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Numerical Methods for Moving Barrier Option Pricing

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Moving barrier options are a class of time-dependent barrier options whose pricing poses significant computational challenges. Focusing on the down-and-out call option, this paper develops a finite difference scheme for such moving barrier options and employs the binomial tree method to solve the pricing problem. Furthermore, the equivalence between these two numerical methods is established. Numerical experiments are also conducted to validate the theoretical results.

Keywords: *moving barrier option pricing; finite difference method; binomial tree method; Black–Scholes model.*

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1. Introduction

1.1. Background

Barrier options are a class of exotic options whose payoff depends not only on the terminal price of the underlying asset, but also on whether the asset price has crossed a predetermined barrier level during the option's life. When the barrier level is a known function of time, the option is referred to as a moving barrier option. These options are widely used in financial markets for hedging and speculative purposes due to their flexibility and lower premium compared to other options [1].

The analytical pricing of moving barrier options presents significant challenges, as the timedependent barrier introduces additional complexity into the Black–Scholes partial differential equation. An explicit closed-form solution exists only when the barrier function takes an exponential form. In most practical situations, numerical methods are required. The most commonly used numerical approaches include the binomial tree method and the finite difference method. While both methods have been well established for plain vanilla options, their relationship in the context of moving barrier options has not yet been thoroughly investigated. Establishing the equivalence between these two methods not only deepens the theoretical understanding but also provides practical guidance for the selection and validation of numerical schemes.

1.2. Literature Review

The pricing of options using numerical methods has a rich history. The binomial tree method was proposed by Cox, Ross and Rubinstein [2] as a simple and intuitive discrete-time model for option pricing. Finite difference methods for option pricing were pioneered by Brennan and Schwartz [3] and have been extensively applied to the numerical solution of the Black–Scholes partial differential equation [4]. For barrier options, Merton [5] was the first to derive closed-form solutions for standard fixed barrier options. Extensions to time-dependent barriers have been studied by numerous researchers; for instance, Roberts and Shortland [6] examined the pricing of options with exponential barriers using transform methods.

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Despite the extensive literature on numerical methods, the equivalence between binomial trees and finite difference schemes has been mainly studied for plain vanilla European options. In the classic textbook by Jiang [7], it is shown that, as the time step approaches zero, the binomial tree method and the explicit finite difference scheme for the Black–Scholes equation coincide up to higher-order infinitesimals. However, a rigorous proof of this equivalence for moving barrier options is still lacking in the literature.

In this paper, we address this gap by focusing on a down-and-out call option with a moving barrier. We first derive the explicit finite difference scheme for this option after a suitable transformation that converts the time-dependent barrier problem into a constant-coefficient PDE. Then we prove that, in the sense of neglecting higher-order infinitesimal terms of the time step Δt, the binomial tree method yields exactly the same recursive formula as the finite difference scheme when the grid parameters satisfy $\omega = \frac{\sigma^2 \Delta t}{(\Delta x)^2} = 1$. Numerical experiments are performed to verify the theoretical findings. The main contribution of this paper lies in establishing a clear theoretical connection between two prevalent numerical approaches for pricing moving barrier options, thus laying a solid foundation for their practical implementation and cross-validation.

2. Model Setup

2.1. Definition and Assumptions

We consider a down-and-out call option with a moving barrier. Let S_t denote the price of the underlying stock at time t. The barrier level $S_B(t)$ is a known function of time; the option is knocked out (becomes worthless) if at any time $[0,T]$, the stock price falls to or below $S_B(t)$. If the barrier is never hit, the payoff at maturity T is $(S-K)^+$, where K is the strike price.

Assume that the price evolution of the underlying stock follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

where $S_t = (S(t))_{t\geqslant 0}$ is the price of the underlying asset, W_t is a standard Brownian motion,

$$E(dW_t) = 0,$$
$$\mathrm{Var}(dW_t) = dt,$$

where μ the constant expected rate of return, σ is the constant volatility. There are no transaction costs or taxes. There are no arbitrage opportunities.

2.2. Notation

The following notation is used throughout the paper: V is the price of the down-and-out call option, C_{BS} is the Black–Scholes value of the down-and-out call option, C_{FEM} is the finite element method value of the down-and-out call option, C_{FDM} is the finite difference method value of the down-and-out call option, C_{BTM} is the binomial tree method value of the down-and-out call option, S_0 is the initial value of the underlying asset, r is the risk-free interest rate, q is the dividend yield, σ is the volatility, K is the strike price, $S_{\mathrm{B}}(\mathrm{t})$ is the barrier level, T is the maturity date.

3. Explicit Finite Difference Scheme for a Moving Barrier Option

We take a down-and-out call option as an example. The mathematical model for a moving barrier option over the solution domain $D = \{(S,t) \mid S_B(t) \leqslant S < \infty, 0 \leqslant t \leqslant T\}$ is given by:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-q)S\dfrac{\partial V}{\partial S} - rV = 0, & \text{in } D, \\ V(S,T) = (S-K)^+, & S_B(T) < S < \infty, \\ V(S_B(t),t) = 0, & 0 \leqslant t \leqslant T. \end{cases}$$

We introduce the transformation as follows:

$$x = \ln \frac{S}{S_B(t)}, \quad V(S,t) = S_B(T)\, u(x,t)$$

Therefore, we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2} \dfrac{\partial^2 u}{\partial x^2} + \left(r - q - \dfrac{\sigma^2}{2} - \alpha(t)\right) \dfrac{\partial u}{\partial x} - ru = 0, & \\ u(x,T) = \left(e^x - K_B\right)^+, & 0 < x < \infty, \\ u(0,t) = 0, & 0 \leqslant t \leqslant T, \end{cases}$$

where $K_B = \frac{K}{S_B(T)}$, $\alpha(t) = \frac{1}{S_B} \cdot \frac{dS_B}{dt}$.

Assume that $\alpha(t) = \alpha$ is constant, i.e. $\frac{dS_B}{dt} = S_B(t)\alpha$, then $S_B(t) = S_B(T) \cdot e^{-\alpha(T-t)}$.

The mathematical model for a moving barrier option can be transformed into:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2} \dfrac{\partial^2 u}{\partial x^2} + \left(r - \widehat{q} - \dfrac{\sigma^2}{2}\right) \dfrac{\partial u}{\partial x} - ru = 0, \quad \widehat{q} = \alpha + q, & \\ u(x,T) = \left(e^x - K_B\right)^+, & 0 \leqslant x < \infty, \\ u(0,t) = 0, & 0 \leqslant t \leqslant T. \end{cases}$$

A grid is constructed over the domain $\{x \geqslant 0, 0 \leqslant t \leqslant T\}$:

$$x_m = m\Delta x, \quad 0 \leqslant m < \infty; \quad t_n = n\Delta t, \quad 0 \leqslant n \leqslant N, \quad N = T/\Delta t.$$

We define the grid values:

$$u_m^n = u\left(V_m^n, t_n\right), \quad V_m^n = V\left(x_m, t_n\right).$$

Next, we discretize the well-posed problem:

$$\begin{cases} \dfrac{u_m^{n+1} - u_m^n}{\Delta t} + \dfrac{\sigma^2}{2} \dfrac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \left(r - \widehat{q} - \dfrac{\sigma^2}{2}\right) \dfrac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} - ru_m^n = 0, \\ u_m^N = \left(e^{m\Delta x} - K_B\right)^+, \\ u(0,t_n) = 0. \end{cases}$$

When $n = N$, the value of u_m^n is known.

If the value of u_m^{n+1} is known ($0 \leqslant m < \infty$), then at $t = t_n = n\Delta t$,

$$\begin{aligned} u_m^n = \frac{1}{1 + r\Delta t} \Bigg[& \left(1 - \frac{\sigma^2 \Delta t}{\Delta x^2}\right) u_m^{n+1} + \left(\frac{\sigma^2 \Delta t}{2\Delta x^2} + \frac{1}{2}\left(r - \widehat{q} - \frac{\sigma^2}{2}\right) \frac{\Delta t}{\Delta x}\right) u_{m+1}^{n+1} \\ & + \left(\frac{\sigma^2 \Delta t}{2\Delta x^2} - \frac{1}{2}\left(r - \widehat{q} - \frac{\sigma^2}{2}\right) \frac{\Delta t}{\Delta x}\right) u_{m-1}^{n+1} \Bigg]. \end{aligned}$$

This yields the (backward) explicit finite difference scheme for the moving barrier option.

Let $\omega = \frac{\sigma^2 \Delta t}{(\Delta x)^2}$, the corresponding discrete form is

$$\begin{aligned} u_m^n = \frac{1}{1 + r\Delta t} \Bigg[& (1 - \omega) u_m^{n+1} + \left(\frac{\omega}{2} + \frac{\sqrt{\omega}}{2\sigma}\left(r - \widehat{q} - \frac{\sigma^2}{2}\right) \sqrt{\Delta t}\right) u_{m+1}^{n+1} \\ & + \left(\frac{\omega}{2} - \frac{\sqrt{\omega}}{2\sigma}\left(r - \widehat{q} - \frac{\sigma^2}{2}\right) \sqrt{\Delta t}\right) u_{m-1}^{n+1} \Bigg] \end{aligned}$$

When $\omega = 1$,

$$u_m^n = \frac{1}{1 + r\Delta t}\left[\frac{1}{2}\left(1 + \frac{\sqrt{\Delta t}}{\sigma}\left(r - \widehat{q} - \frac{\sigma^2}{2}\right)\right) u_{m+1}^{n+1} + \frac{1}{2}\left(1 - \frac{\sqrt{\Delta t}}{\sigma}\left(r - \widehat{q} - \frac{\sigma^2}{2}\right)\right) u_{m-1}^{n+1}\right].$$

4. Relationship Between Binomial Tree Method and Finite Difference Method for a Moving Barrier Option

Theorem 1. *In the sense of neglecting higher-order infinitesimal terms of Δt, the relationship between the binomial tree method and the explicit finite difference scheme of the Black–Scholes equation for moving barrier option pricing is as follows ($\omega = \sigma^2\Delta t/(\Delta x)^2 = 1$):*

$$C_{\mathrm{BTM}} = C_{\mathrm{FDM}}.$$

Remark 1. When $\alpha = 0$, the barrier level is fixed as a constant, which corresponds to the case of a standard barrier option.

Proof. The interval $[0, T]$ is divided into N equal parts:

$$0 = t_0 < t_1 < \ldots < t_N = T.$$

Here, $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$.

Assume that at time $t = t_n$, the price of the underlying asset S_n is known. Then at time $t = t_{n+1}$, S_{n+1} has two possible values:

$$S_{n+1} = \begin{cases} S_n u & \text{up movement,} \\ S_n d & \text{down movement.} \end{cases}$$

In a risk-neutral world, consider the continuous model for the evolution of the underlying asset price with a continuous dividend yield:

$$\begin{cases} \dfrac{dS}{S} = (r-q)\,dt + \sigma\,dW_t, \\ S\,(t_n) = S_n. \end{cases}$$

Applying Itô's lemma, we obtain

$$d\ln S = \left(r - q - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t$$

Furthermore, since $x = \ln\frac{S}{S_B(t)}$ and $S_B(t) = S_B(T)\cdot e^{-\alpha(T-t)}$, we have $d\ln S_B(t) = \alpha dt$. Therefore

$$\begin{aligned} dx = d\ln S - d\ln S_B(t) &= \left(r - q - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t - \alpha\,dt \\ &= \left(r - q - \frac{\sigma^2}{2} - \alpha\right)dt + \sigma\,dW_t = \left(r - \widehat{q} - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t. \end{aligned}$$

Integrating both sides from t_n to t_{n+1}, we have

$$\Delta x = x_{n+1} - x_n = \left(r - \widehat{q} - \frac{\sigma^2}{2}\right)dt + \sigma\,\Delta W_n, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n}.$$

From the properties of Brownian motion, $\Delta W_n \sim N(0, \Delta t)$. Therefore, we obtain

$$\begin{aligned} E(\Delta x) &= \left(r - \widehat{q} - \frac{\sigma^2}{2}\right)\Delta t, \\ \mathrm{Var}(\Delta x) &= \sigma^2\Delta t. \end{aligned} \tag{1}$$

In the binomial tree model,

$$\Delta x = \begin{cases} +\Delta x, & q_u, \\ -\Delta x, & q_d = 1 - q_u. \end{cases} \tag{2}$$

From (2) it follows that,

$$\begin{aligned} E(\Delta x) &= q_u\cdot\Delta x + q_d\cdot(-\Delta x) \\ &= (2q_u - 1)\,\Delta x, \\ \mathrm{Var}(\Delta x) &= 4q_u\,(1 - q_u)\,(\Delta x)^2. \end{aligned} \tag{3}$$

Comparing (1) and (3), we obtain

$$\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)\Delta t=(2q_u-1)\,\Delta x,$$
$$\sigma^2\Delta t=4q_u\,(1-q_u)\,(\Delta x)^2. \tag{4}$$

Let $\Delta x=\beta\sqrt{\Delta t}$, where $\beta>0$ is to be determined. Substituting this into (4):

$$2q_u-1=\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)\frac{\sqrt{\Delta t}}{\beta},$$
$$\beta^2=\sigma^2+\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)^2\Delta t.$$

From the above equation, we obtain

$$\beta=\sigma\left[1+\frac{1}{2\sigma^2}\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)^2\Delta t+O\left(\Delta t^2\right)\right].$$

Neglecting the $O(\Delta t)$ term, we get $\beta=\sigma+O(\Delta t)$. Neglecting higher-order infinitesimal terms of Δt, we have $\Delta x=\sigma\sqrt{\Delta t}$. Therefore, neglecting higher-order infinitesimal terms of Δt,

$$q_u=\frac{1}{2}\left[1+\frac{1}{\sigma}\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)\sqrt{\Delta t}\right],$$
$$q_d=\frac{1}{2}\left[1-\frac{1}{\sigma}\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)\sqrt{\Delta t}\right].$$

Furthermore, since the binomial tree equation for moving barrier option pricing can be expressed as:

$$u(x_m,t_n)=e^{-r\Delta t}\left[q_u\,u(x_{m+1},t_{n+1})+q_d\,u(x_{m-1},t_{n+1})\right]. \tag{5}$$

Substituting q_u and q_d into (5), and using $e^{-r\Delta t}=\frac{1}{1+r\Delta t}+O(\Delta t^2)$, we obtain:

$$u(x_m,t_n)=\frac{1}{1+r\Delta t}\left[\frac{1}{2}\left(1+\frac{\sqrt{\Delta t}}{\sigma}\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)\right)u(x_{m+1},t_{n+1})\right.$$
$$\left.+\frac{1}{2}\left(1-\frac{\sqrt{\Delta t}}{\sigma}\left(r-\widehat{q}-\frac{\sigma^2}{2}\right)\right)u(x_{m-1},t_{n+1})\right].$$

This is exactly the computational formula of the finite difference scheme for the moving barrier option when $\omega=\sigma^2\Delta t/(\Delta x)^2=1$. Thus, the theorem holds. ∎

5. Numerical Results

In this section, numerical examples are provided to validate the theoretical results established in the previous section. To this end, the following parameters are defined:

$$S_0=100,\quad K=100,\quad T=1\text{ year},\quad r=0.05,\quad q=0.02,\quad \sigma=0.2,\quad \alpha=0.1,\quad S_B(T)=100.$$

The computed results are presented in Table 1.

Table 1. Comparison of the C_{BTM} and C_{FDM} results.

N	Δt	C_{BTM}	C_{FDM}	$\lvert C_{\mathrm{BTM}}-C_{\mathrm{FDM}}\rvert$
20	$\frac{1}{20}$	0.0619056	0.0619095	3.86278×10^{-6}
40	$\frac{1}{40}$	0.0620674	0.0620693	1.93802×10^{-6}
80	$\frac{1}{80}$	0.0621446	0.0621456	9.70613×10^{-7}
160	$\frac{1}{160}$	0.0621836	0.0621841	4.8571×10^{-7}
320	$\frac{1}{320}$	0.0622032	0.0622034	2.42956×10^{-7}
640	$\frac{1}{640}$	0.0622129	0.062213	1.21503×10^{-7}

The table lists the initial option prices obtained by the binomial tree method (C_{BTM}) and the explicit finite difference scheme (C_{FDM}) under different time steps N, along with their absolute differences $|C_{\mathrm{BTM}} - C_{\mathrm{FDM}}|$. The results show that as Δt decreases, the computed values from the two methods rapidly converge. When $N = 640$, the difference reduces to the order of 10^{-7}, which fully verifies that, in the sense of neglecting higher-order infinitesimals, the binomial tree method and the explicit finite difference scheme are equivalent for pricing moving barrier options.

6. Conclusion

In this paper, we have investigated the relationship between two fundamental numerical methods for pricing moving barrier options: the binomial tree method and the explicit finite difference scheme. After transforming the time-dependent barrier problem into a constant-coefficient PDE via an exponential barrier assumption, we derived the explicit finite difference scheme and showed that, when the grid parameters satisfy $\omega = \frac{\sigma^2 \Delta t}{(\Delta x)^2} = 1$, its recursive formula coincides with that of the binomial tree method up to higher-order terms in Δt. The theoretical results are verified by numerical experiments.

The main contribution of this work is to establish a rigorous link between two widely used numerical techniques for a class of exotic options with moving barriers. This equivalence not only enhances the theoretical understanding but also provides a practical tool for cross-validation, as either method can be used with confidence, knowing that they produce equivalent results under the stated assumptions. Another interesting direction is to investigate the equivalence between finite element methods and finite difference schemes for moving barrier options, following a similar approach. Finally, the stability and convergence properties of the explicit scheme under the condition $\omega = 1$ deserve further study.

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Чисельні методи оцінки опціонів із рухомим бар'єром

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Опціони з рухомим бар'єром є класом залежних від часу бар'єрних опціонів, оцінювання вартості яких є відносно складним. Зосереджуючи увагу на опціоні "даун-ендаут" кол (із нижнім бар'єром вибивання), у цій роботі розроблено скінченно-різницеву схему для таких опціонів із рухомим бар'єром та використано метод біноміального дерева для розв'язання задачі ціноутворення. Крім того, встановлено еквівалентність між цими двома чисельними методами. Для підтвердження теоретичних результатів також проведено чисельні експерименти.

Ключові слова: *оцінка опціонів з рухомим бар'єром; скінченно-різницевий метод; метод біноміального дерева; модель Блека–Шоулза.*