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The Generalized Principle of Localization for the N-Fold Fourier Integrals

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ABSTRACT

In this paper we investigate the almost everywhere convergence properties of the Riesz means of the eigenfunction expansions (multiple Fourier integrals), associated with Laplace operator. The generalized principle of localization for the Riesz means of the order $s=(N-1)(1/p-1/2),\ \ 1\le p\le 2,$ is proved.

Keywords: Riesz means, Multiple Fourier integral, the almost everywhere convergence, generalized localization principle.

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INTRODUCTION

We consider the N-dimensional Euclidean space R^N, whose elements are denoted by $x=(x_1,x_2,...,x_N), y=(y_1,y_2,...,y_N)$ and we put $(x,y)=x_1y_1+x_2y_2+...+x_Ny_N,\quad |x|^2=(x,x).$

Let us consider the Laplace operator Δ:

$$\Delta u(x)=\frac{\partial^2 u(x)}{\partial x_1^2}+\frac{\partial^2 u(x)}{\partial x_2^2}+...+\frac{\partial^2 u(x)}{\partial x_N^2},\qquad u(x)\in C_0^\infty(R^N),$$

where $C_0^\infty(R^N)$ denotes the space of functions with compact support that are infinitely differentiable in R^N. The Laplace operator with the domain $D_\Delta=C_0^\infty(R^N)$ is symmetric and nonnegative:

1) $(-\Delta u,v)=(u,-\Delta v),\forall u,v\in D_\Delta$;
2) $(-\Delta u,u)=(\nabla u,\nabla u)=\|\nabla u\|^2_{L_2(R^N)}\ge 0,\forall u\in D_\Delta.$

A well-known theorem of Fredrichs (Alimov, Il'in and Nikishin, (1976)) asserts that every symmetric semi-bounded operator has at least one self-adjoint extension with the same lower bound. Let $\hat{A}$ be a self-adjoint extension of the Laplace operator in $L_2(R^N)$. By von Neumann's spectral theorem, the operator $\hat{A}$ has a decomposition of unity $\{E_\lambda\}$, and can be represented in the following form

$$\hat{A} = \int_0^\infty \lambda dE_\lambda.$$

The projections E_λ increase monotonically, are continuous on the left, and tend strongly to the unit operator, that is,

$$\lim_{\lambda\to\infty} || E_\lambda f - f ||_{L_2(R^N)} = 0.$$

The spectral decomposition of any arbitrary element $f \in L_2(R^N)$ is determined by the formula

$$E_\lambda f(x) = (2\pi)^{-N/2} \int_{|\xi|^2<\lambda} \hat{f}(\xi) e^{i(x,\xi)} d\xi,$$

where $\hat{f}$ is the Fourier transform of the function f:

$$\hat{f}(\xi) = (2\pi)^{-N/2} \int_{R^N} f(x) e^{-i(x,\xi)} dx.$$

The Riesz means of order s with $\Re(s) \geq 0,$ is defined by

$$E_\lambda^s f(x) = (2\pi)^{-N/2} \int_{|\xi|^2<\lambda} \left(1 - \frac{|\xi|^2}{\lambda}\right)^s \hat{f}(\xi) e^{i(x,\xi)} d\xi,$$

which can be written as follows

$$E_\lambda^s f(x) = \int_{R^N} \Theta^s(x,y,\lambda) f(y) dy \quad , \tag{1}$$

where

$$\Theta^s(x,y,\lambda) = (2\pi)^{-N} \int_{|\xi|^2<\lambda} \left(1 - \frac{|\xi|^2}{\lambda}\right)^s e^{i(x,\xi)} d\xi.$$

For $s=0$ this kernel is called the spectral function of the Laplace operator for the entire space R^N. The Riesz means of the spectral function can be computed explicitly by introducing polar coordinates, and take the form

$$\Theta^s(x,y,\lambda)=(2\pi)^{-N}2^s\Gamma(s+1)\lambda^N\cdot\frac{J_{\frac{N}{2}+s}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{N}{2}+s}}.$$

The classical Riemann localization theorem (Alimov, Ashurov and Pulatov, (1992)) states that the convergence or divergence of a one dimensional Fourier series at a given point depends only on the behavior of the function $f\in L_1$ in an arbitrary small neighborhood of this point. If we shift to the multidimensional case, the localization principle does not hold in the class $L_1(T^N)$ for spherical partial sums, where T^N is a N-dimensional cube. The principle of generalized localization was introduced by Il'in (Il'in, 1968)), where instead of the uniform convergence of the spectral expansions, it is required the almost everywhere convergence.

We wish to investigate the general localization principle for multiple Fourier integrals, which is related to the Laplace operator in R^N. In this work we establish the principle of generalized localization for Riesz means of order $s=(N-1)(1/p-1/2)$. As starting point for such problems, is the conjuncture: the sufficient conditions for localization can be weakened if we consider the general localization problem.

We first give the definition of the principle of generalized localization. Let Ω be any open domain in R^N.

Definition 1.1. *We say that the principle of generalized localization holds for* $E_\lambda^s f(x)$ *in* L_p, $p\geq 1$, *if for all functions* $f\in L_p(R^N)$

$$\lim_{\lambda\to\infty}E_\lambda^s f(x)\to 0$$

almost everywhere on $R^N\setminus supp(f)$.

We proceed to the formulation of the fundamental results of the paper.

Theorem 1.2. *Let* $f \in L_p(R^N), 1 \le p \le 2,$ *then the Riesz means at index* $s = (N-1)\left(\frac{1}{p} - \frac{1}{2}\right)$ *of the Fourier integrals of the function* f:

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) \to 0$$

almost everywhere on $R^N \setminus supp(f)$.

In other words, the theorem states that for the Riesz means of multiple Fourier Integrals of the order $s = (N-1)(1/p - 1/2)$ the generalized localization principle holds in classes $L_p(R^N)$, $1 \le p \le 2$. We note that, in case of $s = 0, p = 2$ the statement of Theorem 1.2 has been proved in Bastys, (1973). For the partial integrals of Fourier integrals in the classes $L_p(R^N), 2 \le p < 2N/(N-1)$, the principle of generalized localization is proved in Carbery and Soria, (1988). The summability of the Fourier-Laplace series by Cesaro means is investigated in the following works (Anvarjon, (2009), Anvarjon, (2006), Bonami and Clerc, (1973), Bastys, (1983) and Rakhimov, (2004)). The principle of generalized localization in L_1 for the Fourier-Laplace series is established by Rakhimov (Rakhimov, (2004)). In the work Anvarjon, (2009), following the ideas of interpolation of an analytic family of linear operators, it is proved that the principle of generalized localization is valid for Riesz means of the Fourier-Laplace series, on the critical line. We note that the work Anvarjon, (2009) is first result on critical line, all previous works were devoted to the investigation on the above of the critical line. In current work, using the L_2 estimates in Carbery and Soria, (1988) for partial integrals of the Fourier integral and applying again interpolation theorem on analytic family of linear operators (Yoram, (1969)) we have established the problem of generalized localization on critical line for spectral decomposition of the Laplace operator in entire R^N. For more information about the multiple Fourier integrals we refer to Alimov, Il'in and Nikishin, (1976) and Alimov, Ashurov and Pulatov, (1992).

In the study of questions of almost everywhere convergence it is convenient to introduce the maximal operator

$$E_*^s f(x) = \sup_{\lambda>0} | E_\lambda^s f(x) |. \tag{2}$$

The investigation of the almost everywhere convergence of the Riesz means $E_\lambda^s f$ is based on estimates of $E_*^s f$ in L_1 and L_2, and on a subsequent application of interpolation theorem.

Let $|A|$ denote the Lebesgue measure of the set A. The following statement establishes necessary estimate for $E_*^s f$ in L_1.

Theorem 1.3 *Let* $f \in L_1(R^N)$ *and* $f(x) = 0, x \in \Omega \subset R^N$, *then for the Riesz means of the order* s, $\Re(s) = \dfrac{N-1}{2}$ *and for any* $\alpha > 0$,

$$|\Omega \bigcap \{x : E_*^s f(x) > \alpha\}| \leq C \frac{\|f\|_{L_1(R^N)}}{\alpha}. \tag{3}$$

The statement of the Theorem 1.3 we use when $p = 1$. In case of $p = 2$ for partial integrals we have

Theorem 1.4 *Let* $f \in L_2(R^N)$ *and* $f(x) = 0, x \in \Omega \subset R^N$, *then for partial integrals of the Fourier integrals of the function* f,

$$|K \bigcap \{x : E_*^s f(x) > \alpha\}| \leq C \left(\frac{\|f\|_{L_2(R^N)}}{\alpha} \right)^2, \tag{4}$$

where K *is any compact subset of* Ω.

PROOF OF THEOREM 1.3

This section is devoted to the proof of the estimation in L_1. To establish the estimation for Riesz means in L_1 we use asymptotical estimations of the spectral function and properties of the maximal functions of the Hardy-Littlewood. Let $f \in L_1(R^N)$. The following function

$$m_f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

is called the maximal function of the Hardy-littlewood of the f, where

$$B(x,r) = \{y \in R^N : |x - y| < r\}.$$

Lemma 2.1 *Let* $f \in L_1(R^N)$ *and* $f(x)=0, x\in\Omega\subset R^N$. *Then for any* $x\in\Omega$,

$$E_*^s f(x) \le C\cdot m_f(x), \quad \Re(s)\ge \frac{N-1}{2}. \tag{5}$$

Proof. Due to regularity of Riesz means it is sufficient to prove the statement of the Lemma for the case $\Re(s)=\frac{N-1}{2}$. Assume that the function $f\in L_1(R^N)$ with the compact support vanishes in the domain $\Omega\subset R^N$. For an arbitrary compact set $K\subset\Omega$, let denote $\delta = dist(K,\partial\Omega)$. Then for any $x\in K$, and any $y\in\Omega$, we have $|x-y|>\delta$. Thus we obtain

$$E_\lambda^s f(x) = C\lambda^{\frac{N}{2}-s}\int\limits_{R^N\backslash\Omega}\frac{J_{\frac{N}{2}+s}(\lambda|x-y|)}{|x-y|^{\frac{N}{2}+s}}f(y)dy.$$

To estimate the absolute value we have

$$|E_\lambda^s f(x)| \le C\lambda^{\frac{N}{2}-\Re(s)}\int\limits_{R^N\backslash\Omega}|J_{\frac{N}{2}+s}(\lambda|x-y|)|x-y|^{-\frac{N}{2}-\Re(s)}f(y)|dy \le$$

$$\le C\lambda^{\frac{N-1}{2}-\Re(s)}\int\limits_\delta^R r^{-\frac{N+1}{2}-\Re(s)}dF_r f(x) = C\int\limits_\delta^R r^{-N}dF_r f(x).$$

In the last integral integrating by parts gives

$$E_*^s f(x) \le Cr^{-N}F_r f(x) + CN\int\limits_\delta^R r^{-N-1}F_r f(x)dr \le Cm_f(x).$$

Lemma 2.1 is proved.

Lemma 2.2 *Let* f *be a given function defined on* R^N

1) *If* $f\in L_p(R^N), 1\le p\le\infty$, *then the function* m_f *is finite almost everywhere.*
2) *If* $f\in L_1(R^N)$, *then for every* $\alpha>0$

$$|\{x: m_f(x)>\alpha\}| \le \frac{C}{\alpha}\int\limits_{R^N}|f(x)|dx.$$

This lemma is proved in [16]. The proof of Theorem 1.3 can be established as follows: for s with $\Re(s)=\frac{N-1}{2}$, and for all $x\in\Omega$, using the relation

$\{x : E_*^s f(x) > \alpha\} \subset \{x : m_f(x) > \alpha\}$ we obtain:

$|\{x : E_*^s f(x) > \alpha\}| \leq C |\{x : m_f(x) > \alpha\}| \leq \frac{C}{\alpha} \| f \|_{L_1(R^N)}$.

Theorem 1.3 is proved.

PROOF OF THEOREM 1.4

Let $f \in L_2(R^N)$, and $supp(f) \subset \{x \in R^N : |x| \geq 3\}$. We prove that for every $r < 3$ the following inequality holds:

$$\int_{|x|\leq r} \sup_{\lambda>0} |E_\lambda^{i\tau} f(x)|^2 \, dx \leq C_r \int_{|x|\geq 3} |f(x)|^2 \, dx, \qquad (6)$$

with $\tau : -\infty < \tau < \infty$.

Let $\chi_b(t)$ be the characteristic function of the segment $[-b,b]$ and $\phi(t)$ be a smooth function defined for $t \geq 0$, such that $\chi_{(3-r)/3}(t) \leq \phi(t) \leq \chi_{2(3-r)/3}(t)$. Then we define $\psi(x) = \phi(|x|) - \phi(2|x|)$ and $\psi_j(x) = \psi(\frac{x}{2^j})$ for $j = 1, 2, \ldots$. We obtain $\phi(|x|) + \sum_{j\geq 1} \psi_j(x) \equiv 1$.

Let $\Theta_\lambda^{\tau,j} = \Theta_\lambda^{i\tau} \psi_j$. If $supp(f) \subset \{|x| \geq 3\}$, then for all $x : |x| \leq r$ we have,

$$E_\lambda^{i\tau} f(x) = \Theta_\lambda^{i\tau} * f = \sum_{j\geq 1} \Theta_\lambda^{\tau,j} * f(x),$$

because $(\Theta_\lambda^{i\tau} \phi(|\cdot|) * f)(x) = 0$ if $|x| < r, r < 3$.

We investigate the Fourier transform of the "spectral function" $\Theta_t^{\tau,j}(x)$. Let $m_t^{\tau,j}(\xi) = \hat{(\Theta_t^{\tau,j})}(\xi)$. When $\tau = 0$ we use notation m_t^j, i.e. $m_t^{0,j}(\xi) = m_t^j(\xi)$. For $m_t^j(\xi)$ we have,

Lemma 3.1 *For any* $t > 0, \xi \in R^N, j \geq 1$

$$|m_t^j(\xi)| \leq \int_{||\xi|-t|<|y|2^{-j}} |\hat{\psi}(y)| \, dy.$$

This lemma is proved in Carbery and Soria, (1988).

Lemma 3.2 *For any positive integer* n *there exists a constant* C *such that for any* $t>0, \xi \in R^N$, $j=1,2,...$ *we have the following estimate*
$$|m_t^{\alpha,j}(\xi)| \le \frac{C}{(1+||\xi|-t|2^j)^n}.$$

Proof. We consider the following function:
$$m_t^{\alpha,j}(\xi)=\int_0^{\lambda}(1-\frac{t}{\lambda})^{i\alpha}dm_t^j(\xi).$$
The integral we divide integral into two parts as follows
$$m_t^{\alpha,j}(\xi)=\int_0^{\lambda/2}(1-\frac{t}{\lambda})^{i\alpha}dm_t^j(\xi)+\int_{\lambda/2}^{\lambda}(1-\frac{t}{\lambda})^{i\alpha}dm_t^j(\xi)=I_1(\lambda)+I_2(\lambda).$$
Estimation of $I_2(\lambda)$. Using the following formula
$$\frac{d}{dt}m_t^j(\xi)=\frac{N}{t}m_t^j(\xi)+t^N\int_{|y|<2^j}\nabla\hat{\psi}(2^j\xi+yt)ydy$$
we obtain :
$$|\frac{d}{dt}m_t^j(\xi)| \le C2^j\int_{||\xi|-t|2^j<|y|}(|\hat{\psi}(y)|+|\nabla\hat{\psi}(y)|(1+|y|))dy.$$
Then for $I_2(\lambda)$ we have
$$|I_2(\lambda)| \le \frac{c2^j}{\lambda}\int_{\lambda/2}^{\lambda}\int_{||\xi|-t|2^j<|y|}(|\hat{\psi}(y)|+|\nabla\hat{\psi}(y)|(1+|y|))dydt \le$$
$$\le \frac{c2^j}{\lambda}\int_{\lambda/2}^{\lambda}\int_{||\xi|-t|2^j}^{\infty}\frac{r^{N-1}}{(1+r)^l}drdt.$$

By changing the order of integration, and taking into account the relation $\{(t,r):\lambda/2\le t\le\lambda, ||\xi|-t|2^j\le r\le\infty\}\subset\{(t,r):||\xi|-t|2^j\le r\le\infty, |\xi|-r\le t\le|\xi|+r\}$
$$|I_2(\lambda)| \le \frac{c2^j}{\lambda}\int_{||\xi|-t|2^j}^{\infty}\frac{r^{N-1}}{(1+r)^l}\int_{|\xi|-r}^{|\xi|+r}dtdr=\frac{c2^j}{\lambda}\int_{||\xi|-\lambda|2^j}^{\infty}\frac{r^N}{(1+r)^l}dr \le$$
$$\le \frac{c2^j}{\lambda}\frac{1}{(1+||\xi|-\lambda|2^j)^n}, n=l-N-1.$$

Now we consider $I_1(\lambda)$. Using the integration by parts we obtain

$$I_1(\lambda)=\int_0^{\lambda/2}(1-\frac{t}{\lambda})^{i\alpha}dm_t^j(\xi)=(1-\frac{t}{\lambda})^{i\alpha}m_t^j(\xi)|_0^{\lambda/2}+\frac{i\alpha}{\lambda}\int_0^{\lambda/2}(1-\frac{t}{\lambda})^{i\alpha}(1-\frac{t}{\lambda})^{-1}m_t^j(\xi)dt.$$

Using the inequality

$$|m_t^j(\xi)|\le\int_{||\xi|-t|\le|y|}|\hat{\psi}(y)|dy,$$

we get

$$|I_1(\lambda)|\le|m_{\lambda/2}^j(\xi)|+\frac{2|\alpha|}{\lambda}\int_0^{\lambda/2}\int_{||\xi|-t|\le|y|}|\hat{\psi}(y)|dydt.$$

Similarly as in previous case we have

$$\int_0^{\lambda/2}\int_{||\xi|-t|\le|y|}|\hat{\psi}(y)|dydt\le C\int_0^{\lambda/2}\int_{||\xi|-t|}^{\infty}\frac{r^{N-1}}{(1+r)^l}drdt\le$$

$$C\frac{1}{(1+\varepsilon_0||\xi|-\lambda|2^j)^n}.$$

Consequently for $I_1(\lambda)$ we have

$$|I_1(\lambda)|\le\frac{C}{(1+||\xi|-\lambda|2^j)^n}.$$

Finally we obtain

$$|m_\lambda^{\alpha,j}(\xi)|\le\frac{C}{(1+||\xi|-\lambda|2^j)^n}.$$

The lemma is proved.

For the derivative of the function $m_t^{\alpha,j}(\xi)$ we obtain

Lemma 3.3 *For any positive integer* n *there exists a constant* C *such that for any* $t>1$ *and* $\xi\in R^N, j=1,2,\ldots$,

$$|\frac{d}{dt}m_t^{\alpha,j}(\xi)|\le\frac{C2^j}{(1+\varepsilon_0||\xi|-t|2^j)^n}.$$

We proceed to the proof of Theorem 1.4.

For the maximal operator we have

$$E_*^{i\tau} f(x) \le \sup_{\lambda>0} |\Theta_\lambda^{i\tau} * f| \le \sum_{j\ge1} \sup_{\lambda>0} |\Theta_\lambda^{\tau,j} * f(x)|.$$

We see that it suffices to prove the estimate

$$\sup_{\lambda>0} |\Theta_\lambda^{\tau,j} * f(x)| \le C2^{-j} \int |f(x)|^2 \, dx.$$

To prove this inequality we follow the ideas of Alimov, Ashurov and Pulatov, (1992):

$$|\Theta_\lambda^{\tau,j} * f(x)|^2 = 2\int_0^\lambda |\Theta_t^{\tau,j} * f(x) \frac{d}{dt} \Theta_t^{\tau,j} * f(x)| \, dt.$$

By the Cauchy-Schwarz inequality, we obtain

$$\int_{R^N} \sup_{\lambda>0} |\Theta_\lambda^{\tau,j} * f(x)|^2 \, dx \le 2\left(\int_{R^N}\int_0^\infty |\Theta_t^{\tau,j} * f(x)|^2 \, dtdx \right)^{1/2} \left(\int_{R^N}\int_0^\infty |\frac{d}{dt}\Theta_t^{\tau,j} * f(x)|^2 \, dtdx \right)^{1/2}.$$

We show that

$$\left(\int_{R^N}\int_0^\infty |\Theta_t^{\tau,j} * f(x)|^2 \, dtdx \right)^{1/2} \le C2^{-j} \| f \|_{L_2},$$

and

$$\left(\int_{R^N}\int_0^\infty |\frac{d}{dt}\Theta_t^{\tau,j} * f(x)|^2 \, dtdx \right)^{1/2} \le C \| f \|_{L_2}.$$

By duality and Plancherel's theorem, it is not difficult to see that these estimates are equivalent, respectively, to

$$\int_{R^N} |\int_0^\infty m_t^{\tau,j}(\xi) g(\xi,t) dt|^2 \, d\xi \le C2^{-2j} \int_{R^N}\int_0^\infty |g(\xi,t)dt|^2 \, dtd\xi,$$

and

$$\int_{R^N} |\int_0^\infty \frac{d}{dt} m_t^{\tau,j}(\xi) g(\xi,t) dt|^2 \, d\xi \le C \int_{R^N}\int_0^\infty |g(\xi,t)dt|^2 \, dtd\xi.$$

The last inequalities are easy consequence of the Lemma 3.2 and Lemma 3.3.

This completes the proof of Theorem 1.4.

INTERPOLATION THEOREM

For Riesz means we will apply interpolation theorem of an analytic family. The classical Riesz-Thorin interpolation theorem was extended by Hirchman, (1953) and Stein, (1956) to analytic families of operators.

Let $F(z), z = x + iy,$ be analytic function in $0 < \Re(z) < 1$ and continuous in $0 \le \Re(z) \le 1$. $F(z)$ is said to be of admissible growth if:

$$\sup_{0\le x\le 1} \log |F(x+iy)| \le Ae^{a|y|}$$

where $a < \pi$. The significance of this notion is in the following lemma due to Hirchman, (1953):

Lemma 4.1 *If* $F(z)$ *is of admissible growth and if*

$$\log |F(it)| \le a_0(t),$$
$$\log |F(1+it)| \le a_1(t),$$

then

$$\log |F(\vartheta)| \le \int_{-\infty}^{\infty} P_0(\vartheta,t)a_0(t)dt + \int_{-\infty}^{\infty} P_1(\vartheta,t)a_1(t)dt$$

where $P_j(\vartheta,t), j = 1,2,$ *are the values of the Poisson kernel for the strip on* $\Re(z) = 0, \ \Re(z) = 1.$

We next define analytic families of linear operators: Let (M,μ) and (N,ν) be two measure spaces. Let $\{T_z\}$ be a family of linear operators indexed by $z, 0 \le \Re(z \le 1$. So that for each $z, 0 \le \Re(z) \le 1,$ the T_z is a mapping of simple functions on M to measurable functions on $N.$

The family of linear operators $\{T_z\}$ is called an analytic family if for any measurable set E of M of finite measure, for almost every $y \in N$, the function $\Psi_y(z) = T_z(\chi_E)(y)$ is analytic in $0 < \Re(z) < 1$, continuous in

$0 \leq \Re(z) \leq 1$, where χ_E is characteristic function of the set E. The analytic family is of admissible growth if for almost every $y \in N$, $\Psi_y(z)$ is of admissible growth.

We recall the notion of L_{pq} spaces. An exposition of these spaces can be found in Hunt, (1966).

Let f be a measurable function defined on unit sphere. We assume that f is finite valued almost everywhere. We denote,
$E_\alpha = \{x : |f(x)| > \alpha\}, \lambda_f(\alpha) = |E_\alpha|$.

We assume also that for some $\alpha > 0, \lambda_f(\alpha) < \infty$. We define
$f^*(t) = \inf_{\alpha > 0} \{\lambda_f(\alpha) \leq t\}$.

$L_{pq}(R^N)$ is the space of all measurable functions f for which $PfP^*_{pq} < \infty$, where

$$\| f \|^*_{pq} == \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty t^{q/p-1} [f^*(t)]^q \, dt \right)^{1/q}, & 0 < p, q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leq \infty, q = \infty. \end{cases} \tag{7}$$

For $p = q$ these are the usual L_p spaces, while for $q = \infty$ we have the so-called weak L_p spaces, i.e. the spaces of functions which satisfy

$\lambda_f(s) \leq \dfrac{c}{s^p}$.

Theorem 4.2 *If* $\{T_z\}$ *is an analytic family of linear operators, which is of admissible growth, and for all simple functions*

$$\| T_{it} f \||^*_{\bar{p}_0 \bar{q}_0} \leq A_0(t) \| f \|^*_{p_0 q_0}, \tag{8}$$

$$\| T_{1+it} f \||^*_{\bar{p}_1 \bar{q}_1} \leq A_1(t) \| f \|^*_{p_1 q_1}, \tag{9}$$

where $\log A_i(t) \leq A e^{a|t|}, a < \pi$, then for all $\vartheta, 0 < \vartheta < 1$

$$\frac{1}{\overline{p}}=\frac{1-\vartheta}{\overline{p}_0}+\frac{1}{\overline{p}_1},\frac{1}{\overline{q}}=\frac{1-\vartheta}{\overline{q}_0}+\frac{\vartheta}{\overline{q}_1}$$

$$\frac{1}{p}=\frac{1-\vartheta}{p_0}+\frac{1}{p_1},\frac{1}{q}=\frac{1-\vartheta}{q_0}+\frac{\vartheta}{q_1}$$

we have for all simple function f

$$\| T_\vartheta f \|\|^*_{\overline{pq}} \le BA_\vartheta \| f \|\|^*_{pq},$$

where

$$\log | A_\vartheta | \le \int_{-\infty}^{\infty} P_0(\vartheta,t)\log A_0(t)dt + \int_{-\infty}^{\infty} P_1(\vartheta,t)\log A_1(t)dt.$$

The proof of this theorem is given in Yoram, (1969).

Lemma 4.3 *Let* $f\in L_p(R^N), 1\le p\le 2,$ *and* $f(x)=0, x\in\Omega\subset R^N$ *, then for Riesz means of the order* $s=(N-1)(1/p-1/2)$ *one has*

$$|\{x: E_*^s f(x)>\alpha\}| \le \left(C\frac{\| f \|_{L_p}}{\alpha} \right)^p. \qquad (10)$$

Proof. If $f\in L_1(R^N),$ and $f(x)=0, x\in\Omega\subset R^N$, then for the Riesz means of the order $s=\frac{N-1}{2}+i\tau, \tau\ne 0,$ we have:

$$|\{E_*^s f(x)>\alpha\}| \le A_N e^{\pi|\tau|}\frac{\| f \|_{L_1}}{\alpha}.$$

In L_{pq} sense, this is:

$$\| E_*^s f(x) \|^*_{1,\infty} \le A_N e^{\pi|\tau|} \| f \|^*_{1,1}, \ s=\frac{N-1}{2}+i\tau, \tau\ne 0. \qquad (11)$$

In case of L_2, for any τ we have

$$\| E_*^{i\tau} f(x) \|_{L_2(K)} \le c_N e^{\pi|\tau|} \| f \|_{L_2(R^N)}, \forall K\subset\Omega.$$

The weak estimation is

$$|\{x: E_*^{i\tau} f(x)>\alpha\}| \le \left(B_N e^{\pi|\tau|}\frac{\| f \|_{L_2}}{\alpha} \right)^2,$$

and in sense of L_{pq} we have:

$$\| E_*^{i\tau} f(x) \|_{2,\infty}^* \leq B_N e^{\pi|\tau|} \| f \|_{2,2}^* . \tag{12}$$

We apply to (11) and (12) the interpolation theorem on an analytic family of linear operators on L_{pq} space.

Let $\mu(x)$ be a measurable function on R^N such as $0 \leq \mu(x) \leq \mu_0 < \infty$ and $s(z) = \frac{N-1}{2} z, \quad 0 \leq \Re(z) \leq 1.$ We define an analytic family of linear operators:

$$T_z f(x) = E_{\mu(x)}^{s(z)} f(x), \quad 0 \leq \Re(z) \leq 1.$$

From (12) we have

$$\| T_{iy} f(x) \|_{2,\infty}^* \leq \| E_*^{s(iy)} f(x) \|_{2,\infty}^* \leq B e^{\pi|y|/2} \| f \|_{2,2}^* . \tag{13}$$

Secondly on the line $z = 1 + iy$, we have

$$\| T_{1+iy} f(x) \|_{1,\infty}^* \leq \| E_*^{s(1+iy)} f(x) \|_{1,\infty} \leq A e^{\pi|y|/2} \| f \|_{1,1}^* . \tag{14}$$

Here we can write (13) and (14) as follows:

$$\| T_{iy} f(x) \|_{2,\infty}^* \leq K_0(y) \| f \|_{2,2}^*, \tag{15}$$

$$\| T_{1+iy} f(x) \|_{1,\infty}^* \leq K_1(y) \| f \|_{1,1}^*, \tag{16}$$

where $K_0(y) \leq A e^{\pi|y|/2}$ and $K_1(y) \leq B e^{\pi|y|/2}$. Therefore by the interpolation we get

$$\| E_*^{s(t)} f(x) \|_{p,\infty}^* \leq K_t \| f \|_{p,p}^*, \tag{17}$$

where K_t is given by

$$\log K_t \leq \int_{-\infty}^{\infty} \omega(1-t, y) \log K_0(y) dy + \int_{-\infty}^{\infty} \omega(t, y) \log K_1(y) dy,$$

and $\omega(t, y)$ is the Poisson kernel for the strip $0 \leq t \leq 1, -\infty < y < \infty.$ By the properties of Poisson kernel, we have

$$\omega(t,y)\geq 0,$$

$$\int_{-\infty}^{\infty}\omega(1-t,y)dy\leq 1,\ \int_{-\infty}^{\infty}\omega(t,y)dy\leq 1$$

and

$$\log K_t\leq C.$$

Note that if,

$$\frac{1}{p}=\frac{1-t}{2}+\frac{t}{1},$$

then

$$s(t)=\frac{N-1}{2}t=\frac{N-1}{2}\left(\frac{2}{p}-1\right)=(N-1)\left(\frac{1}{p}-\frac{1}{2}\right).$$

Hence (17) becomes

$$\| E_*^{s(t)}f(x)\|_{p,\infty}^*\leq C\| f\|_{p,p}^* .$$

This means for $s=(N-1)\left(\frac{1}{p}-\frac{1}{2}\right), 1<p<2,$ we have

$$|\{x: E_*^s f(x)>\alpha\}|\leq\left(C\frac{\| f\|_{L_p}}{\alpha}\right)^p .$$

Lemma 4.3 is proved.

The proof of Theorem 1.2.

Let us denote by $\Lambda f(x)$ the fluctuation of $E_\lambda^s f(x)$:

$$\Lambda f(x)=|\limsup_{\lambda\to\infty}E_\lambda^s f(x)-\liminf_{\lambda\to\infty}E_\lambda^s f(x)|.$$

It is obvious, that

$$\Lambda f(x)\leq 2E_*^s f(x).$$

From density of C^∞ in $L_p, p\geq 1,$ we have for any $\varepsilon>0$ every function $f\in L_p$ can be represented as the sum of two functions:

$$f(x)=f_1(x)+f_2(x),$$

where $f_1(x) \in C^{\infty}$, and $\| f_2 \|_{L_p} \le \varepsilon^{1+1/p}$. Then we have

$$|\{x : \Lambda f(x) > \varepsilon\}| = |\{x : |\limsup_{\lambda\to\infty} E_\lambda^s f_2(x) - \liminf_{\lambda\to\infty} E_\lambda^s f_2(x)| > \varepsilon\}| \le$$

$$\le \{x : E_*^s f(x) > \frac{\varepsilon}{2}\}| \le C \left(\frac{\| f \|_{L_p}}{\varepsilon} \right)^p \le \varepsilon.$$

Therefore almost everywhere $\Lambda f(x) = 0.$ Consequently for Riesz means of order $s = (N-1)\left(\frac{1}{p} - \frac{1}{2}\right), 1 \le p \le 2,$ we have

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0,$$

almost everywhere in $\Omega.$

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REFERENCES

Alimov, Sh.A., Il'in, V.A. and Nikishin, E.M. 1976. Convergence Problems of Multiple Trigonometric series and Spectral Decompositions. *RussianMath. Surv.,* **31**(6) : 29- 86.

Alimov, Sh.A., Ashurov, R.R. and Pulatov, A.K. 1992. *Multiple Fourier Series and Fourier Integrals Commutative Harmonic Analysis -IV,* Springer-Verlag:1-97.

Anvarjon, Akhmedov. 2006. Conditions of localization of Riesz means of the spectral decomposition of the distrubitions, related to Laplace – Beltrami operator on sphere. *International Science Journal, Science, Education, Technology*, **4**: 36-40.

Anvarjon, Akhmedov. 2006. About convergence of the spectral expansions of functions from Liuovill class. *Natural and Technical Science*, **25**(5):12-17.

Anvarjon, Ahmedov. 2009. *The principle of generalized localization on unit sphere.* Journal of Mathematical Analysis and Applications. **356**(1): 310-321.

Bastys, A. J. 1973. Generalized localization of fourier series with respect to the eigenfunctions of the Laplace operator in the classes L_p. *Litovskii Matematicheskii Sbornik*, **31**(3): 387-405.

Bastys, A.J. 1983. The almost everywhere convergence of expansions in eigenfunctions of the Laplace operator on the sphere. *Mathematical Notes*, **33**(6): 777-778.

Bonami, A. and Clerc, J.,L. 1973. Sommes De Cesàro Et Multiplicateurs Des Développements en Harmoniques Sphériques. *Transactions of the American Mathematical Society*, **183**: 223-263.

Carbery, A. and Soria, F. 1988. Almost everywhere convergence of Fourier integrals for functions in Sobolev spaces, and an L_2-localisation principle. *Revista Mat.Iberoamericana*, **4**(2): 319-337.

Hirchman, I.I. (1953). A convexity for certain groups of transformations. *J. Analyse Math.,* **2**: 209-218.

Hunt, R.A. 1966. On $L(p,q)$ space. *L'Enseignement Math.,* **12**: 249-276.

Il'in, V.A. 1968. On a generalized interpretation of the principle of localization for Fourier series with respect to fundamental systems of functions. *Sib. Mat. Zh.,* **9**(5): 1093-1106.

Rakhimov, A. 2004. *Summability of the spectral expansions of distributions, connected with the elliptic partial differential operators,* Thesis of Doctor Science Degree, Tashkent.

Stein, E. M. 1956. Interpolation of linear operators. *Trans. Amer. Math. Soc.*, **83**: 482-492.

Stein, E. M. 1970. Singular Integrals and Differentiability Properties of Functions. *Princeton Math. Series,* **30,** Princeton University Press Princeton, New Jersey.

Yoram, Sagher. 1969. On analytic families of operators. *Israel Journal of Mathematics,***7**(4): 777-778.