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Classification of 3-Dimensional Complex Diassociative Algebras

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ABSTRACT

The paper deals with the classification problems of a subclass of finite-dimensional algebras. One considers a class of algebras having two algebraic operations with five identities. They have been called diassociative algebras by Loday. In this paper we describe all diassociative algebra structure in complex vector space of dimension at most three.

Keywords: Associative algebra, diassociative algebra, isomorphism.

INTRODUCTION

In 1993 Loday (Loday, (1993)) introduced several classes of algebras. These classes of algebras initially have arisen from some problems of algebraic *K*-theory. However, later on it turned out they have some geometrical and physical applications as well. Let us take one of them to motivate the research problems of this paper.

It is well known that any associative algebra gives rise to a Lie algebras, with bracket $[a,b] := ab - ba$. Let D be an algebra, equipped with two binary operations, $\dashv$ - the left product and $\vdash$ the right product, satisfying the following five axioms:

$$(a \dashv b) \dashv c = a \dashv (b \dashv c),$$
$$(a \dashv b) \dashv c = a \dashv (b \vdash c),$$
$$(a \vdash b) \dashv c = a \vdash (b \dashv c),$$
$$(a \dashv b) \vdash c = a \vdash (b \vdash c),$$
$$(a \vdash b) \vdash c = a \vdash (b \vdash c),$$

$\forall a,b,c \in D$.

Then D, according to Loday (Loday *et al.*, (2001)), is said to be an associative dialgebra (or a diassociative algebra). In fact, these axioms are variations of the associative law. Therefore associative algebras are dialgebras for which the two products coincide. The peculiar point is that the bracket $[a, b] = a \dashv b - b \vdash a$ defines a structure in D, called Leibniz algebra structure, with identity

$$[[a,b],c]=[[a,c],b]+[a,[b,c]].$$

The operation $[\cdot,\cdot]$ in D is not antisymmetric, unless the left and right products coincide. If we require the antisymmetricity of $[\cdot,\cdot]$ then $(D,[\cdot,\cdot])$ becomes a Lie algebra. The main motivation of Loday to introduce the class of Leibniz algebras was the search of an obstruction to the periodicity in algebraic *K*-theory. Besides this purely algebraic motivation some relationships with classical geometry, non-commutative geometry and physics have been recently discovered.

We will briefly discuss the categories of Loday's algebras and interrelations between them in Section 2 (Loday Diagram). The goal of this paper is to give a complete classification of complex diassociative algebras in dimensions at most three.

The outline of the paper is as follows. Section 2 (Loday Diagram) and Section 3 (On Algebraic Variety Dias) deal with the definitions and simple properties of the Loday algebras. The main result of the paper is in Section 4 (Classification of Low-Dimensional Complex Diassociative Algebras), where we describe all diassociative algebra structures on two and three dimensional complex vector spaces. Further all algebras are assumed to be over complex numbers.

LODAY DIAGRAM

Definition 2.1 *Leibniz algebra* L *is an algebra with a binary operation* $[\cdot,\cdot]:L\times L\rightarrow L$ *satisfying the following Leibniz identity*:

$$[[a,b],c]=[[a,c],b]+[a,[b,c]], \quad \forall a,b,c\in L.$$

When the bracket happens to be skew-symmetric, we get a Lie algebra since the Leibniz identity becomes the Jacobi identity.

Any associative algebra gives rise to a Lie algebra by $[a,b]=ab-ba.$ In 1993 Loday proposed to introduce a new notion of algebra which gives, by a similar procedure, a Leibniz algebra. He started with two distinct operations for the product ab and the product ba, so that the bracket is not necessarily skew-symmetric. Explicitly, he defined an associative dialgebra (or a diassociative algebra) as a vector space D equipped with two associative operations $\dashv$ and $\vdash$ called respectively left and right product, satisfying three more axioms:

$$(a \dashv b) \dashv c = a \dashv (b \vdash c),$$
$$(a \vdash b) \dashv c = a \vdash (b \dashv c),$$
$$(a \dashv b) \vdash c = a \vdash (b \vdash c),$$

$\forall a, b, c \in D.$

It is immediate to check that $[a,b]=a \dashv b - b \vdash a$ defines a Leibniz bracket. Hence any diassociative algebra gives rise to a Leibniz algebra.

Definition 2.2 *Let* $(D_1, \vdash_1, \dashv_1)$ *and* $(D_2, \vdash_2, \dashv_2)$ *be diassociative algebras. Then a homomorphism of dialgebras* D_1 *and* D_2 *is a linear mapping* $f\ :\ D_1 \to D_2$ *such that:*

$$f(a \vdash_1 b)=f(a) \vdash_2 f(b),$$
$$f(a \dashv_1 b)=f(a) \dashv_2 f(b),$$

$\forall a,b \in D_1.$

Bijective homomorphism is said to be isomorphism.

Loday and his colleagues have constructed and studied a (co)homology theory for diassociative algebras (Loday *et al.*, (2001)). Since an associative algebra is a particular case of diassociative algebra, we get a new (co)homology theory for associative algebras as well.

Moreover, Loday introduced another class of algebras, called *dendriform algebras*, which are closely related to the above defined classes of algebras in (co)homological manner.

Definition 2.3 *Dendriform algebra* E *is an algebra with two binary operations*

$$\succ: E \times E \to E \ , \ \prec: E \times E \to E$$

satisfying the following axioms:

$$(a \prec b) \prec c = (a \prec c) \prec b + a \prec (b \succ c) \ ,$$

$$(a \succ b) \prec c = a \succ (b \prec c) \ ,$$

$$(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c) \ ,$$

$\forall a, b, c \in E.$

The result intertwining diassociative and dendriform algebras can be expressed in the framework of algebraic operads. In order to illustrate it, Loday defines a class of *Zinbiel algebras*, which is Koszul dual to the category of Leibniz algebras.

Definition 2.4 *Zinbiel algebra* R *is an algebra with a binary operation* $\cdot: R \times R \to R$*, satisfying the condition:*

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) + a \cdot (c \cdot b) \ , \quad \forall a, b, c \in R \ .$$

Each one of these types of algebras defines a binary quadratic operad. For these operads, there is a well-defined notion of Koszul duality theory devised by Ginzburg and Kapranov. Let $\mathsf{P}^!$ be the dual of the operad P (note that $\mathsf{P}^{!!} = \mathsf{P}$).

The notion of diassociative algebra defines an algebraic operad *Dias*, which is binary and quadratic. By the theory of Ginzburg and Kapranov, there is a well-defined dual operad $Dias^!$. Loday has shown that this is precisely the operad *Dend* of the dendriform algebras. In other words a dual diassociative algebra is nothing but a dendriform algebra.

The categories of algebras over these operads assemble into a commutative diagram of functors below which reflects the Koszul duality (Ginzburg and Kapranov, (1994)).

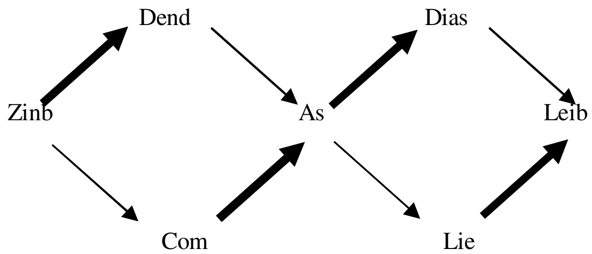

Figure 1: Loday diagram.

In the diagram *Zinb*, *Com*, *As*, *Lie*, *Leib* stand for *Zinbiel, Commutative, Associative, Lie* and *Leibniz* algebras categories, respectively, and the Koszul duality in it corresponds to symmetry around the vertical axis passing through *As*:

$$As^! = As, \quad Com^! = Lie, \quad Zinb^! = Leib, \quad Dend^! = Dias.$$

Observe that classification of complex low dimensional Lie, Leibniz and Zinbiel algebras has been given in (Jacobson, (1962)), (Ayupov and Omirov, (1999)) and (Dzhumadildaev and Tulenbaev, (2005)) respectively.

ON ALGEBRAIC VARIETY *DIAS*

In this section we recall some elementary facts on diassociative algebras that will be used later on.

Let V be an n-dimensional vector space and $e_1, e_2, \ldots, e_n$ be a basis of V. Then a diassociative structure on V can be defined as two bilinear mappings:

$$\lambda : V \times V \to V$$

representing the left product $\dashv$ and

$$\mu : V \times V \to V$$

representing the right product $\vdash$, consented via diassociative algebra axioms.

Hence, an n-dimensional diassociative algebra D can be seen as a triple $D=(V,\lambda,\mu)$ where λ and μ are associative laws on V. We will denote by $Dias$ the set of diassociative algebra laws on V.

Let us denote by γ_{ij}^k and δ_{st}^q, where $i,j,k,s,t,q=1,2,3,...,n,$ the structure constants of a diassociative algebra with respect to the basis $e_1,e_2,...,e_n$ of V, where

$$e_i \dashv e_j=\sum_k \gamma_{ij}^k e_k \text{ and } e_i \vdash e_j=\sum_k \delta_{ij}^k e_k \text{ for } i,j,k=1,2,3,\dots,n.$$

Then $Dias$ can be considered as a closed subset of $2n^3$-dimensional affine space specified by the following system of polynomial equations with respect to the structure constants γ_{ij}^k and δ_{st}^q :

$$\begin{aligned}
\gamma_{ij}^t\gamma_{tk}^s &= \gamma_{it}^s\gamma_{jk}^t,\\
\gamma_{ij}^t\gamma_{tk}^s &= \gamma_{it}^s\delta_{jk}^t,\\
\delta_{ij}^t\gamma_{tk}^s &= \delta_{it}^s\gamma_{jk}^t,\\
\gamma_{ij}^t\delta_{tk}^s &= \delta_{it}^s\delta_{jk}^t,\\
\delta_{ij}^t\delta_{tk}^s &= \delta_{it}^s\delta_{jk}^t.
\end{aligned}$$

Thus $Dias$ can be considered as a subvariety of $2n^3$-dimensional affine space. On $Dias$ the linear matrix group GL_n acts by changing of basis. This action can be expressed as follows: if $g=[g_i^j]\in GL_n$ and $D=\{\gamma_{ij}^k,\delta_{st}^r\}$, then

$$\{(g*D)_{ij}^k,(g*D)_{st}^r\}=\{g_i^p\cdot g_j^q\cdot(g_l^k)^{-1}\cdot\gamma_{pq}^t, g_s^p\cdot g_t^q\cdot(g_l^r)^{-1}\cdot\gamma_{pq}^t\}.$$

CLASSIFICATION OF LOW-DIMENSIONAL COMPLEX DIASSOCIATIVE ALGEBRAS

In this section we discuss a classification of low-dimensional diassociative algebras. Our strategy consists of the following two steps. For the first step we consider associative algebra with the operation $\vdash$. Getting associative algebras classification, we fix one of them and instead of the action of GL_n, consider the same action of the chosen associative algebras stabilizer. In the second step we take structure constants with respect to algebraic operation $\vdash$ and elements of stabilizer group as variables and then examine diassociative algebra axioms. Although this procedure makes it slightly easier to solve the classification problem in low dimensional cases, the problem of classification in general, however, remains a big problem.

From now and what follows we use the following notations: As_n^q- and $Dias_n^q$- stand for q-th associative and q-th diassociative algebra structures in n-dimensional vector space, respectively.

Two Dimensional Diassociative Algebras

Theorem 4.1 *Let A be a 2-dimensional complex associative algebra. Then it is isomorphic to one of the following pairwise non-isomorphic associative algebras*:

As_2^1 : *Abelian*;
As_2^2 : $e_1e_1 = e_1$;
As_2^3 : $e_1e_1 = e_2$;
As_2^4 : $e_1e_1 = e_1$, $e_1e_2 = e_2$;
As_2^5 : $e_1e_1 = e_1$, $e_2e_1 = e_2$;
As_2^6 : $e_1e_1 = e_1$, $e_1e_2 = e_2$, $e_2e_2 = e_2$;
As_2^7 : $e_1e_1 = e_1$, $e_2e_2 = e_2$.

For 2-dimensional complex diassociative algebras the following structural result holds.

Theorem 4.2 *Any 2-dimensional complex diassociative algebra either is associative or isomorphic to one of the following pairwise non-isomorphic diassociative algebras*:

$Dias_2^1$: $e_1 \vdash e_1=e_1$, $e_1 \dashv e_1=e_1$, $e_2 \dashv e_1=e_2$;
$Dias_2^2$: $e_1 \vdash e_1=e_1$, $e_1 \vdash e_2=e_2$, $e_1 \dashv e_1=e_1$;
$Dias_2^3$: $e_1 \vdash e_1=e_2$, $e_1 \dashv e_1=\alpha e_2$;
$Dias_2^4$: $e_1 \vdash e_1=e_1$, $e_1 \vdash e_2=e_2$, $e_1 \dashv e_1=e_1$, $e_2 \dashv e_1=e_2$.

Proof. Let D be a two-dimensional vector space. To determine a diassociative algebra structure on D, we consider D with respect to one associative operation. It is one of algebras from the list of Theorem 4.1.

Let $A_1=(D, \vdash)$ be the algebra

$$e_1 \vdash e_1 = e_1.$$

The second multiplication operation $\dashv$ in D, we define as follows:

$$\begin{aligned} e_1 \dashv e_1 &= \alpha_1 e_1 + \alpha_2 e_2, \\ e_1 \dashv e_2 &= \alpha_3 e_1 + \alpha_4 e_2, \\ e_2 \dashv e_1 &= \alpha_5 e_1 + \alpha_6 e_2, \\ e_2 \dashv e_2 &= \alpha_7 e_1 + \alpha_8 e_2. \end{aligned}$$

Now verifying diassociative algebra axioms, we get several constraints for the coefficients α_i where $i = 1, 2, \ldots, 8$.

Applying $(e_1 \dashv e_1) \vdash e_1 = e_1 \vdash (e_1 \vdash e_1)$, we get $(\alpha_1 e_1 + \alpha_2 e_2) \vdash e_1 = e_1 \vdash e_1$ and then $\alpha_1 e_1 = e_1$. Therefore $\alpha_1 = 1$.

The verification of $(e_1 \vdash e_1) \dashv e_1 = e_1 \vdash (e_1 \dashv e_1)$ leads to $e_1 \dashv e_1 = e_1 \vdash (e_1 + \alpha_2 e_2)$ and from this we get $e_1 + \alpha_2 e_2 = e_1$. Hence we obtain $\alpha_2 = 0$.

Consider $(e_1 \dashv e_1) \dashv e_2 = e_1 \dashv (e_1 \vdash e_2)$. It implies that $e_1 \dashv e_2 = 0$, therefore $\alpha_3 = 0$ and $\alpha_4 = 0$.

The next relation to consider is $(e_1 \dashv e_2) \dashv e_1 = e_1 \dashv (e_2 \dashv e_1)$. It implies that $0 = e_1 \dashv (\alpha_5 e_1 + \alpha_6 e_2)$ and we get $\alpha_5 = 0$.

By the following observation we find $\alpha_6(\alpha_6 - 1) = 0$. Therefore α_6 is either equal to 0 or 1. Indeed, $(e_2 \dashv e_1) \dashv e_1 = e_2 \dashv (e_1 \vdash e_1) \Rightarrow \alpha_6(e_2 \dashv e_1) = e_2 \dashv e_1$.

To find α_7 and α_8, we note that $(e_2 \dashv e_2) \dashv e_1 = e_2 \dashv (e_2 \vdash e_1) \Rightarrow (\alpha_7 e_1 + \alpha_8 e_2) \dashv e_1 = 0 \Rightarrow \alpha_7 e_1 + \alpha_6 \alpha_8 e_2 = 0$. Hence we have $\alpha_7 = 0$, $\alpha_6 \alpha_8 = 0$.

Finally, we apply $(e_2 \dashv e_2) \dashv e_2 = e_2 \dashv (e_2 \vdash e_2) \Rightarrow \alpha_8(e_2 \dashv e_2) = 0$, and get $\alpha_8 = 0$.

The verification of all other cases leads to the obtained constraints.

Thus, in this case we come to the diassociative algebra with the multiplication table:

$$e_1 \vdash e_1 = e_1,\ e_1 \dashv e_1 = e_1,\ e_2 \dashv e_1 = \alpha_6 e_2$$

where $\alpha_6(\alpha_6 - 1) = 0$.

If $\alpha_6 = 0$, then the right and left products coincide and we get the associative algebra.

If $\alpha_6 = 1$, one obtains the diassociative algebra $Dias_2^1$.
The other algebras of the list of Theorem 4.2 can be obtained by a minor modification of the observation above.

The Classification of 3-Dimensional Complex Associative Algebras

As mentioned above, to classify the low-dimensional diassociative algebras we need complete list of associative algebras in respective dimension. By the following theorem we give a result from (Basri and Rikhsiboev, (2007)) on classification of 3-dimensional complex associative algebras.

Theorem 4.3 *Any* 3*-dimensional non decomposable complex associative algebra* A *is isomorphic to one of the following pairwise non-isomorphic algebras*

As_3^1 : $e_1e_2 = e_1$, $e_2e_2 = e_2$, $e_3e_1 = e_1$, $e_3e_3 = e_3$;
As_3^2 : $e_1e_3 = e_2$, $e_2e_3 = e_2$, $e_3e_1 = e_2$, $e_3e_2 = e_2$, $e_3e_3 = e_3$;
As_3^3 : $e_1e_3 = e_2$, $e_2e_3 = e_2$, $e_3e_3 = e_3$;
As_3^4 : $e_3e_1 = e_2$, $e_3e_2 = e_2$, $e_3e_3 = e_3$;
As_3^5 : $e_1e_3 = e_1$, $e_2e_3 = e_2$, $e_3e_1 = e_1$, $e_3e_2 = e_2$, $e_3e_3 = e_3$;
As_3^6 : $e_2e_3 = e_2$, $e_3e_1 = e_1$, $e_3e_2 = e_2$, $e_3e_3 = e_3$;
As_3^7 : $e_2e_3 = e_2$, $e_3e_1 = e_1$, $e_3e_3 = e_3$;
As_3^8 : $e_1e_3 = e_1$, $e_2e_3 = e_2$, $e_3e_1 = e_1$, $e_3e_3 = e_3$;
As_3^9 : $e_1e_1 = e_2$, $e_1e_3 = e_1$, $e_2e_3 = e_2$, $e_3e_1 = e_1$, $e_3e_2 = e_2$, $e_3e_3 = e_3$;
As_3^{10} : $e_1e_3 = e_2$, $e_3e_1 = e_2$;
As_3^{11} : $e_1e_3 = e_2$, $e_3e_1 = ke_2$, $k \in \mathbb{C}\setminus\{1\}$;
As_3^{12} : $e_1e_1 = e_2$, $e_1e_2 = e_3$, $e_2e_1 = e_3$;

Remark 4.1 *There exist the following pairwise non-isomorphic decomposable associative algebras in dimension three:*

As_3^{13} : $e_1e_1 = e_1$, $e_2e_2 = e_2$, $e_3e_3 = e_3$;
As_3^{14} : $e_1e_2 = e_1$, $e_2e_1 = e_1$, $e_2e_2 = e_2$, $e_3e_3 = e_3$;
As_3^{15} : $e_1e_2 = e_1$, $e_2e_2 = e_2$, $e_3e_3 = e_3$;
As_3^{16} : $e_2e_1 = e_1$, $e_2e_2 = e_2$, $e_3e_3 = e_3$;
As_3^{17} : $e_1e_1 = e_2$, $e_3e_3 = e_3$.

The Classification of 3-Dimensional Complex Diassociative Algebras

Using the result from Theorem 4.3, we have the following

Theorem 4.4 *Any* 3*-dimensional complex diassociative algebra* D *is either associative or isomorphic to one of the following pairwise non-isomorphic algebras.*

$Dias_3^1$: $e_1 \dashv e_2=e_1$, $e_2 \dashv e_2=e_2$, $e_3 \dashv e_3=e_3$, $e_2 \vdash e_2=e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^2$: $e_1 \dashv e_2=e_1$, $e_2 \dashv e_2=e_2$, $e_3 \dashv e_3=e_3$, $e_2 \vdash e_1=e_1$, $e_2 \vdash e_2=e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^3$: $e_1 \dashv e_2=e_1$, $e_2 \dashv e_2=e_2$, $e_3 \dashv e_3=e_3$, $e_2 \vdash e_2=e_2$, $e_3 \vdash e_1=e_1$;

$Dias_3^4$: $e_1 \dashv e_3=e_2$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_3=e_3$;

$Dias_3^5$: $e_1 \dashv e_3=e_2$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_1- e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^6$: $e_1 \dashv e_3=e_2$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_1$, $e_3 \vdash e_2=e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^7$: $e_1 \dashv e_3=e_2$, $e_2 \dashv e_3= e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_2$, $e_3 \vdash e_2=e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^8$: $e_1 \dashv e_3=e_2$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_2$, $e_2 \vdash e_3=e_2$, $e_3 \vdash e_1=e_1- e_2$,
$e_3 \vdash e_3= e_3$;

$Dias_3^9$: $e_3 \dashv e_1=e_2$, $e_3 \dashv e_2=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_1$, $e_3 \vdash e_2=e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^{10}$: $e_3 \dashv e_1=e_1$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_1$, $e_3 \vdash e_3=e_3$;

$Dias_3^{11}$: $e_3 \dashv e_1=e_1$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_3=e_3$, $e_3 \vdash e_1=e_1$, $e_3 \vdash e_2=e_2$, $e_3 \vdash e_3=e_3$;

$Dias_3^{12}$: $e_1 \dashv e_3=e_1$, $e_2 \dashv e_3=e_2$, $e_3 \dashv e_1=e_1$, $e_3 \dashv e_3=e_3$, $e_1 \vdash e_3=e_1$, $e_3 \vdash e_1=e_1$,
$e_3 \vdash e_3=e_3$;

$Dias_3^{13}$: $e_1 \dashv e_3=e_1,\ e_2 \dashv e_3=e_2,\ e_3 \dashv e_1=e_1,\ e_3 \dashv e_3=e_3,\ e_1 \vdash e_3=e_1,\ e_3 \vdash e_1=e_1,$
$e_3 \vdash e_2=e_2,\ e_3 \vdash e_3=e_3;$

$Dias_3^{14}$: $e_1 \dashv e_3=e_1,\ e_2 \dashv e_3=e_2,\ e_3 \dashv e_1=e_1,\ e_3 \dashv e_3=e_3,\ e_1 \vdash e_3=e_1+e_2,\ e_3 \vdash e_1=e_1,$
$e_3 \vdash e_2=e_2,\ e_3 \vdash e_3=e_3;$

$Dias_3^{15}$: $e_1 \dashv e_1=e_2,\ e_3 \dashv e_3=e_3,\ e_3 \vdash e_3=e_3;$

$Dias_3^{16}$: $e_1 \dashv e_3=e_2,\ e_3 \dashv e_1=ke_2,\ e_1 \vdash e_1=me_2,\ e_1 \vdash e_3=ne_2,\ e_3 \vdash e_1=pe_2,$
$e_3 \vdash e_3=qe_2;$

$Dias_3^{17}$: $e_1 \dashv e_3=e_2,\ e_1 \dashv e_2=e_3,\ e_2 \dashv e_1=e_3,\ e_1 \vdash e_1=e_2+e_3;$

where $k, m, n, p, q \in \mathbb{C}$.

Proof. We give the proof only for one case. The other cases can be carried out by a minor changing.

Suppose that associative algebra $A_1=(D,\dashv)$ has the following multiplication table (this is the algebra As_3^7 from Theorem 4.3):

$$e_2 \dashv e_3=e_2,\ \ e_3 \dashv e_1=e_1,\ \ e_3 \dashv e_3=e_3.$$

We define $A_2=(D, \vdash)$ by the multiplication table:

$$e_1 \vdash e_1=\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3,\ \ e_1 \vdash e_2=\alpha_4 e_1+\alpha_5 e_2+\alpha_6 e_3,\ e_1 \vdash e_3=\alpha_7 e_1+\alpha_8 e_2+\alpha_9 e_3,$$
$$e_2 \vdash e_1=\beta_1 e_1+\beta_2 e_2+\beta_3 e_3,\ \ e_2 \vdash e_2=\beta_4 e_1+\beta_5 e_2+\beta_6 e_3,\ e_2 \vdash e_3=\beta_7 e_1+\beta_8 e_2+\beta_9 e_3,$$
$$e_3 \vdash e_1=\gamma_1 e_1+\gamma_2 e_2+\gamma_3 e_3,\ \ e_3 \vdash e_2=\gamma_4 e_1+\gamma_5 e_2+\gamma_6 e_3,\ e_3 \vdash e_3=\gamma_7 e_1+\gamma_8 e_2+\gamma_9 e_3,$$

where α_i, β_j and γ_k are unknowns ($i, j, k = 1, 2, \ldots, 9$).

Verifying the diassociative algebra axioms, we get the following constraints for the structure constants:

$$\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=\alpha_7=\alpha_8=\alpha_9=0,$$
$$\beta_1=\beta_2=\beta_3=\beta_4=\beta_5=\beta_6=\beta_7=\beta_9=0,$$
$$\gamma_2=\gamma_3=\gamma_4=\gamma_6=\gamma_7=0,\ \ \gamma_1=\gamma_9=1$$

and

$$\beta_8=\beta_8^2,\ \ \beta_8\gamma_5=0,\ \ \beta_8\gamma_8=0,\ \ \gamma_5\gamma_8=0,\ \ \gamma_5=\gamma_5^2.$$

After some simple computations we get a list of algebras as follows:

Case 1. If $\beta_8 \neq 0$ then $\beta_8 = 1$ and $\gamma_5 = \gamma_8 = 0$. We then have

$$e_2 \dashv e_3 = e_2,\ e_3 \dashv e_1 = e_1,\ e_3 \dashv e_3 = e_3,$$
$$e_2 \vdash e_3 = e_2,\ e_3 \vdash e_1 = e_1,\ e_3 \vdash e_3 = e_3.$$

It is an associative algebra.

Case 2. Suppose that $\beta_8 = 0$ then we have

$$\gamma_5\gamma_8 = 0,\ \gamma_5 = \gamma_5^2.$$

Now let us consider the following:

Case 2.1. If $\gamma_5 \neq 0$ then $\gamma_5 = 1$ and $\gamma_8 = 0$. In this case we obtain the algebra $Dias_3^{11}$:

$$e_2 \dashv e_3 = e_2,\ e_3 \dashv e_1 = e_1,\ e_3 \dashv e_3 = e_3,$$
$$e_3 \vdash e_1 = e_1,\ e_3 \vdash e_2 = e_2,\ e_3 \vdash e_3 = e_3.$$

Case 2.2. If $\gamma_5 = 0$ then it gives the following multiplication table:

$$e_3 \vdash e_1 = e_1,\quad e_3 \vdash e_3 = \gamma_8 e_2 + e_3.$$

Taking the following base change: $e_3' = \gamma_8 e_2 + e_3$, $e_2' = e_2$, $e_1' = e_1$ we find the table of multiplication as follows:

$$e_3 \dashv e_1 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_3 = e_3,\ e_3 \vdash e_1 = e_1,\ e_3 \vdash e_3 = e_3.$$

This is the algebra $Dias_3^{10}$.
Similar observations can be applied for the other cases.

The final results then can be listed out as follows:

Associative algebras	Corresponding diassociative algebras
$A_1 = As_3^1$, $A_2 = (D, \vdash)$	Trivial algebra*
$A_1 = As_3^2$, $A_2 = (D, \vdash)$	Trivial algebra
$A_1 = As_3^3$, $A_2 = (D, \vdash)$	Trivial algebra, $Dias_3^4, Dias_3^5, Dias_3^6, Dias_3^7, Dias_3^8$

Associative algebras	**Corresponding diassociative algebras**
$A_1 = As_3^4$, $A_2=(D, \vdash)$	Trivial algebra, $Dias_3^9$
$A_1 = As_3^5$, $A_2=(D, \vdash)$	Trivial algebra
$A_1 = As_3^6$, $A_2=(D, \vdash)$	Trivial algebra
$A_1 = As_3^8$, $A_2=(D, \vdash)$	Trivial algebra, $Dias_3^{12}, Dias_3^{13}, Dias_3^{14}$
$A_1 = As_3^9$, $A_2=(D, \vdash)$	Trivial algebra
$A_1 = As_3^{10}$, $A_2=(D, \vdash)$	$Dias_3^{16}$
$A_1 = As_3^{11}$, $A_2=(D, \vdash)$	$Dias_3^{16}$
$A_1 = As_3^{12}$, $A_2=(D, \vdash)$	Trivial algebra, $Dias_3^{17}$
$A_1 = As_3^{13}$, $A_2=(D, \vdash)$	Trivial algebra
$A_1 = As_3^{14}$, $A_2=(D, \vdash)$	Trivial algebra
$A_1 = As_3^{15}$, $A_2=(D, \vdash)$	Trivial algebra, $Dias_3^1, Dias_3^2, Dias_3^3$
$A_1 = As_3^{16}$, $A_2=(D, \vdash)$	Trivial algebra
$A_1 = As_3^{17}$, $A_2=(D, \vdash)$	Trivial algebra, $Dias_3^{15}$

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