then $x_+^{\lambda} * x_+^{\mu} = B(\lambda+1, \mu+1)x_+^{\lambda+\mu+1}$. Equivalently, $f_+^{\lambda} * f_+^{\mu} = f_+^{\lambda+\mu+1}$. In particular,

$$x_+^{\lambda} * H(x) = \frac{x_+^{\lambda+1}}{\lambda+1} = \int_{-\infty}^{x} x_+^{\lambda}\, dx.$$

Further, if $\lambda, \mu > -1 > \lambda + \mu$, then $x_-^\lambda * x_+^\mu = B(\lambda+1, -\lambda-\mu-1)x_+^{\lambda+\mu+1} + B(\mu+1, -\lambda-\mu-1)x_-^{\lambda+\mu+1}$.

Now let f, g be locally summable functions and suppose that $\operatorname{supp} f \subseteq [a, b]$. Then if G is a primitive of g and $[c, d]$ is any interval,

$$\int_c^d g(x-t)\,dx = G(d-t) - G(c-t).$$

This implies that the function $\int_c^d g(x-t)\,dx$ is bounded on the interval $[a, b]$ and so $f(t)\int_c^d g(x-t)\,dx$ is a locally summable function. This proves that

$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \\ &= \int_a^b f(t)g(x-t)\,dt\end{aligned}$$

exists and further

$$\begin{aligned}\int_c^d (f*g)(x)\,dx &= \int_c^d \Big\{\int_a^b f(t)g(x-t)\,dt\Big\}dx \\ &= \int_a^b f(t)\Big\{\int_c^d g(x-t)\,dx\Big\}dt,\end{aligned}$$

proving that $f*g$ is a locally summable function if f has compact support. Similarly, $f*g$ is a locally summable function if g has compact support and in either case $f*g = g*f$.

Finally suppose that $(f*g)(x) \neq 0$. Then there exists a point t_0 such that $f(t_0)g(x-t_0) \neq 0$ which implies that $t_0 \in \operatorname{supp} f$ and $x - t_0 \in \operatorname{supp} g$. Thus $x \in \operatorname{supp} f + \operatorname{supp} g$ or

$$\operatorname{supp}(f*g) \subseteq \operatorname{supp} f + \operatorname{supp} g.$$

We now consider the problem of defining the convolution $f * g$ of two distributions f and g. First of all suppose that f, g are locally summable functions and that $f * g$ exists. Then for arbitrary $\phi \in D$ we can write

$$\begin{aligned} \langle f * g, \phi \rangle &= \int_{\infty}^{\infty} \Big\{ \int_{-\infty}^{\infty} f(t) g(x-t)\, dt \Big\} \phi(x)\, dx \\ &= \int_{-\infty}^{\infty} f(t) \Big\{ \int_{-\infty}^{\infty} g(x-t) \phi(x)\, dx \Big\} dt. \end{aligned}$$

We put

$$\begin{aligned} \psi(t) &= \int_{-\infty}^{\infty} g(x-t) \phi(x)\, dx \\ &= \int_{-\infty}^{\infty} g(x) \phi(x+t)\, dx = \langle g(x), \phi(x+t) \rangle, \end{aligned}$$

where $\phi(x+t) \in D$ as a function of x, and $\psi(t) = \langle g(x), \phi(x+t) \rangle$ in fact exists for every distribution g and for all t since ϕ has compact support. It is easy to prove that $\psi(t)$ is a continuous function for every distribution $g \in D$. To see this, let $\{t_n\}$ be an arbitrary sequence converging to t_0. Then $\phi(x+t_n)$ converges uniformly to $\phi(x+t_0)$ together with all its derivatives and each $\phi(x+t_n)$ has its support contained in some fixed bounded interval. Since g is a continuous linear functional,

$$\psi(t_n) = \langle g(x), \phi(x+t_n) \rangle \to \langle g(x), \phi(x+t_0) \rangle = \psi(t_0),$$

proving the continuity of ψ. Further,

$$\begin{aligned} \frac{\psi(t_n) - \psi(t_0)}{t_n - t_0} &= \Big\langle g(x), \frac{\phi(x+t_n) - \psi(x+t_0)}{t_n - t_0} \Big\rangle \\ &\to \langle g(x), \phi'(x+t_0) \rangle, \end{aligned}$$

proving that ψ is differentiable with derivative

$$\psi'(t) = \langle g(x), \phi'(x+t) \rangle.$$

Definition 20. Let f, g be distributions in D' and suppose that either f or g has bounded support or that f and g are bounded on the same side. Then $f * g$, the convolution of f and g, is defined by

$$\langle f * g, \phi \rangle = \langle f(t), \langle g(x), \phi(x+t) \rangle \rangle,$$

for all $\phi \in D$.

Example 21. Let f be an arbitrary distribution. Then

$$f * \delta^{(r)} = f^{(r)}, \quad r = 0, 1, 2, \ldots.$$

and $x_+^\lambda * x_+^\mu = B(\lambda+1, \mu+1) x_+^{\lambda+\mu+1}$ for $\lambda, \mu, \lambda+\mu+1 \neq -1, -2, \ldots$.

Multiplication of Distributions

If f is a distribution in D' and g is an infinitely differentiable function then the product $fg = gf$ is defined by

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ in D and satisfies the rule

$$f^{(r)} g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[F g^{(i)} \right]^{(r-i)}$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}$$

for $r = 1, 2, \ldots$. The first extension of the product of a distribution and an infinitely differentiable function is the following definition, see for example [**9**].

Definition 22. Let f and g be distributions in D' for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$

and $g^{(k)}$ is a locally summable function in $L^q(a,b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a,b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

In the literature, many attempts have been made to define a product of distributions. Konig was the first to develop a systematic treatment of the subject in an abstract way and showed that there are actually many possible product theories if one gives up some requirements such as associativity, see Konig [**44**].

Konig theory was followed by Güttinger on noticing that certain products of distributions could be defined on subspaces of D'. For example, if

$$D_0 = \{\phi \in D : \phi(0) = 0\}$$

and $\phi \in D_0$, then $x^{-1}\phi(x) = \psi(x)$ is a continuous function. He then defined the product $\delta(x) \cdot x^{-1}$ on D_0 by the equation

$$\langle \delta(x) \cdot x^{-1}, \phi(x) \rangle = \psi(0) = \phi'(0) = -\langle \delta'(x), \phi(x) \rangle.$$

It follows that $\delta(x) \cdot x^{-1} = -\delta'(x)$ on D_0, see [**17**]. Extending the linear functional to the whole of D' by the Hahn–Banach Theorem, he obtained the equation

$$\delta(x) \cdot x^{-1} = -\delta'(x) + c_0\delta(x),$$

where c_0 is an arbitrary constant. More generally,

$$\delta^{(r)}(x) \cdot x^{-1} \quad = \quad (-1)^{r+1}\delta^{(r+1)}(x) + \sum_{i=0}^{r} c_i \delta^{(i)}(x), \tag{22}$$

where $c_0, c_1, \ldots, c_r$ are arbitrary constants. Formal differentiation of Equation (22) gives

$$\delta^{(r+1)}(x) \cdot x^{-1} - \delta^{(r)}(x) \cdot x^{-2} = (-1)^{r+1}\delta^{(r+2)}(x) + \sum_{i=0}^{r} c_i \delta^{(i+1)}(x)$$

and so $\delta^{(r)}(x) \cdot x^{-2}$ is defined by

$$\begin{aligned} \delta^{(r)}(x) \cdot x^{-2} &= \delta^{(r+1)} \cdot x^{-1} + (-1)^r \delta^{(r+2)}(x) - \sum_{i=0}^{r} c_i \delta^{(i+1)}(x) \\ &= 2(-1)^r \delta^{(r+2)}(x) + c_0 \delta(x) + \sum_{i=1}^{r+1} (c_i - c_{i-1}) \delta^{(i)}(x). \quad (23) \end{aligned}$$

Note that the $c_i - c_{i-1}$ are not considered to be further independent constants and expression for the product $\delta^{(r)}(x) \cdot x^{-3}$ can now be found by formally differentiating equation (23).

In a similar way, Güttinger obtained the product

$$\delta^{(r)}(x) \cdot H(x) = -\sum_{i=0}^{r} b_i \delta^{(r-i)}(x), \tag{24}$$

where $b_0, b_1, \ldots, b_r$ are again arbitrary constants. Formal differentiation of equation (24) gives

$$\delta^{(r+1)}(x) \cdot H(x) + \delta^{(r)}(x) \cdot \delta(x) = -\sum_{i=0}^{r} b_i \delta^{(r-i+1)}(x)$$

and so $\delta^{(r)}(x) \cdot \delta(x)$ is defined by

$$\delta^{(r)}(x) \cdot \delta(x) = -\delta^{(r+1)}(x) \cdot H(x) - \sum_{i=0}^{r} b_i \delta^{(r-i+1)}(x) = b_{r+1}\delta(x).$$

The constants in equations (23) and (24) are completely independent and they would also be completely independent of any new constants introduced to define further products, unless of course they are obtained by formal differentiation.

However the products which result from this approach are generally neither commutative nor associative, and there is an inherent arbitrariness as the presence of the arbitrary constants in the examples given above makes clear. For these reasons Konig and Güttinger approach has generally found less favor than the sequential treatments developed by Mikusiński in [**46**]. These have been generally guided by the idea of developing a product which remains consistent with the Schwartz product and which further extends its domain of definition, see Hoskins [**23**].

However, the products of some singular distributions very important to applications, but does not exist in the sense of definition 22. Then there are some further extension of this definition in order to apply the product to the wide range of distributions. In order to include the singular distributions the definition of product was extended in two directions as follows:

One way is the Fourier Transform method by using the convolution method one can define the product of distributions which is known as the Fourier Transform method. For given two distributions $f, g \in D'$ assume that their Fourier transforms $F(f)$, $F(g)$ exists. Then the product of two distributions f and g is defined by following equation

$$f.g = F^{-1}\left(F(f) * F(g)\right), \tag{25}$$

see for example Bremermenn [**4**].

The second method is the regularization and passage to the limit. Sometimes it is known as the methods of the sequential completion for the product of distributions that is also compatible with the ordinary product. This method was first used by Mikusiński and Sikorski in [**46**] for a wide range of irregular distributions. To deal with the sequential completion approach we need the following concept of delta sequences.

Delta Sequences and Convergence

Definition 23. A sequence $\delta_n : \mathbb{R} \to \mathbb{R}$ is called a delta sequence of ordinary functions which converges to the singular distribution $\delta(x)$ and satisfy the following conditions:

(i) $\delta_n(x) \geq 0$ for all $x \in \mathbb{R}$,

(ii) δ_n is continuous and integrable over $\mathbb{R}$ with $\displaystyle\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$,

(iii) Given any $\epsilon > 0$,

$$\lim_{n\to\infty} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \delta_n(x)\,d\,x = 0.$$

For example, $\delta_n(xt) = \dfrac{n}{\pi(n^2t^2+1)}$ then

$$\begin{aligned}\int_a^b \delta_n(t)\ dt &= \int_a^b \frac{n}{\pi(n^2t^2+1)}\ dt \\ &= \frac{1}{\pi}\left[\arctan(nb) - \arctan(an)\right]\end{aligned}$$

now if we let $n \to \infty$ than it follows that δ_n is a delta sequence. In general, if ϕ is a continuous, nonnegative, $\phi(x) = 0$ for all $|x| \geq 1$ and $\displaystyle\int_{-1}^{1} \phi(x)\,dx = 1$, if we set $\delta_n(x) = n\phi(nx)$. Then one can show that δ_n is a delta sequence.

Thus the above examples show that there are several ways to construct a delta sequence. For our next definition we let $\rho(x)$ be a fixed infinitely differentiable function in D having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1.$

The function δ_n is then defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the delta function δ. If now f is an arbitrary distribution in D', the function f_n is defined by

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution f.

Let $f, g \in D'$ then the convolutions $f_n = f * \delta_n$ and $g_n = g * \delta_n$ always exists as infinitely smooth functions for each n if (δ_n) are delta sequences. By using the above regular sequence idea we can develop an alternative generalization of definition 22 and based on the product $f\phi = \phi f \in D'$ for a distribution $f \in D'$ and infinitely differentiable function $\phi \in C^\infty$ which was introduced by Schwartz in [**50**].

Therefore in the literature there are several product of distributions by using delta sequences. In summary, if we let $f, g \in D'$ be two distributions then product of two distributions can be defined either of the following equations:

$$\begin{aligned} f.(g_n) &= \lim_{n\to\infty} f(g*\delta_n) & (26)\\ (f_n).g &= \lim_{n\to\infty} (f*\delta_n)g & (27)\\ (f_n).(g_n) &= \lim_{n\to\infty} (f*\delta_n)(g*\epsilon_n) & (28)\\ (f.g)_n &= \lim_{n\to\infty} (f*\delta_n)(g*\delta_n) & (29) \end{aligned}$$

provided that the limits exist in the space D' for arbitrary delta sequences (δ_n) and (ϵ_n). Equation (26) is due to Mikusinski and Sikorski [**46**], equations (27) and (28) to Hirata and Ogata [**21**] required both simultaneously, (29) is due to Fisher [**9**]. It should be noted that if the respective limits exist in the above definitions then they are independent of the choice of the sequence defining the δ function. By using above definitions one can propose several results for $\delta^2(x)$ such as

$$\delta^2(x) = 0\,, \;\; c_1\,\delta(x)\,, \;\; c_1\,\delta(x) + \frac{1}{2\pi i}\delta'(x)\,, \;\; c_1\,\delta(x) + c_2\,\delta'(x)$$

with arbitrary constants c_1 and c_2. Later Tysk in [**54**] gave a comparison between equations (28) and (29).

However one can combine these two non symmetric equations and generate another new commutative product as follows:

$$\begin{aligned} \langle f.g, \phi\rangle &= \lim_{n\to\infty} \frac{1}{2}\langle f.(g_n) + (f_n).g, \phi\rangle \\ &= \lim_{n\to\infty} \langle f(g*\delta_n) + (f*\delta_n)g, \phi\rangle & (30) \end{aligned}$$

for all $\phi \in D$, see Kılıçman [**35**].

APPLICATIONS OF DISTRIBUTIONS

With admission of the delta function (or distribution) we can also have a solution for the following equation

$$x^n \cdot f(x) = g(x),$$

where $g(x)$ assumed to be ordinary function. In fact this idea can also be extended further as follows. Consider that $p(x)$ a polynomial having the zeros at $x = a_1$, $x = a_2$, $\dots x = a_n$ that is

$$p(x) = (x - a_1)\ (x - a_2)\ (x - a_3)\ \dots\ (x - a_n) = \prod_{i=1}^{n} (x - a_i).$$

Then the equation $p(x)\ f(x) = g(x)$ has the distributional solutions

$$f(x) \quad = \quad \frac{g(x)}{p(x)} + \sum_{i=0}^{n-1} c_i \delta^{(i)}(x - a_i), \tag{31}$$

for any constants $c_1, c_2, c_3, \dots, c_n$ and $a_1 \neq a_2 \neq a_3 \neq \dots \neq a_n$, see Kılıçman [**41**].

To solve a differential equation there are several methods and each method requires different techniques and there are no general method that will solve all the differential equations.

To solve a differential equation there are several methods and each method requires different techniques and there are no general method that will solve all the differential equations. We list the common methods by using the some sophisticated software such as Scientific Work Place or MAPLE:

(a) Exact Solutions Method, In this method return exact solutions to a differential equation. This method is a more general method that it can work for some nonlinear differential equations as well. Each of these options recognizes some functions that the other may not.

(b) Integral Transform Methods, Laplace transforms, Fourier transform, Mellin transform, sin and cos transforms that solve either homogeneous or non homogeneous systems in which the coefficients are all constants. Initial conditions appear explicitly in the solution.

(c) Numerical Solutions, Some Appropriate systems can be solved numerically. These numeric solutions are functions that can be evaluated at points or plotted. The method implemented by Maple for numerical solutions is a Fehlberg fourth-fifth order Runge-Kutta method.

(d) Series Solutions, For many applications, a few terms of a Taylor series solution are sufficient. We can also control the number of terms that appear in the solution by changing series order.

Example 24. Consider the following differential equations

$$\frac{dy}{dx} = x \sin \frac{1}{x}$$

and the exact solution is given by :

$$y(x) = \frac{1}{2}\left(\sin\frac{1}{x}\right)x^2 + \frac{1}{2}\left(\cos\frac{1}{x}\right)x + \frac{1}{2}\operatorname{Si}\left(\frac{1}{x}\right) + C_1.$$

Example 25. Similar to the previous example, consider to find the general solution of differential equation

$$x^2\frac{dy}{dx} + xy = \sin x$$

then exact solution is given by

$$y\,(x) = \frac{1}{x}\left(\mathrm{Si}\,(x) + C_1\right).$$

Distributional Solutions

Consider the initial-value problem

$$\frac{d^2y}{dx^2} + y = \sum_{k=0}^{\infty} \delta(x - k\pi), \qquad y(0) = y'(0) = 0$$

then we give the solution as

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} (-1)^k (H(x - k\pi)) \sin(x) \\ &+ C_1 \sin(x) + C_2 \cos(x). \end{aligned}$$

However there are no serial solution for these differential equations, see Kılıçman and Hassan [**40**].

In fact when we try to solve the differential equation

$$P(D)\,y = f(x)$$

we might have either of the following cases, see the details by Kanwal [**25**].

(i) The solution y is a smooth function such that the operation can be performed in the classical sense and the resulting equation is an identity. Then y is a classical solution.

(ii) The solution y is not smooth enough, so that the operation can not be performed but satisfies as a distributions.

(iii) The solution y is a singular distribution then the solution is a distributional solution.

Example 26. Let f be the given distribution and if we can find a fundamental solution g, then we are able to solve the equation

$$P(D)\, g = f$$

when $f * g$ is defined. Now consider to find the general solutions of the following ordinary differential equations:

$$g'' + g = \delta, \qquad\qquad f'' + f = \delta'$$

then on using the delta sequences we have

$$y_n'' + y_n = \delta_n \longrightarrow y'' + y = \delta \quad \text{and} \quad \mathrm{y_n''} + \mathrm{y_n} = (\delta_\mathrm{n})' \longrightarrow \mathrm{y}'' + \mathrm{y} = \delta'$$

where $y_n'' = y'' * \delta_n$ and $y_n = y * \delta_n$ as n tends to ∞. Then we can take

$$\delta_n(t) = n - nH\left(t - \frac{1}{n}\right) = \begin{cases} n & 0 < t < \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

By using the Laplace Transform one can easily show that

$$\begin{aligned} y_n(t) &= n - n\cos t - \left(n - n\cos\left(t - \frac{1}{n}\right)\right) H\left(t - \frac{1}{n}\right) \\ &= \begin{cases} n - n\cos t & 0 < t < \frac{1}{n} \\ n\cos\left(t - \frac{1}{n}\right) - n\cos t & \frac{1}{n} < t \end{cases} \end{aligned}$$

for fix $t > 0$, then for large enough n we have

$$\lim_{n\to\infty} y_n(t) = \sin t$$

and also for $t = 0$,

$$\lim_{n\to\infty} y_n(0) = 0 = \sin 0$$

therefore

$$\lim_{n\to\infty} y_n(t) = \sin t$$

as a symbolic solution. We can also have same result by using the Laplace Transform directly, see Nagle and Saff [**47**].

Example 27. But if we have differential equations in the form of

$$x\, f' = 1$$

then this equation has distributional solution

$$f = c_1\, \ln|x| + c_1 H(x) + c_3$$

of course this not classical solution since f is not differentiable at zero, see Kılıçman [**41**]. Now if we replace 1 by δ then we try to find the fundamental solution for

$$x\, f' = \delta \quad \Rightarrow \quad f' = x^{-1}\, \delta$$

or more general form of

$$x^s\, f' = \delta^{(r)} \quad \Rightarrow \quad f' = x^{-s}\, \delta^{(r)}, \;\; \text{for} \;\; \mathrm{r, s} = 0, 1, 2, 3, 4, \ldots. \tag{32}$$

Now we can ask the following question: What is the interpretation of this?

There is no general method that can solve all the differential equations. Each might require different methods. Now consider to find the fundamental solution for

$$x\, y' = \delta(x) \Longrightarrow \quad y' = x^{-1}\, \delta(x)$$

or more general form of

$$x^s\, y^{(n)} = \delta^{(r)}(x) \Longrightarrow y^{(n)} = x^{-s}\; \delta^{(r)}(x), \tag{33}$$

for $n, r, s = 0, 1, 2, 3, 4, \ldots,$.

The equation $y'' = x^{-1}\,\delta$ has no classical solution on $(-1, 1)$. However on using the distributional approach then we have

$$\begin{aligned} y(x) &= (f * g)(x) \\ &= (x^{-1}\,\delta(x)) * (\text{Heaviside}(x)x + C_1 x + C_2) \\ y(x) &= \int \left(\int \left(\frac{\delta(x)}{x} \right) dx + x\,C_1\,dx \right) + C_2 \end{aligned}$$

as a distributional solution since

$$y(x) = \text{Heaviside}(x)x + C_1 x + C_2$$

is a solution for elementary equation

$$y''(x) = \delta(x).$$

In general, if we have the equation as $y' = f\,g$ where $f, g \in D'$ then we have the following three interpretations to solve it, $y' * \delta_n = (f * \delta)\,g$, $y' * \delta_n = f\,(g * \delta_n)$ and $y' * \delta_n = (f * \delta_n)\,(g * \delta_n)$ by using the neutrix limit

$$\underset{n\to\infty}{\text{N}-\lim}\, y' * \delta_n = \underset{n\to\infty}{\text{N}-\lim} (f * \delta_n)\,g \tag{34}$$

$$\underset{n\to\infty}{\text{N}-\lim}\, y' * \delta_n = \underset{n\to\infty}{\text{N}-\lim}\, f\,(g * \delta_n) \tag{35}$$

$$\underset{n\to\infty}{\text{N}-\lim}\, y' * \delta_n = \underset{n\to\infty}{\text{N}-\lim} (f * \delta_n)\,(g * \delta_n). \tag{36}$$

So we can easily see that solving the differential equation is reduced to existence of the distributional products. The same procedure applies in the case of more general differential equations.

For example, suppose that we want to find the distribution g satisfying

$$P(D)g = f, \tag{37}$$

where $P(D)$ is the generalized differential operator given by

$$P(D) = a_0(x)\frac{d^s}{dx^s} + a_1(x)\frac{d^{s-1}}{dx^{s-1}} + \ldots + a_s(x)$$

Note if f is a regular distribution generated by a locally integrable function but not continuous or if it is a singular distribution then equation (37) has no meaning in the classical sense. The solution in this case is called a weak or distributional solution.

While it is possible to add distributions, it is not possible to multiply distributions easily, especially when they have coinciding singular support. Despite this, it is possible to take the derivative of a distribution, to get another distribution. Consequently, they may satisfy a linear partial differential equation, in which case the distribution is called a weak solution. For example, given any locally integrable function f it makes sense to ask for solutions u of Poisson's equation

$$\bigtriangledown^2 u = f$$

by only requiring the equation to hold in the sense of distributions, that is, both sides are the same distribution. For example, the problem for the Greens function is as follows. We scale cylindrical coordinates (r, θ, z) so that the boundary conditions are imposed on $r = 1$. The Greens function satisfies

$$\bigtriangledown^2 G = -4\pi\delta(x)$$

and the boundary condition $\dfrac{\partial G}{\partial r} = 0$ on $r = 1$. We know that $u(x;t) = H(x - ct)$ solves the wave equation. This area still need some more research we only list Friedman [**15**] for the introductory level and more recently Farassat [**7**].

We can extend the single Laplace transform of delta function to double Laplace transform as follows:

$$\begin{aligned} L_x L_t \left[\delta(t-a)\delta(x-b)\right] &= \int_0^\infty e^{-px} \int_0^\infty e^{-st}\delta(t-a)\delta(x-b)dtdx \\ &= e^{-sa-pb} \end{aligned}$$

and also double laplace transform of the partial derivative with respect to x and t as

$$\begin{aligned} L_x L_t \left[\frac{\partial}{\partial t}\delta(t-a)\frac{\partial}{\partial x}\delta(x-b)\right] &= \left[\frac{\partial^2}{\partial x \partial t}\left(e^{-st-px}\right)\right]_{t=a,x=b} \\ &= pse^{-sa-pb}. \end{aligned}$$

In general multiple Laplace transform of delta function in n dimensional given by

$$L_{t_n}[\delta(t_1 - a_1)\delta(t_2 - a_2)...\delta(t_n - a_n)] = e^{-s_1 a_1 - s_2 a_2 ... - s_n a_n}$$

where L_{t_n} means multiple Laplace transform in n dimensional. Kanwal, (2004) defined the classical derivative of a function

$$f(t) = \begin{cases} g_2(t), & t > a \\ g_1(t), & t < a \end{cases}$$

$$f(t) = g_1(t)H(a-t) + g_2(t)H(t-a)$$

where $a > 0$ and g_1, g_2 are continuously differentiable function by

$$f'(t) = g_1'(t)H(a-t) + g_2'(t)H(t-a)$$

for all $t \neq a$ see [3]. We try to extend Kanwal's result from single variable to two variables as

$$f(x,t) = \begin{cases} g_2(x,t), & x > a,\ t > b \\ g_1(x,t), & x < a,\ t < b \end{cases} \tag{38}$$

The above function can be written in the form

$$f(x,t) = g_1(x,t)H(a-x)H(b-t) + g_2(x,t)H(x-a)H(t-b) \tag{39}$$

where $a > 0$ and $b > 0$ and g_1, g_2 are continuously differentiable function the classical partial derivative respect to t, x given by

$$f_t = \frac{\partial g_1(x,t)}{\partial t}H(a-x)H(b-t) + \frac{\partial g_2(x,t)}{\partial t}H(x-a)H(t-b). \tag{40}$$

If we take the derivative with respect to x in equation (40) we obtain

$$f_{tx} = \frac{\partial^2 g_1(x,t)}{\partial t \partial x}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial t \partial x}H(x-a)H(t-b) \tag{41}$$

now if we take second partial derivative with respect to x we get

$$f_{xx} = \frac{\partial^2 g_1(x,t)}{\partial x^2}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial x^2}H(x-a)H(t-b) \tag{42}$$

similarly, we take second partial derivative with respect to t

$$f_{tt} = \frac{\partial^2 g_1(x,t)}{\partial t^2}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial t^2}H(x-a)H(t-b) \tag{43}$$

for all $x \neq a$ and $t \neq b$. The generalized partial derivative of equation (39) with respect to x follows

$$\begin{aligned} \overline{f}_x(x,t) &= \frac{\partial g_1(x,t)}{\partial x}H(a-x)H(b-t) - g_1(x,t)\delta(a-x)H(b-t) \\ &\quad + \frac{\partial g_2(x,t)}{\partial x}H(x-a)H(t-b) + g_2(x,t)\delta(x-a)H(t-b) \end{aligned} \tag{44}$$

and the generalized partial derivative of equation (44) with respect to t given by

$$\begin{aligned} \overline{f}_{xt}(x,t) &= \frac{\partial^2 g_1(x,t)}{\partial x \partial t}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial x \partial t}H(x-a)H(t-b) \\ &\quad + \frac{\partial g_2(x,t)}{\partial x}H(x-a)\delta(t-b) - \frac{\partial g_1(x,t)}{\partial x}H(a-x)\delta(b-t) \\ &\quad + \frac{\partial g_2(x,t)}{\partial t}\delta(x-a)H(t-b) - \frac{\partial g_1(x,t)}{\partial t}\delta(a-x)H(b-t) \\ &\quad + g_2(x,t)\delta(x-a)\delta(t-b) + g_1(x,t)\delta(a-x)\delta(b-t). \end{aligned} \tag{45}$$

Similar to the previous equation the generalized second partial derivative with respect to x follows

$$\begin{aligned} \overline{f}_{xx}(x,t) &= \frac{\partial^2 g_1(x,t)}{\partial x^2}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial x^2}H(x-a)H(t-b) \\ &\quad - 2\frac{\partial g_1(x,t)}{\partial x}\delta(a-x)H(b-t) + 2\frac{\partial g_2(x,t)}{\partial x}\delta(x-a)H(t-b) \\ &\quad + g_1(x,t)\frac{\partial \delta(a-x)}{\partial x}H(b-t) + g_2(x,t)\frac{\partial \delta(x-a)}{\partial x}H(t-b) \end{aligned} \tag{46}$$

and generalized second partial derivative with respect to t given by

$$\begin{aligned}\overline{f}_{tt}(x,t) &= \frac{\partial^2 g_1(x,t)}{\partial t^2}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial t^2}H(x-a)H(t-b) \\ &\quad -2\frac{\partial g_1(x,t)}{\partial t}H(a-x)\delta(b-t) + 2\frac{\partial g_2(x,t)}{\partial t}H(x-a)\delta(t-b) \\ &\quad +g_1(x,t)H(a-x)\frac{\partial\delta(b-t)}{\partial t} + g_2(x,t)H(x-a)\frac{\partial\delta(t-b)}{\partial t}. \end{aligned} \tag{47}$$

Now we use double Laplace transform for equation (42)

$$f_{xx} = \frac{\partial^2 g_1(x,t)}{\partial x^2}H(a-x)H(b-t) + \frac{\partial^2 g_2(x,t)}{\partial x^2}H(x-a)H(t-b)$$

form definition of equation (38) and we take Laplace transform with respect to x equation (42) becomes

$$L_x[f_{xx}] = H(b-t)\int_0^a e^{-px}\frac{\partial^2 g_1(x,t)}{\partial x^2}dx + H(t-b)\int_a^\infty e^{-px}\frac{\partial^2 g_2(x,t)}{\partial x^2}dx \tag{48}$$

if we integrate by part the first and second terms of equation (48), then we obtain

$$\begin{aligned}L_x[f_{xx}] &= H(b-t)\left[e^{-pa}\frac{\partial g_1(a,t)}{\partial x} - \frac{\partial g_1(0,t)}{\partial x} + pe^{-pa}g_1(a,t)\right] \\ &\quad +H(b-t)\left[-pg_1(0,t) + p^2\int_0^a e^{-px}g_1(x,t)dx\right] \\ &\quad +H(t-b)\left[-e^{-pa}\frac{\partial g_2(a,t)}{\partial x} - pe^{-pa}g_2(a,t)\right] \\ &\quad +H(t-b)p^2\int_a^\infty e^{-px}g_2(x,t)dx. \end{aligned} \tag{49}$$

By taking Laplace transform with respect to t for equation (49), then we obtain double Laplace Transform for equation (42) as

$$\begin{aligned} L_tL_x[f_{xx}] &= pe^{-pa}\left[\int_0^b e^{-st}g_1(a,t)dt - \int_b^\infty e^{-st}g_2(a,t)dt\right] \\ &\quad + e^{-pa}\left[\int_0^b e^{-st}\frac{\partial g_1(a,t)}{\partial x}dt - \int_b^\infty e^{-st}\frac{\partial g_2(a,t)}{\partial x}dt\right] \\ &\quad - \int_0^b e^{-st}\frac{\partial g_1(0,t)}{\partial x}dt - p\int_0^b e^{-st}g_1(0,t)dt + p^2F(p,s) \end{aligned} \tag{50}$$

where we assume that the integral exists. In particular if we substitute $a = 0$, $b = 0$ and $x, t > 0$ in equation (50), it is easy to see that the equation (50) gives double Laplace Transform of second order partial derivative with respect to x in classical sense as

$$L_tL_x[f_{xx}] = p^2F(p,s) - \frac{\partial g_2(0,s)}{\partial x} - pg_2(0,s) \tag{51}$$

by the same way we take double Laplace transform with respect to x, t, for equation (43) and we obtain

$$\begin{aligned} L_xL_t[f_{tt}] &= se^{-sb}\left[\int_0^a e^{-px}g_1(x,b)dx - \int_a^\infty e^{-px}g_2(x,b)dx\right] \\ &\quad + e^{-sb}\left[\int_0^a e^{-px}\frac{\partial g_1(x,b)}{\partial t}dx - \int_a^\infty e^{-px}\frac{\partial g_2(x,b)}{\partial t}dx\right] \\ &\quad - \int_0^a e^{-px}\frac{\partial g_1(x,0)}{\partial t}dx - s\int_0^a e^{-px}g_1(x,0)dt + s^2F(p,s) \end{aligned} \tag{52}$$

provided that the integrals exist. In particular if we substitute $a = 0$, $b = 0$ and $x, t > 0$ in equation (52) give double Laplace Transform of second order partial derivative with respect to t in classical sense as

$$L_x L_t[f_{tt}] = s^2 F(p,s) - s g_2(p,0) - \frac{\partial g_2(p,0)}{\partial t}$$

double Laplace transform of a mixed partial derivative of equation (41) by similar way we obtain double Laplace transform for mixed partial derivatives sa follows

$$\begin{aligned} L_t L_x\left[f_{tx}\right] &= e^{-pa} e^{-sb}\left[g_1(a,b) + g_2(a,b)\right] + g_1(0,0) - e^{-pa} g_1(a,0) \\ &\quad - e^{-sb} g_1(0,b) + s e^{-pa}\left[\int_0^b e^{-st} g_1(a,t)dt - \int_b^\infty e^{-st} g_2(a,t)dt\right] \\ &\quad + p e^{-sb}\left[\int_0^a e^{-px} g_1(x,b)dx - \int_a^\infty e^{-px} g_2(x,b)dx\right] \\ &\quad - s\int_0^b e^{-st} g_1(0,t)dt - p\int_0^a e^{-px} g_1(x,0)dx + psF(p,s) \qquad (53) \end{aligned}$$

In particular if we substitute $a = 0$,$b = 0$ and $x, t > 0$ equation (53) becomes

$$\begin{aligned} L_t L_x\left[f_{tx}\right] &= psF(p,s) + g_2(0,0) - s\int_0^\infty e^{-st} g_2(0,t)dt \\ &\quad - p\int_0^\infty e^{-px} g_2(x,0)dx \qquad (54) \end{aligned}$$

equation (54) can be written in the form

$$L_t L_x\left[f_{tx}\right] = psF(p,s) - sF(0,s) - pF(p,0) + g_2(0,0) \qquad (55)$$

equation (55) give double Laplace Transform in classical sense for mixed partial derivative with respect to x, t.

Distribution Defined by Divergent Integrals

In this section we try to extend the idea of one dimension pseudo-function to two dimensional. Now if we examine the function in the form

$$\begin{aligned} f(x,y) &= \begin{cases} x^{-n}\ y^{-n}, & x,y>0 \\ 0, & x,y<0 \end{cases} \\ &= \frac{H(x,y)}{x^n y^n} \end{aligned} \tag{56}$$

where n is positive integer and $H(x,y) = \begin{cases} 1, & x,y>0 \\ 0, & x,y<0 \end{cases}$ then we can write in the form of tensor product as $H(x,y) = H(x) \otimes H(y)$. We first consider the simple case $n = 1$ there for study the integral

$$\begin{aligned} \langle f(x,y), \phi(x,y) \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H(x) \otimes H(y)}{xy} \phi(x,y) dx dy \\ &= \int_0^{\infty} \frac{1}{y} \left[\int_0^{\infty} \frac{1}{x} \phi(x,y) dx \right] dy. \end{aligned} \tag{57}$$

Now if we consider Taylor series as

$$\phi(x,y) = \phi(0.0) + y\phi_y(0,0) + x\phi_x(0,0) + xy\psi(x,y) \tag{58}$$

where $\psi(x,y)$ defined by

$$\begin{aligned} \psi(x,y) &= \frac{1}{2} x y^{-1} \phi_{xx}(0,0) + \phi_{xy}(0,0) + \frac{1}{2} y x^{-1} \phi_{yy}(0,0) + \\ &\quad + \ldots + \frac{x^{n-1-k} y^{k-1}}{(n-k)!k!} \frac{\partial^n \phi(t,t)}{\partial x^{n-k} \partial y^k} \quad \text{for} \quad 0<t<1 \end{aligned} \tag{59}$$

is continuous function for $x,y>0$, now further consider that the supp$\phi(x,y) \subset [0,a] \times [0,b]$ and $a,b>0$. Let us going back to the integral inside bracket in equation (57) have singularity at $x = 0$, for $\varepsilon > 0$ can be write it in the form of improper integral

$$\begin{aligned}
\int_0^\infty \frac{1}{x}\phi(x,y)dx &= \lim_{\varepsilon\to 0}\int_\varepsilon^a \frac{1}{x}\phi(x,y)dx \\
&= \lim_{\varepsilon\to 0}\left[\phi(0,0)\ln a - \phi(0.0)\ln\varepsilon + y\phi_y(0,0)\ln a\right] \\
&\quad + \lim_{\varepsilon\to 0}\left[-y\phi_y(0,0)\ln\varepsilon + a\phi_x(0,0) - \varepsilon\phi_x(0,0)\right] \\
&\quad + \lim_{\varepsilon\to 0}\int_\varepsilon^a y\psi(x,y)dx. \qquad (60)
\end{aligned}$$

Then it follows

$$\begin{aligned}
\int_0^\infty \frac{1}{x}\phi(x,y)dx &= \lim_{\varepsilon\to 0}\left[\phi(0,0)\ln a - \phi(0.0)\ln\varepsilon + y\phi_y(0,0)\ln a\right] \\
&\quad + \lim_{\varepsilon\to 0}\left[-y\phi_y(0,0)\ln\varepsilon + a\phi_x(0,0) - \varepsilon\phi_x(0,0)\right] \\
&\quad + \lim_{\varepsilon\to 0}\int_\varepsilon^a y\psi(x,y)dx. \qquad (61)
\end{aligned}$$

We substitute (61) into (57) and apply the similar technique that we used in above, for $\beta > 0$, and calculating the integrals and taking the limit yields the Hadamard finite part of the divergent of equation (57) in the form of

$$pf\left(\frac{H(x)\otimes H(y)}{xy}\right) = \frac{\partial^2}{\partial x\partial y}\ln(x)\ln(y).$$

In the next we study the pseudo-function, see Kanwal (2004) in case $n = 2$ as

$$\begin{aligned}
f(x,y) &= \begin{cases} 0, & x,y<0 \\ x^{-2}y^{-2}, & x,y>0 \end{cases} \\
&= \frac{H(x,y)}{x^2y^2}. \qquad (62)
\end{aligned}$$

Let us now examine the above function

$$\begin{aligned} \langle f(x,y), \phi(x,y)\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H(x,y)}{x^2y^2}\phi(x,y)dxdy \\ &= \int_{0}^{\infty} \frac{1}{y^2}\left[\int_{0}^{\infty}\frac{1}{x^2}\phi(x,y)dx\right]dy. \end{aligned} \tag{63}$$

By similar way we obtain Hadamard finite part for two dimensional of above equation as follows

$$\begin{aligned} FP\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H(x,y)}{x^2y^2}\phi(x,y)dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{xy}(x,y)\left(\frac{H(x,y)}{xy}\right)dxdy \\ &+ \phi_{xy}(0,0) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{xy}(x,y)\left(\frac{H(x,y)}{xy}\right)dxdy \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\delta_{xy}(x,y)dxdy. \end{aligned}$$

Finally yields the required the relation

$$pf\left(\frac{H(x,y)}{x^2y^2}\right) = \frac{\partial^2}{\partial x\partial y}\left[pf\left(\frac{H(x,y)}{xy}\right)\right] + \delta_{xy}(x,y). \tag{64}$$

We can continue the above analysis to generalized equation (64) as

$$pf\left(\frac{H(x,y)}{(xy)^{m+1}}\right) = \frac{\partial^2}{\partial x\partial y}pf\left(\frac{H(x,y)}{m^2\,(xy)^m}\right) + \left(\frac{1}{m!}\right)^2 \frac{\partial^{2m}}{\partial x^m\partial y^m}\delta(x,y),\ m \geq 1. \tag{65}$$

We can generalize the distributional derivative from one dimension pseudo-function see R. F. Hoskins(1979) and Kanwal (2004) and to two dimensional cases.

Question: Now consider the equation

$$P(D)\,u = f(x,y)$$

and multiply the differential operator by a function then what will happen to the classification. Since convolution compatible with differentiation then we can ask the question what will happen the new classification problem of the

$$(Q(x,t) ** P(D)\,)\,u = F(x,t).$$

For example,

$$(Q(x,t) ** P(\text{Elliptic})\,)\,u = F(x,t)$$

when it will be elliptic and on what conditions. Similarly,

$$(Q(x,t) ** P(\text{Hyperbolic})\,)\,u = F(x,t)$$

when it will be Hyperbolic and on what conditions.

We note that in the literature there is no systematic way to generate a partial differential equation with variable coefficients from the PDE with constant coefficients, however the most of the partial differential equations with variable coefficients depend on nature of particular problems, see Kılıçman and Eltayeb [**8**] and Kılıçman [**13**].

In particular, consider the differential equation in the form of

$$\begin{aligned} y''' - y'' + 4y' - 4y &= 2\cos(2t) - \sin(2t) \\ y(0) &= 1,\ y'(0) = 4,\ y''(0) = 1. \end{aligned} \tag{66}$$

Then, by taking the Sumudu transform, we obtain:

$$Y(u) = \frac{u^3\,(2u+1)}{(4u^2+1)(1-u+4u^2-4u^3)} + \frac{(u^2+3u+1)}{(1-u+4u^2-4u^3)}. \tag{67}$$

Replacing the complex variable u by $\frac{1}{s}$, Eq. (67) turns to:

$$Y\left(\frac{1}{s}\right) = \frac{s(s+2)}{(s^2+4)(s^2+4)(s-1)} + \frac{s(s^2+3s+1)}{(s^2+4)(s-1)}. \tag{68}$$

Now in order to obtain the inverse Sumudu transform for Eq.(68), we use

$$S^{-1}(Y(s)) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} Y\left(\frac{1}{s}\right)\frac{ds}{s} = \sum \text{residues}\left[e^{st}\frac{Y(\frac{1}{s})}{s}\right].$$

Thus, the solution of Eq. (66) is given by:

$$y(t) = \frac{13}{8}\sin(2t) - \frac{1}{4}t\cos(2t) + e^t.$$

Now, if we consider to multiply the left hand side of Eq. (66) by the non constant coefficient t^2, then Eq. (66) becomes

$$\begin{aligned} t^2 * \left(y''' - y'' + 4y' - 4y\right) &= 2\cos(2t) - \sin(2t) \\ y(0) &= 1,\ y'(0) = 4,\ y''(0) = 1. \end{aligned} \tag{69}$$

By applying a similar method, we obtain the solution of Eq. (69) in the form:

$$y_1(t) = \cos(2t) - t\sin(2t) + \frac{3}{2}\sin(2t).$$

Now in order to see the effect of the convolutions we can see the difference as $||y - y_1||$ on $[0, 1]$, see the detail [**42**].

Question: How to generate a PDE with variable coefficients from the PDE with constants coefficients. For example, if we consider the wave equation in the following example

$$\begin{aligned} u_{tt} - u_{xx} &= G(x,t) \qquad\qquad (x,t) \in \mathbb{R}^2_+ \\ u(x,0) &= f_1(x),\ u_t(x,0) = g_1(x) \\ u(0,t) &= f_2(t),\ u_x(0,t) = g_2(t). \end{aligned} \tag{70}$$

Now, if we consider to multiply the left hand side equation of the above equation by non-constant coefficient $Q(x,t)$ by using the double convolution with respect to x and t respectively, then the equation becomes

$$Q(x,t) ** (u_{tt} - u_{xx}) = G(x,t) \qquad (t,x) \in \mathbb{R}^2_+ \tag{71}$$

$$\begin{aligned} u(x,0) &= f_1(x), \quad u_t(x,0) = g_1(x) \\ u(0,t) &= f_2(t), \quad u_x(0,t) = g_2(t). \end{aligned} \tag{72}$$

Thus the relationship between the solutions partial differential equations with constant coefficients and non constant coefficients was studied in [**13**].

Application to Probability Theory

In order to present the application of generalized function to the theory of probability and Random processes we assume the basic concepts are well known for the probability space. For a random variable X we define its probability distribution function $F(x)$ as

$$F(x) = P\{X < x\} = P\{X^{-1}(-\infty, x)\},\ x \in \mathbb{R}$$

and the function F has the following properties

- F is monotone
- F is continuous from left and
- $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

Now if the function $F(x)$ being a locally integrable function and defines a generalized function

$$\langle F(x), \phi(x) \rangle = \int_{-\infty}^{\infty} F(x)\phi(x)dx$$

where ϕ is infinitely differentiable function. Accordingly,

$$\begin{aligned}
\langle F'(x), \phi(x) \rangle &= -\langle F(x), \phi'(x) \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x)dx \\
&= -|\phi(x)F(x)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(x)\ d(F(x)) \\
&= \langle f(x), \phi(x) \rangle
\end{aligned}$$

where $f(x) = \dfrac{dF}{dx}$ is called the probability density function. The density function $f(x)$ has the following properties:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$.
- $f(-\infty) = 0,\ f(\infty) = 1$ we find that

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Tossing a Coin Let us consider the tossing of a coin. We assign the value $x = 0$ if we obtain heads and the value $x = 1$ if we get tails. In order to evaluate the probability distribution $F(x)$ we have the following:

(i) If head $x = 0$, if tails we have $x = 1$. Then the probability we have only two cases, either $\{x = 0\}$ or $\{x = 1\}$ so that

$$x < 0 \quad \text{yields} \quad F(x) = 0, \quad x \leq 0.$$

(ii) $0 < x \leq 1 \Rightarrow F(x) = \frac{1}{2}$ and

(iii) If $x > 1$, $F(x) = 1$ thus we find

$$\begin{aligned} F(x) &= \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{2} & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \\ &= \frac{1}{2}\ [H(x) + H(x-1)] \end{aligned}$$

where H is the Heaviside function. Then the probability density function is

$$f(x) = \frac{1}{2}\ [\delta(x) + \delta(x-1)]\,.$$

In the case of Random variable X takes the values $a_1, a_2, a_3\ , \ldots, a_n$ with the probabilities $p_1, p_2, p_3, \ldots p_n$ respective such that

$$\sum_{k=0}^{n} p_k = 1$$

The generalized function

$$f(x) = \sum_{k=1}^{n} p_k \delta(x - a_k)$$

is the probability density function.

- If X has the Binomial distribution then

$$f(x) = \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} \delta(x-k), 0 \leq p \leq 1,\ q = 1 - p$$

then the Probability function

$$F(x) = \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} H(x-k).$$

- If X has the Poisson distribution then the

$$f(x) = e^{-\lambda} \sum_{k=1}^{n} \binom{\lambda^k}{k!} \delta(x-k), \lambda > 0$$

then the

$$F(x) = e^{-\lambda} \sum_{k=1}^{\infty} \binom{\lambda^k}{k!} H(x-k).$$

The Characterization of Random Variables

- **Expectation Value of** X

$$E(x) = \int_G X(u) dP(u) = \int_{-\infty}^{\infty} x dF(x) = \langle x, f \rangle.$$

- **The variance of** X

$$\begin{aligned} D(x) &= \int_G (X(u) - E(x))^2 \, dP(u) \\ &= \int_{-\infty}^{\infty} (x - E(x))^2 \, dF(x) \\ &= \langle (x - E(x))^2, f(x) \rangle. \end{aligned}$$

- **The m-moment of** X

$$\begin{aligned} E(x^m) &= \int_G (X(u))^m \, dP(u) \\ &= \int_{-\infty}^{\infty} x^m \, dF(u) = \langle x^m, f(x) \rangle. \end{aligned}$$

Since the m-moment function of $f(x)$ is defined by

$$\langle f(x), x^m \rangle = \int_{-\infty}^{\infty} f(x) x^m \, dx.$$

Then we consider a test function $\phi(x)$ and its Taylor series is

$$\phi(x) = \sum_{m=0}^{\infty} \phi^{(m)}(0) \frac{x^m}{m!}$$

then it follows by putting that $\phi^{(m)}(0) = (-1)^m \langle \delta^{(m)}(x), \phi(x) \rangle$ then easily we see that

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m E(x^m)}{m!} \delta^{(m)}(x).$$

Now if we apply this to the asymptotic analysis we have

$$f(\lambda x) \sim \sum_{m=0}^{\infty} \frac{(-1)^m E(x^m)}{m! \lambda^{m+1}} \delta^{(m)}(x).$$

If $f(x) = e^{-x} H(x)$ then the moments are

$$E(x^m) = \int_0^{\infty} e^{-x} x^m = \Gamma(m+1) = m!$$

the moment expansion is

$$H(x)e^{-x} = \sum_{m=0}^{\infty} (-1)^m \delta^{(m)}(x)$$

then the asymptotic expansion is

$$H(x)e^{-\lambda x} \sim \sum_{m=0}^{\infty} \frac{(-1)^m \delta^{(m)}(x)}{\lambda^{n+1}}, \quad \lambda \to \infty$$

now if we set $\lambda = \frac{1}{\epsilon}$ then we have

$$H(x)e^{-\frac{x}{\epsilon}} = \sum_{m=0}^{\infty} (-1)^m \delta^{(m)}(x) \epsilon^{m+1}, \quad \epsilon \to 0$$

in fact this is the case in the boundary layer problem.

The Characteristic function of a Random variable

Since the probability distribution $f(x)$ is a generalized function we can find its Fourier Transform. Thus

$$E(e^{iux}) = \int_{-\infty}^{\infty} e^{iux} f(x) \, dx = F(f) = \chi(u)$$

and the

$$f(x) = F^{-1}\left(\chi(u)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\chi(u)\ du.$$

Now let us take $\chi(u) = e^{\lambda iu}$ then

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\chi(u)\ du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}e^{\lambda iu}\ du \\ &= \int_{-\infty}^{\infty} e^{-iu(x-\lambda)}\ du = \delta(x-\lambda). \end{aligned}$$

Application to Economics Most models of the dynamical behavior of an economic system usually assume that the variables of the system are the function of the time. In fact this not general but reasonable assumption, such as the price of the certain commodity or the prevailing interest rate.

The size of the capital stock can be observed at almost all times, but it might suffer jumps when additions are made in a very short period. A Dirac delta function placed at the instant of the jump is the best description of the investment. The basic dynamic model for the investment decision of the firm postulates that the investment schedule $I(t)$ for $t \geq t_0$ is chosen at the time $t = t_0$ in such a way as to maximize the present value of the future stream of profits.

$$\phi = \phi(I) = \int_{t_0}^{\infty} V(t-t_0)\ \{R(t, K(t)) - r(I(t)))\}dt$$

where

- $K(t)$ is the capital stock at the time t, then
  $K'(t) = \dfrac{dK}{dt} = I(t)$

- $R(t, K)$ is the expected quasi-rent to be obtained from capital stock of size K at time t,
- $r(I)$ is the cost of adjustment; and
- $V(t)$ is the discount factor.

Then the cost of the adjustment is a non linear functional of the investment $I(t)$ given by

$$C(V, I) = \int_{t_0}^{\infty} V(t - t_0) r(I(t)) dt$$

for a given discount factor $V(t)$.

The Radon Transform and Tomography

Recently, impact of computer technology has informed us that there is a great need of further developments of distribution theory in Applied Sciences.

It turns out that all the conclusions about distributions in D can be extended to the distributions on multidimensional spaces. One encounter the two-and three dimensional impulse symbols $\delta(x, y)$ and $\delta(x, y, z)$ as a natural expansion of δ. For example we can interpret

- the $\delta(x, y)$ describes the pressure distribution over the (x, y)- plane when a concentrated unit force is applied at the origin.
- the $\delta(x, y, z)$ describes the charge density in a volume containing a unit charge at the point $(0, 0, 0)$.

Then we have

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dx\ dy &= 1\\
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y,z)dx\ dy\ dz &= 1\\
\delta(x,y,z) &= \delta(x,y)\ \delta(z) = \delta(x)\ \delta(y)\ \delta(z).
\end{aligned}
$$

The basic problem of tomography is given a set of 1-D projections and the angles at which these projections were taken, then the problem is how to reconstruct the 2-D image from which these projections were taken.

In recent years the Radon transform have received much attention which is able to transform two dimensional images with lines into a domain of possible line parameters, where each line in the image will give a peak positioned at the corresponding line parameters.

This have lead to many line detection applications within image processing, computer vision, and seismic. There are several definitions of the Radon transform in the literature, but they are related, and a very popular form expresses lines in the form $R = x\cos(\theta) + y\sin(\theta)$, where θ is the angle and R the smallest distance to the origin of the coordinate system.

The Radon transform for a set of parameters (R, θ) is the line integral through the image $f(x, y)$, where the line is positioned corresponding to the value of (R, θ). The delta δ is the Dirac delta function which is infinite for argument 0 and zero for all other arguments (it integrates to one).

If the density distribution is $f(x, y)$ which is not symmetrical but depends on two coordinates, the scans may still be taken but they will depend on the direction of the scanning θ, Calling the abscissa for each scan R we define the Radon Transform

$$Q(\theta, R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(R - x\cos\theta - y\sin\theta)dx\ dy.$$

The factor $\delta(R - x\cos\theta - y\sin\theta)$ is zero everywhere except where its argument is zero, which is along the straight line $x\cos\theta + y\sin\theta = R$. In fact this is equivalent with

$$Q(\theta, R) = \int_{-\infty}^{\infty} f(R\cos\theta - t\sin\theta, R\sin\theta + t\cos\theta)\ dt.$$

The collection of these $Q(\theta, R)$ at all θ is called the Radon Transform of image $f(x, y)$.

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BIOGRAHPY

Adem Kılıçman was born in Hassa, Hatay, Turkey on 3rd February 1966. He obtained his early education in Tiyek Koyu Ilkokulu He then completed his secondary school at Hassa Ortaokulu and continued his upper secondary in Hassa Lisesi.

Kılıçman graduated with Bachelor of Science (Hons.) in Mathematics at Hacettepe University, Ankara, Turkey in 1989 and stared to work in Ondokuz Mayıs University, in the Black Sea region. He then continued his Master in the Hacettepe University and graduated in 1991. Kılıçman was sent to do his PhD in University Leicester and obtained his PhD in 1995 from University of Leicester, England.

Currently, Adem Kılıçman is a Professor in the Department of Mathematics at University Putra Malaysia. His research interest includes Functional Analysis, Topology as well as Differential Equations. He has published several research papers in international journals and actively involved in several activities in Department of Mathematics, Institute of Mathematical Research(INSPEM) and Institute of Advanced Technology(ITMA). Adem Kılıçman is also the authors of three books Applied Mathematics for Business and Economics, An Introduction to Real Analysis and Distributions Theory and Neutrix Calculus.

Adem Kılıçman is a Member of Several Associations, for example, he is a Life Member of Malaysian Mathematical Sciences Society(PERSAMA), Member of American Mathematical Society(AMS), Member of Society for

Industrial Applications of Mathematics(SIAM), member of New York Academy of Sciences. He is an Editor-In-Chief in two international Journals and also member on the Editorial Board of International Journals and Bulletins which some in the ISI listing and Kılıçman is a Co-Editors in some proceedings of International Conferences as well as National Conferences. His research contributions are recognized and his name is listed in Who's Who in Science and Engineering (MARQUIS)(USA), The Contemporary Who's Who(USA), Great Minds of The 21st Century(UK), 2000 Outstanding Intellectuals of the 21st Century by International Biographical Center, Cambridge, UK.

Further, some of Kılıçman's research were listed in the Top 8/25 Hottest articles, such as in Topology and its Applications(Elsevier), April - June 2007, and in the Top 5/25 Hottest articles, Applied Mathematics Letters (Elsevier), October - December 2008. Apart from being a reviewer in the several international Journals, Kılıçman is also a Reviewer for American Mathematical Society, Reviewer for Computing Reviews(USA) as well as Reviewer for Zentralblatt Mathematik(Germany) since 2002.

ACKNOWLEDGEMENT

And with Him all things have their end.

That is, human beings are sent to this world, which is the realm of trial and examination, with the important duties of trading and acting as officials. After they have concluded their trading, accomplished their duties, and completed their service, they will return and meet once more with their Generous Master and Glorious Creator Who sent them forth in the first place. Leaving this transient realm, they will be honoured and elevated to the presence of grandeur in the realm of permanence. That is to say, being delivered from the turbulence of causes and from the obscure veils of intermediaries, they will meet with their Merciful Sustainer without veil at the seat of His eternal majesty. Everyone will find his Creator, True Object of Worship, Sustainer, Lord, and Owner and will know Him directly. Thus, this phrase proclaims the following joyful news, which is greater than all the rest:

"O mankind! Do you know where you are going and to where you are being impelled? As is stated at the end of the Thirty-Second Word, a thousand years of happy life in this world cannot be compared to one hour of life in Paradise. And a thousand years of life in Paradise cannot be compared to one hour's vision of the sheer loveliness of the Beauteous One of Glory. And you are going to the sphere of His mercy, and to His presence.

That is, everything will return to the realm of permanence from the transient realm, and will go to the seat of post-eternal sovereignty of the Sempiternal Ever-Enduring One. They will go from the multiplicity of causes

to the sphere of power of the All-Glorious One of Unity, and will be transferred from this world to the Hereafter. Your place of recourse is His Court, therefore, and your place of refuge, His mercy.

Since this world is transitory, and since life is short, and since the truly essential duties are many, and since eternal life will be gained here, and since the world is not without an owner, and since this guest-house of the world has a most Wise and Generous director, and since neither good nor bad will remain without recompense, and since according to the verse,

> On no soul does God place a burden greater than it can bear
>
> Qur'an 2: 286

there is no obligation that cannot be borne, and since a safe way is preferable to a harmful way, and since worldly friends and ranks last only till the door of the grave, then surely the most fortunate is he who does not forget the hereafter for this world, and does not sacrifice the hereafter for this world, and does not destroy the life of the hereafter for worldly life, and does not waste his life on trivial things, but considers himself to be a guest and acts in accordance with the commands of the guest-house's Owner, then opens the door of the grave in confidence and enters upon eternal happiness...

Thus first of all I am very grateful to Allah (s.w.t) that provided the life with faith and all praise to be upon Muhammad(s.a.w) who is a guidance for all the universe.

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O God! Grant blessings and peace to our master Muhammad(s.a.w) to the number of the particles of the universe, and to all his Family and Companions. And all praise be to God, the Sustainer of All the Worlds. May Allah grant all of us happiness in this world and in the hereafter(Amen).

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