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GUARANTEED PURSUIT AND EVASION TIMES IN HILBERT SPACE DIFFERENTIAL GAMES

(*Masa Terjamin untuk Pengejaran dan Pengelakan dalam Permainan Pembezaan Ruang Hilbert*)

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ABSTRACT

We analyze a differential game involving two competitors - a pursuing player and an evading player - who control the state of a dynamic system evolving in the Hilbert space ℓ_2. The system dynamics are governed by an infinite system of first-order ordinary differential equations (ISFODE), and the players' controls are geometrically constrained, with the pursuing player holding a significant control resource advantage over the evading player. The pursuing player aims to drive the system's state to the zero state of ℓ_2 after some time, whereas the evading player's objective is to keep the state away from the zero state indefinitely. We derive an explicit expression for the guaranteed pursuit time and construct a strategy that ensures pursuit completion. Similarly, we establish a formula for the guaranteed evasion time and identify a corresponding evasion strategy. Moreover, we provide necessary and sufficient conditions on the players' control resources limit under which these guaranteed times exist.

Keywords: pursuit-evasion; differential game; infinite system of differential equations; guaranteed pursuit time; guaranteed evasion time; control; strategy

ABSTRAK

Kami menganalisis sebuah permainan pembezaan antara dua pesaing - pemain pengejar dan pemain pengelak - yang mengawal evolusi keadaan sistem dinamik dalam ruang Hilbert ℓ_2. Dinamika sistem dikawal oleh sistem tak terhingga persamaan pembezaan biasa orde pertama (ISFODE), dan kawalan pemain dihadkan secara geometri, di mana pemain pengejar mempunyai kelebihan sumber kawalan yang ketara berbanding pemain pengelak. Pemain pengejar bertujuan memacu keadaan sistem ke keadaan sifar dalam ℓ_2 selepas suatu tempoh tertentu, manakala pemain pengelak berusaha memastikan keadaan sistem kekal menjauhi keadaan sifar tanpa had masa. Kami menurunkan ungkapan eksplisit bagi masa pengejaran terjamin dan membina strategi yang memastikan pengejaran selesai. Begitu juga, kami memperoleh formula bagi masa pengelakan terjamin serta mengenal pasti strategi pengelakan yang sepadan. Selain itu, kami mengemukakan syarat perlu dan mencukupi ke atas had sumber kawalan pemain di mana masa-masa terjamin ini wujud.

Kata kunci: pengejaran-pengelakan; permainan pembezaan; sistem tak terhingga persamaan pembezaan; masa pengejaran terjamin; masa pengelakan terjamin; kawalan; strategi

1. Introduction

Numerous studies have focused on pursuit-evasion differential games in spaces of finite dimension like the Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$ (see, for instance, Azamov and Ibaydullayev (2021); Azamov & Holboyev (2024); Azamov *et al.* (2024); Badakaya *et al.* (2024); Dorothy *et al.* (2024); Fu *et al.* (2025); Haruna *et al.* (2023); Huang *et al.* (2025); Li *et al.* (2025); Lozano *et al.* (2024); Rilwan *et al.* (2023); Wang *et al.* (2025); Wei *et al.* (2025); Xu *et al.* (2025); Zhang *et al.* (2025)). Similarly, considerable attention has been given to pursuit-evasion differential games in spaces of infinite dimension (see Aminov and Ruziboev (2024); Badakaya *et al.* (2021, 2025); Ibragimov *et al.* (2022a, b); Kazimirova *et al.* (2024); Sharifi *et al.* (2022)).

Satimov and Tukhtasinov (2006) explored pursuit-evasion differential games with parabolic-type partial differential equations (PDEs), where player control inputs are added as terms on the equations' right-hand side. They analyzed four games with different combinations of control constraints (geometric, integral) and identified distinct initial state sets: one ensures pursuit, while the other allows evasion. The works of Allahabi and Mahiub (2019) and Satimov and Tukhtasinov (2006) cover differential games governed by PDEs.

Kazimirova *et al.* (2024) analyzed an evasion differential game in the Hilbert space ℓ_2, involving one evading player and several (m) pursuing players. The game is modelled by m infinite systems of binary first-order ODEs, where the state variables represent the system's dynamics. The objective of the m pursuing players is to achieve a finite-time transition of at least one of the state of the m systems to the origin of ℓ_2, while the evading player aims to keep all the m states away from the origin. The players' control actions are geometrically constrained in the study. The authors derived sufficient conditions based on the players' control resource bounds and constructed explicit evasion strategy to ensure evasion.

Ibragimov *et al.* (2022b) considered a finite-time optimal pursuit-evasion differential game involving a pursuing player versus an evading player with integrally constrained control functions. The authors found the optimal time for pursuit in the game and explicitly constructed optimal strategies for the pursuing player and the evading player under the condition that the pursuing player's control resources exceed those of the evading player. Furthermore, Azamov and Ruziboyev (2013) addressed the time-optimal control problem for an evolution-type distributed-parameter system (DPS) and derived an upper bound for the optimal time needed to steer the system to the zero state.

2. Motivation

Early work on the time-optimal control of PDEs was initiated in Fattorini (1964). A comprehensive treatment of optimal control problems for DPS can be found in Fursikov (2000). The pioneering studies addressing differential game problems governed by PDEs are presented in Lions (1968) and Osipov (1975). In the analysis of control and differential game problems for DPS, the decomposition technique has become a widely adopted tool. This technique reformulates the original problem as one modelled by an infinite system of ODEs (for example, see Allahabi and Mahiub (2019); Azamov and Ruziboyev (2013); Ibragimov *et al.* (2022a, b); Kazimirova *et al.* (2024); Satimov and Tukhtasinov (2006; 2007)).

Indeed, consider a controlled system modelled by the parabolic-type PDE below:

$$\frac{\partial z}{\partial t} + Bz = \omega, \quad z(x,\,0) = z^0(x),\; x \in \Omega, \quad z\big|_{S_\tau} = 0, \tag{1}$$

where $x = (x_1,\, x_2,\, \ldots,\, x_n) \in \Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, z is the unknown spatiotemporal function, $0 \le t \le \tau$, Ω is a bounded region with a piecewise smooth boundary Γ, $\tau > 0$ is arbitrarily given, $z^0(x) \in L_2(\Omega)$, and $\omega = \omega(x,\,t) \in L_2(Q_\tau)$ is the control function, where

$$Q_\tau = \{(x,\,t) \mid x \in \Omega,\; 0 < t < \tau\}$$

is an open cylindrical domain in $\mathbb{R}^{n+1}$ with

$$S_\tau = \{(x,\,t) \mid x \in \Gamma,\; 0 < t < \tau\}$$

as its lateral surface,

$$Bz = -\sum_{j,\,k=1}^{n} \frac{\partial}{\partial x_j}\left(b_{jk}(x)\frac{\partial z}{\partial x_k}\right), \quad b_{jk}(x) = b_{kj}(x),$$

and $b_{jk}(\cdot)$ is a bounded and measurable function of its argument. Moreover, there exists $\mu > 0$ satisfying

$$\sum_{j,\,k=1}^{n} b_{jk}(x)\chi_j\chi_k \geq \mu \sum_{j=1}^{n} \chi_j^2, \quad \text{for every } (\chi_1,\, \chi_2,\, \ldots,\, \chi_n) \in \mathbb{R}^n \text{ and } x \in \Omega.$$

Furthermore, recall the following: $W_2^1(\Omega)$ is the Hilbert space comprising functions in $L_2(\Omega)$ whose first-order generalized (distributional) derivatives are also square-integrable over Ω; $\mathring{W}_2^1(\Omega)$ is defined as the closure of the set of smooth functions with compact support in the W_2^1-norm, making these smooth, compactly supported functions dense in $\mathring{W}_2^1(\Omega)$; $W_2^{1,0}(Q_\tau)$ consists of functions in $L_2(Q_\tau)$ that possess square-integrable (with respect to Q_τ) generalized partial derivatives $\frac{\partial z}{\partial x_j}$ (for $j = 1, 2, \ldots, n$); the subspace $\mathring{W}_2^{1,0}(Q_\tau)$ is then the closure in $W_2^{1,0}(Q_\tau)$ of those smooth functions that vanish in a neighborhood of S_τ (Ladyzhenskaya 1985). The spaces $L_2(\Omega)$ and $W_2^1(\Omega)$ have inner products defined respectively by the formulas

$$\langle u,\, v\rangle_{L_2(\Omega)} = \int_\Omega u(x)v(x)dx, \quad \langle u,\, v\rangle_{W_2^1(\Omega)} = \int_\Omega \left[u(x)v(x) + \nabla u(x)\cdot\nabla v(x)\right] dx,$$

with respective associated norms defined by

$$\|u\|_{L_2(\Omega)} = \sqrt{\langle u,\, u\rangle_{L_2(\Omega)}}, \quad \|u\|_{W_2^1(\Omega)} = \sqrt{\langle u,\, u\rangle_{W_2^1(\Omega)}}.$$

In the above, $\nabla u(x)\cdot\nabla v(x)$ denotes the standard inner product in $\mathbb{R}^n$, that is,

$$\nabla u(x)\cdot\nabla v(x) = \sum_{j=1}^{n} \frac{\partial u(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_j}.$$

In Ladyzhenskaya (1985), it was shown that, under the stated constraints, problem (1) possesses a unique generalized (in distributional sense) solution $z(x,t)$ within the space $\mathring{W}_2^{1,0}(Q_\tau)$ for every $\omega(x,t) \in L_2(Q_\tau)$ and $z^0(x) \in L_2(\Omega)$. Moreover, such a solution takes the form (see (Ladyzhenskaya 1985, Chap. III, Sec. 3)):

$$z(x,\, t) = \sum_{k=1}^{\infty} z_k(t)\phi_k(x) \tag{2}$$

where the functions $z_k(t)$, $t \in [0,\, \tau]$, $k = 1,\, 2,\, \ldots$, are solutions to the Cauchy problem for the ISFODE and initial conditions

$$\dot{z}_k + \alpha_k z_k = \omega_k(t), \quad z_k(0) = z_k^0, \quad k = 1,\, 2,\, \ldots, \tag{3}$$

and $\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_k,\, \ldots$, are generalized eigenvalues of the operator B (Chernousko 1992), which are all positive, and

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq \cdots, \quad \alpha_k \to +\infty \text{ as } k \to \infty.$$

Furthermore, the functions $\phi_1(x),\, \phi_2(x),\, \ldots,\, \phi_k(x),\, \ldots$ form a complete (in $L_2(\Omega)$) orthonormal system of generalized eigenfunctions of B; $z_k(0) = z_k^0$ and $\omega_k(t)$ are respectively, the

Fourier coefficients of $z^0(x)$ and $\omega(x,\, t)$ in the system $\{\phi_k(x)\}$, that is,

$$z(x,\, 0) = z^0(x) = \sum_{k=1}^{\infty} z_k(0)\phi_k(x), \quad \omega(x,\, t) = \sum_{k=1}^{\infty} \omega_k(t)\phi_k(x).$$

Additionally, the series in Eq. (2) converges in the $L_2(Q_\tau)$ norm and, for every $t \in [0, \tau]$, its sum $z(x, t)$ lies in the space $\mathring{W}_2^1(\Omega)$ and is a continuous (in the norm of $\mathring{W}_2^1(\Omega)$) function of t (Ladyzhenskaya 1985).

Therefore, a significant connection exists between control problems modelled by PDEs as in Eq. (1) and those modelled by an ISFODE as in Eq. (3). For instance, the studies in Allahabi and Mahiub (2019), Ibragimov *et al.* (2022a), Satimov and Tukhtasinov (2006) and Tukhtasinov (1995) considered a differential game governed by a linear PDE having the form

$$\frac{\partial z}{\partial t} = Bz + u - v, \quad Bz = \sum_{j,\, k=1}^{n} \frac{\partial}{\partial x_j}\left(b_{jk}(x)\frac{\partial z}{\partial x_k}\right), \quad b_{jk}(x) = b_{kj}(x),$$

where u and v respectively represent the pursuing player's and the evading player's control inputs, and the function $z(x,\, t)$ is scalar-valued. A decomposition technique was then applied to transform this game into one modelled by the ISFODE:

$$\dot{z}_k = \alpha_k z_k + u_k(t) - v_k(t), \quad k = 1,\, 2,\, \ldots, \tag{4}$$

where the variables $z_k,\, u_k,\, v_k$ are real numbers, u_k and v_k respectively represent the kth component of the pursuing player's and the evading player's control inputs, and the coefficients α_k, $k = 1,\, 2,\, \ldots$ satisfy

$$0 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \cdots \to -\infty.$$

Allahabi and Mahiub (2019), Ibragimov *et al.* (2022a), Satimov and Tukhtasinov (2006) and Tukhtasinov (1995) propose examining differential game problems formulated as an ISFODE (4) within a unified theoretical framework that is independent of PDEs; in this approach, the parameters α_k (for $k = 1,\, 2,\, \ldots$) in (4) are allowed to be any real numbers.

For these games modelled using ISFODE, evasion game problems (Ibragimov & Hasim 2010) and pursuit game problems (Ibragimov *et al.* 2022a) have been studied for α_k (in Eq. (5)) a sequence of negative numbers.

For instance, Ibragimov *et al.* (2022a) studied a two-player differential game modelled by the ISFODE:

$$\dot{z}_k = \alpha_k z_k + u_k - v_k, \quad z_k(0) = z_k^0, \quad k = 1,\, 2,\, \ldots, \tag{5}$$

where the variables $z_k,\, u_k,\, v_k$ are real numbers, the sequence $(\alpha_k)_{k\in\mathbb{N}}$ consists of negative numbers, the geometrically constrained control parameters for the pursuing player (respectively, evading player) is $u = (u_1,\, u_2,\, \ldots) \in \ell_2$ (respectively, $v = (v_1,\, v_2,\, \ldots) \in \ell_2$), and the initial state $z^0 \in \ell_2$, $z^0 \neq 0$. The authors obtained the guaranteed times for pursuit and evasion in the game.

In the current work, we examine a pursuit game modelled by Eq. (5) under the assumption that the sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ is bounded and nonnegative. We establish the existence of guaranteed times for pursuit and evasion in this game, explicitly construct permissible strategies for the players and find necessary and sufficient conditions on the players' control resources limit that yield these times.

3. Statement of the Problem

Recall that

$$\ell_2 = \left\{ \beta = (\beta_1,\, \beta_2,\, \ldots),\, \beta_k \in \mathbb{R} \mid \sum_{k=1}^{\infty} \beta_k^2 < \infty \right\}$$

is a Hilbert space endowed with the standard inner product and norm given by

$$\langle \beta,\, \gamma \rangle_{\ell_2} = \sum_{k=1}^{\infty} \beta_k \gamma_k \quad \text{and} \quad \|\beta\|_{\ell_2} = \sqrt{\sum_{k=1}^{\infty} \beta_k^2},$$

respectively, where $\beta,\, \gamma \in \ell_2$.

Let the evolution of the state $z(t) = (z_1(t),\, z_2(t),\, \ldots)$ of a dynamic system in the ℓ_2 space - controlled by a pursuing player and an evading player - be modelled by the ISFODE

$$\dot{z}_k = \alpha_k z_k + u_k - v_k, \quad z_k(0) = z_k^0, \quad k = 1,\, 2,\, \ldots, \tag{6}$$

where $z_k,\, u_k,\, v_k \in \mathbb{R}$ and $(\alpha_k)_{k\in\mathbb{N}}$ is a bounded sequence of nonnegative numbers. The control inputs for the pursuing player and the evading player are, respectively,

$$u = (u_1,\, u_2,\, \ldots) \in \ell_2 \quad \text{and} \quad v = (v_1,\, v_2,\, \ldots) \in \ell_2.$$

Throughout the work, we consider $t \in [0,\, \tau]$ for sufficiently large $\tau > 0$), which may be chosen arbitrarily large when needed.

The system in Eq. (6) has a nonzero initial state given by $z^0 = (z_1^0,\, z_2^0,\, \ldots) \in \ell_2$.

Let $\varrho,\, \sigma > 0$. The following definitions provide a foundational framework for this study.

Definition 3.1. A permissible control for the pursuing player is a vector-valued function

$$u : [0,\, \tau] \to \ell_2, \quad u(t) = (u_1(t), u_2(t), \ldots),$$

where each component function $u_k(t)$ is measurable, and the following geometric constraint holds:

$$\|u(t)\|_{\ell_2} \leq \varrho, \quad t \in [0,\, \tau].$$

Definition 3.2. A permissible control for the evading player is a vector-valued function

$$v : [0,\, \tau] \to \ell_2, \quad v(t) = (v_1(t), v_2(t), \ldots),$$

where each component function $v_k(t)$ is measurable, and the following geometric constraint is satisfied:

$$\|v(t)\|_{\ell_2} \leq \sigma, \quad t \in [0,\, \tau].$$

Remark 3.3. The quantity ϱ^2 (respectively, σ^2) is known as the pursuing player's (respectively, evading player's) instantaneous control resources limit.

In this study, we assume that $\varrho > \sigma$.

Definition 3.4. A strategy for the pursuing player is a function

$$U : [0,\, \tau] \times \ell_2 \to \ell_2, \quad U(t,\, v(t)) = \big(U_1(t,\, v_1(t)),\, U_2(t,\, v_2(t)),\, \ldots\big),$$

where for every permissible evading player control $v(\cdot)$ with $v(t) = (v_1(t), v_2(t), \ldots) \in \ell_2$, each component function $U_k(t,\, v_k(t))$ is Borel measurable and, for all $t \in [0,\, \tau]$, the strategy satisfies the constraint

$$\|U(t,\, v(t))\|_{\ell_2} \leq \varrho.$$

In this formulation, the pursuing player is assumed to have instantaneous knowledge of the evading player's control input $v(t)$ at each time t. This information is utilized to determine the pursuing player's strategy $U(t, v(t))$. The pursuit commences at the time $t = 0$ from the initial state $z(0) \equiv z^0 = (z_1^0,\, z_2^0,\, \ldots) \in \ell_2$.

Since the control parameters u_k and v_k in Eq. (6) are time-dependent, we express them as measurable functions $u_k(t)$ and $v_k(t)$ for $t \in [0,\, \tau]$. Then, the unique solution of the ISFODE in Eq. (6) is given by the state $z(t) = (z_1(t),\, z_2(t),\, \ldots)$, where

$$z_k(t) = e^{\alpha_k t} z_k^0 + \int_0^t e^{\alpha_k (t-s)} (u_k(s) - v_k(s)) ds, \quad k = 1,\, 2,\, \ldots.$$

Let $C([0,\, \tau], \ell_2)$ denote the space of functions $f : [0,\, \tau] \to \ell_2$ that are continuous with respect to the ℓ_2-norm. It is known (Odiliobi *et al.* 2024) that the solution $z(t)$ exists uniquely in the space $C([0,\, \tau],\, \ell_2)$. Consequently, for the game problems, we analyze the unique trajectory $z(t)$ within $C([0,\, \tau], \ell_2)$.

An alternate representation of the solution is given by

$$z_k(t) = e^{\alpha_k t} \left(z_k^0 + \int_0^t e^{-\alpha_k s} \big(u_k(s) - v_k(s)\big) ds \right), \quad k = 1,\, 2,\, \ldots. \tag{7}$$

Defining $y(t) = (y_1(t),\, y_2(t),\, \ldots)$ where

$$y_k(t) = z_k^0 + \int_0^t e^{-\alpha_k s} \big(u_k(s) - v_k(s)\big) ds, \quad k = 1,\, 2,\, \ldots, \tag{8}$$

we observe that $z(t) = 0$ if and only if $y(t) = 0$.

In the differential game, the pursuing player aims for a finite-time transfer of the system's state $z(t)$ to the zero state of ℓ_2, while the evading player's objective is to obstruct this outcome by keeping the state away from the origin for as long as possible. If the pursuing player's objective is achieved, we say that the pursuit has been completed.

Definition 3.5. We refer to the number τ_p, where $\tau_p \leq \tau$, as a guaranteed time for pursuit in the game in Eq. (6) if we can find a strategy $U^0(t,\, v(t))$ for the pursuing player such that $z(\theta) = 0$ (or equivalently, $z_k(\theta) = 0$ for all $k = 1,\, 2,\, \ldots$) at some instant $\theta \in [0,\, \tau_p]$ for every permissible control $v(t)$ of the evading player, where $z(t) = (z_1(t),\, z_2(t),\, \ldots)$ is the solution to the IVP in Eq. (6) at $u(t) = U^0(t,\, v(t))$, $t \in [0,\, \tau]$.

Observe that within the interval $[0,\, \tau_p]$, the pursuing player adopts the strategy $U^0(t,\, v(t))$, while the evading player operates under an arbitrary control $v(t)$. The pursuing player's interest is to minimize τ_p and hence, complete the pursuit as soon as possible. On the other hand, the evading player's interest is to maximize τ_p and hence, delay pursuit completion for as long as possible.

For pursuit completion, the questions guiding the research are

- Can an explicit formula be derived for the guaranteed time for pursuit τ_p for any nonzero initial state z^0?
- Can a permissible strategy that guarantees pursuit completion at τ_p be formulated for the pursuing player?

Definition 3.6. We call the number $\tau_e \in (0, +\infty]$, a guaranteed time for evasion in the game in Eq. (6) if for any finite time ϑ, $\vartheta \in [0, \tau_e)$, we can find a permissible evading player's control $v^0(t)$ such that, for any permissible control $u(t)$ of the pursuing player, the solution $z(t) = (z_1(t), z_2(t), \ldots)$ of the IVP in Eq. (6) remains nonzero for all $t \in [0, \vartheta]$. In the case where $\tau_e = +\infty$, this means that for any arbitrarily large finite time $T \geq 0$, the evading player has a strategy to keep $z(t) \neq 0$ for all $t \in [0, T]$.

Observe that on the interval $[0, \tau_e)$, the evading player adopts the control $v^0(t)$, while the pursuing player uses an arbitrary control $u(t)$. The evading player's interest is to maximize the guaranteed time for evasion. For evasion, the questions guiding the research are

- For any nonzero initial state z^0, can we obtain a guaranteed time for evasion τ_e?
- Is it possible to formulate a permissible control for the evading player that yields this guaranteed time for evasion τ_e?

4. Guaranteed Time for Pursuit

Here, we prove a theorem concerning the guaranteed time for pursuit τ_p. The proof involves formulating a permissible strategy for the pursuing player, such that the state $z(t)$ reaches the origin at time τ_p, regardless of any permissible control chosen by the evading player. The theorem that follows embodies the key finding in this section.

Theorem 4.1. *The number*

$$\tau_p = \theta = \begin{cases} \frac{\|z^0\|}{\varrho-\sigma}, & I_0 = \emptyset, \\ -\frac{1}{\alpha}\ln\left(1 - \frac{\alpha\|z^0\|}{\varrho-\sigma}\right), & I_0 \neq \emptyset, \end{cases}, \tag{9}$$

where $I_0 = \{k \mid \alpha_k > 0\}$, $\alpha = \sup_{k\in\mathbb{N}} \alpha_k$, is a guaranteed time for pursuit in the game (6). *Moreover, if $I_0 \neq \emptyset$, then τ_p is finite only if $\varrho > \sigma + \alpha\|z^0\|$.*

Proof. Note that since $(\alpha_k)_{k\in\mathbb{N}}$ is a bounded sequence of nonnegative real numbers, its supremum α is finite. We provide the pursuing player with the strategy

$$U^0(t, v(t)) = (U_1^0(t, v_1(t)), U_2^0(t, v_2(t)), \ldots),$$

where

$$U_k^0(t, v_k(t)) = \begin{cases} v_k(t) - (\varrho-\sigma)\frac{z_k^0}{\|z^0\|}, & \text{if } t \in [0, \theta_k], \\ v_k(t), & \text{if } t > \theta_k, \end{cases}, \qquad k = 1, 2, \ldots, \tag{10}$$

and

$$\theta_k = \begin{cases} \frac{\|z^0\|}{\varrho-\sigma}, & \alpha_k = 0, \\ -\frac{1}{\alpha_k}\ln\left(1 - \frac{\alpha_k\|z^0\|}{\varrho-\sigma}\right), & \alpha_k > 0. \end{cases} \tag{11}$$

Here, $v_k(t)$ represents the kth component of a permissible control $v(t) = (v_1(t), v_2(t), \ldots)$ chosen by the evading player. To verify the permissibility of the strategy in Eq. (10), we proceed as follows.

For each coordinate k, define the index sets

$$I_1(t) = \{k : t \in [0, \theta_k]\} \quad \text{and} \quad I_2(t) = \{k : t > \theta_k\}.$$

Then, the strategy is given by

$$U_k^0(t, v_k(t)) = \begin{cases} v_k(t) - (\varrho - \sigma)\dfrac{z_k^0}{\|z^0\|}, & k \in I_1(t), \\ v_k(t), & k \in I_2(t). \end{cases}$$

Hence, the overall ℓ_2-square norm of $U(t, v(t))$ is

$$\begin{aligned} \|U^0(t, v(t))\|_{\ell_2}^2 &= \sum_{k\in I_1} \left| v_k(t) - (\varrho - \sigma)\frac{z_k^0}{\|z^0\|} \right|^2 + \sum_{k\in I_2} |v_k(t)|^2 \\ &= \sum_{k\in I_1} \left[|v_k(t)|^2 + (\varrho - \sigma)^2 \frac{|z_k^0|^2}{\|z^0\|^2} - \frac{2(\varrho - \sigma)}{\|z^0\|} v_k(t) z_k^0 \right] + \sum_{k\in I_2} |v_k(t)|^2 \\ &= \sum_{k=1}^{\infty} |v_k(t)|^2 + (\varrho - \sigma)^2 \sum_{k\in I_1} \frac{|z_k^0|^2}{\|z^0\|^2} + \frac{2(\varrho - \sigma)}{\|z^0\|} \sum_{k\in I_1} \left(-v_k(t) z_k^0\right). \end{aligned} \tag{12}$$

Note from Eq. (12) that

$$\sum_{k=1}^{\infty} |v_k(t)|^2 = \|v(t)\|_{\ell_2}^2 \le \sigma^2, \quad \sum_{k\in I_1} \frac{|z_k^0|^2}{\|z^0\|^2} \le 1,$$

and by Cauchy-Schwarz inequality,

$$\left| \sum_{k\in I_1} \left(-v_k(t) z_k^0\right) \right| \le \|v(t)\|_{\ell_2} \|z^0\|_{\ell_2} \le \sigma \|z^0\|.$$

Hence, using the above in Eq. (12), we obtain the bound

$$\|U^0(t,\, v(t))\|_{\ell_2}^2 \le \sigma^2 + (\varrho - \sigma)^2 + 2(\varrho - \sigma)\sigma = \varrho^2.$$

Hence, $\|U^0(t,\, v(t))\|_{\ell_2} \le \varrho$. Therefore, the permissibility condition is verified.

Next, we show that θ as defined in the theorem is a guaranteed time for pursuit. We start by showing that

$$z_k(t) = 0, \quad \text{for all } t \ge \theta_k \text{ and } k = 1,\, 2,\, \ldots. \tag{13}$$

Indeed, setting $u(s) = U^0(s,\, v(s))$ in Eq. (7) and using Eq. (10), we find that for any $k \in \mathbb{N}$,

$$\begin{aligned} z_k(\theta_k) &= e^{\alpha_k t} \left[z_k^0 + \int_0^{\theta_k} e^{-\alpha_k s} \left(U_k^0(s,\, v_k(s)) - v_k(s) \right) ds \right] \\ &= e^{\alpha_k t} \left[z_k^0 - (\varrho - \sigma)\frac{z_k^0}{\|z^0\|} \int_0^{\theta_k} e^{-\alpha_k s} ds \right]. \end{aligned} \tag{14}$$

But then, from our definition of θ_k in Eq. (11), we have that

$$\int_0^{\theta_k} e^{-\alpha_k s} ds = \frac{\|z^0\|}{\varrho - \sigma}.$$

Using this in Eq. (14) yields

$$z_k(\theta_k) = 0, \quad k = 1, 2, \dots .$$

Next, we show that $z_k(t) = 0$ (or equivalently, $y_k(t) = 0$) for $t > \theta_k$. It is sufficient to show that for $k = 1, 2, \dots$,

$$y_k(t) = z_k^0 + \int_0^t e^{-\alpha_k s}\big(U_k^0(s, v_k(s)) - v_k(s)\big)ds = 0, \quad t > \theta_k.$$

Of course, since $y_k(\theta_k) = 0$ and $U_k^0(t, v_k(t)) = v_k(t)$ for $t > \theta_k$, we find that

$$\begin{aligned} y_k(t) &= z_k^0 + \int_0^t e^{-\alpha_k s}\big(U_k^0(s, v_k(s)) - v_k(s)\big)ds \\ &= y_k(\theta_k) + \int_{\theta_k}^t e^{-\alpha_k s}\big(v_k(s) - v_k(s)\big)ds = 0. \end{aligned}$$

Thus, we have established Eq. (13), and the number

$$\tau_p = \sup_{k\in\mathbb{N}} \theta_k = \sup_{k\in\mathbb{N}} \begin{cases} -\frac{1}{\alpha_k} \ln\left(1 - \frac{\alpha_k\|z^0\|}{\varrho-\sigma}\right), & \alpha_k > 0, \\ \frac{\|z^0\|}{\varrho-\sigma}, & \alpha_k = 0, \end{cases}$$

is a guaranteed time for pursuit in the game in Eq. (6), since from Eq. (13), we find that $z_k(\tau_p) = 0$, for all $k = 1, 2, \dots$.

What remains is to establish that

$$\tau_p = \theta = \begin{cases} \frac{\|z^0\|}{\varrho-\sigma}, & I_0 = \emptyset, \\ -\frac{1}{\alpha} \ln\left(1 - \frac{\alpha\|z^0\|}{\varrho-\sigma}\right), & I_0 \neq \emptyset, \end{cases} \tag{15}$$

where $\alpha = \sup_{k\in\mathbb{N}} \alpha_k$.

For each coordinate k, define the index sets

$$I_3 = \{k : \alpha_k = 0\} \quad \text{and} \quad I_4 = \{k : \alpha_k > 0\}.$$

If $\alpha_k = 0$ for all $k = 1, 2, \dots$, then I_4 is empty and

$$\tau_p = \theta = \sup_{k\in\mathbb{N}} \theta_k = \sup_{k\in I_3} \theta_k = \frac{\|z^0\|}{\varrho - \sigma}.$$

On the other hand, suppose at least one $j \in \mathbb{N}$ exists such that $\alpha_j > 0$. Then, I_4 is not empty, $\alpha > 0$ and since

$$-\frac{1}{\alpha} \ln\left(1 - \frac{\alpha\|z^0\|}{\varrho - \sigma}\right) > \frac{\|z^0\|}{\varrho - \sigma},$$

we have

$$\tau_p = \theta = \sup_{k\in\mathbb{N}} \theta_k = \sup_{k\in I_4} \theta_k = \sup_{k\in I_4} \left\{ -\frac{1}{\alpha_k} \ln\left(1 - \frac{\alpha_k\|z^0\|}{\varrho-\sigma}\right)\right\}.$$

The final step is to establish that if I_4 is not empty, then

$$\tau_p = \theta = \sup_{k\in I_4} \left\{ -\frac{1}{\alpha_k} \ln\left(1 - \frac{\alpha_k\|z^0\|}{\varrho-\sigma}\right)\right\} = -\frac{1}{\alpha} \ln\left(1 - \frac{\alpha\|z^0\|}{\varrho-\sigma}\right), \tag{16}$$

where $\alpha = \sup_{k\in\mathbb{N}} \alpha_k = \sup_{k\in I_4} \alpha_k > 0$. To establish this, we consider the function

$$\varphi(x) = -\frac{\ln(1-\gamma x)}{x}, \quad 0 < x < \frac{1}{\gamma},$$

where we set $\gamma = \frac{\|z_0\|}{\varrho-\sigma} > 0$, for brevity. Clearly, $\varphi'(x) = \frac{1}{x^2}\psi(x)$, where

$$\psi(x) = \frac{\gamma x}{1-\gamma x} + \ln(1-\gamma x), \quad 0 < x < \frac{1}{\gamma},$$

and

$$\psi'(x) = \frac{\gamma^2}{(1-\gamma x)^2} x > 0, \quad 0 < x < \frac{1}{\gamma}.$$

Thus, the function $\psi(x)$ is strictly increasing. Since ψ is strictly increasing and $\lim_{x\to 0^+} \psi(x) = 0$, we have $\psi(x) > 0$, $0 < x < \frac{1}{\gamma}$. This implies that $\varphi'(x) > 0$, $0 < x < \frac{1}{\gamma}$. As a result, $\varphi(x)$ is increasing, and Eq. (16) holds. Hence, θ is uniquely determined by the formula in Eq. (15).

Now, suppose $I_0 \neq \emptyset$. Then τ_p is given by the second case in (15). Observe that the expression $-\frac{1}{\alpha}\ln(1 - \frac{\alpha\|z^0\|}{\varrho-\sigma})$ is finite if and only if $1 - \frac{\alpha\|z^0\|}{\varrho-\sigma} > 0$, or equivalently, $\varrho > \sigma + \alpha\|z^0\|$.$\square$

5. Guaranteed Time for Evasion

This section presents a theorem concerning the guaranteed time for evasion τ_e. The proof involves designing a permissible control strategy for the evading player, ensuring that for every permissible control chosen by the pursuing player, the trajectory of the state $z(t)$ avoids passing through the zero (i.e., origin) of ℓ_2 within the time interval $[0,\ \tau_e]$. The following theorem is then established.

Theorem 5.1. *Given any initial state $z^0 = (z_1^0,\ z_2^0,\ \ldots) \in \ell_2$, the number*

$$\tau_e = \vartheta = \sup_{k\in\mathbb{N}} \vartheta_k, \quad \vartheta_k = \begin{cases} \frac{|z_k^0|}{\varrho-\sigma}, & \alpha_k = 0, \\ -\frac{1}{\alpha_k}\ln\left(1 - \frac{\alpha_k|z_k^0|}{\varrho-\sigma}\right), & \alpha_k > 0 \quad \text{and} \quad \alpha_k|z_k^0| < \varrho-\sigma, \\ +\infty, & \alpha_k > 0 \quad \text{and} \quad \alpha_k|z_k^0| \geq \varrho-\sigma, \end{cases} \tag{17}$$

is a guaranteed time for evasion in the game in Eq. (6). *Moreover, if $\tau_e = +\infty$, then the evader can guarantee evasion indefinitely.*

Proof. Let $u = u(t)$ be an arbitrary permissible control adopted by the pursuing player. Suppose that ϑ' be an arbitrary time such that $\vartheta' \in (0,\ \vartheta)$. To establish the theorem, we design a permissible control for the evading player ensuring that $z(t) \neq 0$ for all $t \in (0, \vartheta')$. Of course, by the definition of ϑ, we can find an index $j \in \mathbb{N}$ such that $\vartheta' < \vartheta_j$. We demonstrate that we can find a permissible control for the evading player satisfying $z(t) \neq 0$ for all $t \in [0, \vartheta_j)$. Furthermore, note that $z_j^0 \neq 0$; otherwise, $\vartheta_j = 0$, which would contradict the positivity of ϑ_j. We offer the control $v^0(t) = (v_1^0(t), v_2^0(t), \ldots)$ to the evading player, where

$$v_k^0(t) = -q_j \sigma \, \delta_{jk}, \quad q_j = \frac{z_j^0}{|z_j^0|}, \quad k = 1, 2, \ldots, \quad t \in [0, \vartheta_j), \tag{18}$$

and δ_{jk} is the Kronecker delta function. Obviously,

$$\|v^0(t)\|_{\ell_2} = \left(\sum_{k=1}^{\infty} |v_k^0(t)|^2 \right)^{1/2} = |v_j^0(t)| = |-q\sigma| = \sigma,$$

since $|q_j| = 1$. Thus, the control $v^0(t)$ is permissible.

From Eqs. (7) and (8), $z(t) \neq 0$ if and only if $y(t) \neq 0$. Therefore, it is sufficient to demonstrate that for any permissible control $u(\cdot)$ of the pursuing player, the inequality $y(t) \neq 0$ holds for all $t \in [0, \vartheta_j)$. Of course,

$$\begin{aligned} y_j(t) &= z_j^0 + \int_0^t e^{-\alpha_j s}(u_j(s) - v_j(s))ds \\ &= q_j|z_j^0| + \int_0^t e^{-\alpha_j s}(u_j(s) + q_j\sigma)ds. \end{aligned} \tag{19}$$

Multiplying both sides by q_j (noting that $q_j = \pm 1$), we get

$$\begin{aligned} q_j y_j(t) &= |z_j^0| + \int_0^t e^{-\alpha_j s} q_j(u_j(s) + q_j\sigma)ds \\ &= |z_j^0| + \int_0^t e^{-\alpha_j s}(q_j u_j(s) + \sigma)ds. \end{aligned}$$

Since $|u_j(s)| \leq \|u(s)\| \leq \varrho$, we have $q_j u_j(s) \geq -|u_j(s)| \geq -\varrho$, so

$$q_j u_j(s) + \sigma \geq -(\varrho - \sigma).$$

Thus,

$$q_j y_j(t) \geq |z_j^0| - (\varrho - \sigma) \int_0^t e^{-\alpha_j s} ds. \tag{20}$$

Now, consider cases based on α_j:

- Case 1: $\alpha_j = 0$. Then, $\vartheta_j = \frac{|z_j^0|}{\varrho - \sigma}$ and $\int_0^t e^{-\alpha_j s} ds = t$. For $t \in [0, \vartheta_j)$, we have from Eq. (20) that:

  $$q_j y_j(t) \geq |z_j^0| - (\varrho - \sigma)t.$$

  Since $t < \vartheta_j = \frac{|z_j^0|}{\varrho - \sigma}$, it follows that $q_j y_j(t) > 0$.

- Case 2: $\alpha_j > 0$. Then, $\int_0^t e^{-\alpha_j s} ds = \frac{1-e^{-\alpha_j t}}{\alpha_j}$. So, from Eq. (20), we obtain

$$q_j y_j(t) \geq |z_j^0| - (\varrho - \sigma)\frac{1 - e^{-\alpha_j t}}{\alpha_j}. \tag{21}$$

  We now consider subcases based on the value of $\alpha_j|z_j^0|$ relative to $\varrho - \sigma$:

  - Subcase 2a: $\alpha_j|z_j^0| < \varrho - \sigma$. Then, $\vartheta_j = -\frac{1}{\alpha_j}\ln\left(1 - \frac{\alpha_j|z_j^0|}{\varrho-\sigma}\right)$. For $t < \vartheta_j$, we have

$$t < -\frac{1}{\alpha_j}\ln\left(1 - \frac{\alpha_j|z_j^0|}{\varrho - \sigma}\right) \Longleftrightarrow e^{-\alpha_j t} > 1 - \frac{\alpha_j|z_j^0|}{\varrho - \sigma}.$$

    Then, from Eq. (21):

$$q_j y_j(t) > |z_j^0| - \frac{\varrho - \sigma}{\alpha_j}\left(\frac{\alpha_j|z_j^0|}{\varrho - \sigma}\right) = 0.$$

    That is, $q_j y_j(t) > 0$.
  - Subcase 2b: $\alpha_j|z_j^0| \geq \varrho - \sigma$. Then, $\vartheta_j = +\infty$. For any finite t, we find from Eq. (21) that

$$q_j y_j(t) \geq |z_j^0| - \frac{\varrho - \sigma}{\alpha_j}(1 - e^{-\alpha_j t}) = \left(|z_j^0| - \frac{\varrho - \sigma}{\alpha_j}\right) + \frac{\varrho - \sigma}{\alpha_j}e^{-\alpha_j t}.$$

    Since $|z_j^0| - \frac{\varrho-\sigma}{\alpha_j} \geq 0$ and $\frac{\varrho-\sigma}{\alpha_j}e^{-\alpha_j t} > 0$, we get that $q_j y_j(t) > 0$.

In all cases, $q y_j(t) > 0$ for all $t \in [0, \vartheta_j)$, so $y_j(t) \neq 0$, and from Eq. (7), we must have that $z_j(t) \neq 0$. Consequently, we infer that $z(t) \neq 0$ for $t \in [0, \vartheta_j)$. Specifically, this implies that $z(t) \neq 0$ for $t \in [0, \vartheta']$.

If $\tau_e = +\infty$, then for any finite (but arbitrary) time T, we can find j such that $T < \vartheta_j$, and the same strategy in Eq. (18) ensures $z(t) \neq 0$ for all $t \in [0, T]$. Hence, the evader guarantees evasion indefinitely.

This concludes the proof. □

Remark 5.2. Let $\alpha > 0$, where τ_e is defined as in Eq. (17).

(a) $\tau_e < \infty$ if and only if $\sup_{k\in\mathbb{N}} \alpha_k|z_k^0| < \varrho - \sigma$. Equivalently, since $z^0 \in \ell_2$ and $\alpha < \infty$ imply $\alpha_k|z_k^0| \to 0$, the supremum is attained and one may write the criterion as $\max_{k\in\mathbb{N}} \alpha_k|z_k^0| < \varrho - \sigma$.

(b) $\tau_e = \infty$ if and only if there is an index k satisfying $\alpha_k|z_k^0| \geq \varrho - \sigma$ with $\alpha_k > 0$. In that case the k-th coordinate alone yields $\vartheta_k = +\infty$, hence $\tau_e = +\infty$.

6. Illustrative Examples

To enhance clarity, we present the following illustrative examples to examine the guaranteed times for pursuit and evasion.

Example 6.1. Consider the differential game described by Eq. (6) with the parameters $z^0 = (z_k^0)_{k\in\mathbb{N}}$ where $z_k^0 = 2^{-k/2}$, $\alpha_k = \frac{2k}{k+1}$, $k = 1, 2, \ldots$, and $\varrho - \sigma > 2$. We obtain τ_p and τ_e as specified in Theorems 4.1 and 5.1. Since $\|z^0\| = 1$ and $\alpha = \sup_{k\in\mathbb{N}} \alpha_k = 2$, the guaranteed time for pursuit is given by the formula

$$\tau_p = -\frac{1}{2}\ln\left(1 - \frac{2}{\varrho - \sigma}\right).$$

On the other hand, observe that the maximum value of $\alpha_k|z_k^0|$ occurs at $k = 1$, where $\alpha_1|z_1^0| = \frac{1}{\sqrt{2}}$. Since $\varrho - \sigma > 2$ and $\alpha_1|z_1^0| < 2$, it follows that $\alpha_k|z_k^0| < \varrho - \sigma$ for all k.

Now, observe from the equation

$$\tau_e = \sup_{k\in\mathbb{N}} \vartheta_k, \quad \vartheta_k = -\frac{1}{\alpha_k}\ln\left(1 - \frac{\alpha_k|z_k^0|}{\varrho - \sigma}\right),$$

that with the initial values given, $(\vartheta_k)_{k\in\mathbb{N}}$ take its supremum at $k = 1$. Thus, the guaranteed time for evasion is given by the formula

$$\tau_e = -\ln\left(1 - \frac{1}{\sqrt{2}(\varrho - \sigma)}\right).$$

Clearly, $\tau_p > \tau_e$.

Example 6.2. Consider the differential game modelled by Eq. (6) with the parameters $z^0 = (\beta, 0, 0, 0, \ldots) \in \ell_2$, $\beta > 0$, and the bounded sequence (α_k) be such that $\alpha_1 \geq \alpha_2 \geq \cdots > 0$, where $\varrho > \sigma + \alpha_1\beta$. Then, $\alpha = \sup_{k\in\mathbb{N}} \alpha_k = \alpha_1$, and from the formula in Eq. (9), the guaranteed time for pursuit becomes

$$\tau_p = -\frac{1}{\alpha_1}\ln\left(1 - \frac{\alpha_1\beta}{\varrho - \sigma}\right).$$

Next, observe that the maximum value of $\alpha_k|z_k^0|$ is $\alpha_1\beta$ and the condition $\varrho > \sigma + \alpha_1\beta$ ensures that $\alpha_k|z_k^0| < \varrho - \sigma$ for all k. From the formula in Eq. (17), we compute the guaranteed time for evasion. Since $\alpha_k > 0$ and $\alpha_k|z_k^0| < \varrho - \sigma$, then

$$\vartheta_k = -\frac{1}{\alpha_k}\ln\left(1 - \frac{\alpha_k|z_k^0|}{\varrho - \sigma}\right) \quad \text{and} \quad \alpha_k|z_k^0| = \begin{cases} \alpha_1\beta, & k = 1, \\ 0, & k > 1, \end{cases},$$

and so,

$$\vartheta_k = \begin{cases} -\frac{1}{\alpha_1}\ln\left(1 - \frac{\alpha_1\beta}{\varrho-\sigma}\right), & k = 1, \\ 0, & k > 1, \end{cases}; \quad \tau_e = \sup_{k\in\mathbb{N}} \vartheta_k = -\frac{1}{\alpha_1}\ln\left(1 - \frac{\alpha_1\beta}{\varrho - \sigma}\right).$$

Thus, for this example, we have $\tau_p = \tau_e$.

Example 6.3. Consider the differential game modelled by Eq. (6), with parameters defined as follows: $z^0 = (z_k^0)_{k\in\mathbb{N}}$, where $z_k^0 = \frac{1}{k}$; $(\alpha_k)_{k\in\mathbb{N}}$, where $\alpha_k = \frac{k}{k+1}$; $\varrho = 2$; and $\sigma = 1$. Observe that

$$\|z^0\| = \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} = \sqrt{\zeta(2)} = \frac{\pi\sqrt{6}}{6},$$

where $\zeta(\cdot)$ is the Riemann zeta function. This implies that $z^0 \in \ell_2$. Now, $\alpha = \sup_{k\in\mathbb{N}} \alpha_k = 1$. Moreover, $\varrho > \sigma$ but $\varrho \leq \sigma + \alpha\|z^0\|$.

Using Eq. (9), we find that the guaranteed time for pursuit (from the strategy in Eq. (10)) does not exist even though $\varrho > \sigma$.

Now, observe that $\alpha_k > 0$ and $\alpha_k|z_k^0| = \frac{1}{k+1} < \varrho - \sigma$ (since $\varrho - \sigma = 1$). Hence, from Eq. (17), the guaranteed time for evasion is given by the formula

$$\tau_e = \sup_{k\in\mathbb{N}} \vartheta_k, \quad \vartheta_k = \frac{k+1}{k} \ln\left(\frac{k+1}{k}\right)$$

from which we find that $\tau_e = \vartheta_1 = 2\ln 2$.

Example 6.4. Consider the differential game modelled by Eq. (6), with parameters defined as follows: $z^0 = (z_k^0)_{k\in\mathbb{N}}$, where $z_k^0 = \frac{1}{k}$; $(\alpha_k)_{k\in\mathbb{N}}$, where $\alpha_k = 1$; $\varrho = 2$; and $\sigma = 1$. Observe that $z^0 \in \ell_2$ since $\|z^0\| = \frac{\pi\sqrt{6}}{6}$. Now, $\alpha = \sup_{k\in\mathbb{N}} \alpha_k = 1$. Moreover, $\varrho > \sigma$ but $\varrho \leq \sigma + \alpha\|z^0\|$. From Eq. (9), we find that the guaranteed time for pursuit (from the strategy in Eq. (10)) does not exist even though $\varrho > \sigma$.

Now, observe that $\alpha_k > 0$ for all $k \in \mathbb{N}$.

For $k = 1$: $\alpha_1|z_1^0| = 1 = \varrho - \sigma$, so $\vartheta_1 = +\infty$;

For $k \geq 2$: $\alpha_k|z_k^0| = \frac{1}{k} < \varrho - \sigma$, so $\vartheta_k = -\ln(1 - \frac{1}{k})$, which is finite for all $k \geq 2$.

Thus, $\tau_e = \sup_{k\in\mathbb{N}} \vartheta_k = +\infty$.

Example 6.4 above demonstrates a pursuit-evasion game scenario where evasion is guaranteed indefinitely ($\tau_e = +\infty$) due to at least one coordinate (here, $k = 1$) satisfying $\alpha_k|z_k^0| \geq \varrho - \sigma$, while pursuit cannot be guaranteed in finite time.

7. Discussion

In Ibragimov *et al.* (2022a), guaranteed times for pursuit and evasion were derived for the game modelled by Eq. (6) under the assumption that the sequence $(\alpha_k)_{k\in\mathbb{N}}$ consists of negative numbers. In contrast, our study addresses the case where $(\alpha_k)_{k\in\mathbb{N}}$ is a bounded sequence of nonnegative numbers - a restriction imposed to ensure that the system's states remain in ℓ_2 and that the game is well-posed. For the game problems in Ibragimov *et al.* (2022a), the sufficient condition for the existence of guaranteed times for pursuit and evasion is $\varrho > \sigma$. However, in our setting with $\alpha > 0$, this condition is necessary but not sufficient (see Example 6.3). Specifically, when $\alpha > 0$, the uncontrolled system associated with Eq. (6) is unstable and drifts away from the origin; thus, to overcome this instability and drift, the pursuing player requires additional control resources, quantified by $\alpha\|z^0\|$ for the strategy in Eq. (10). Consequently and more precisely, the sufficient condition for pursuit completion using the strategy in Eq. (10) becomes $\varrho > \sigma + \alpha\|z^0\|$ (see Theorem 4.1). For more emphasis on how our research and findings compare with that of Ibragimov *et al.* (2022a), we present the following Table 1 below.

Example 6.2 demonstrates that if the initial state is of the form

$$z^0 = (\beta, 0, 0, \dots) \quad (\beta > 0),$$

that is, $z_1^0 = \beta$ and $z_k^0 = 0$ for $k \geq 2$, then the guaranteed time for pursuit equals the guaranteed time for evasion ($\tau_p = \tau_e$); in this case, τ_p is considered the optimal time for pursuit (see, for example, Ibragimov *et al.* (2022b)). More generally, if $\varrho > \sigma + \alpha\|z^0\|$ holds, then τ_p and τ_e serve as upper and lower bounds, respectively, on the optimal time for pursuit τ_o, so that $\tau_p = \tau_e$ implies optimality.

Table 1: Comparison of Geometric Constraint Pursuit-Evasion Game Models in Hilbert Space: Unstable vs Stable Dynamics

No.	Comparison Criteria	This Research	Ibragimov *et al.* (2022a)
1.	System Coefficient and Stability	$0 \leq \alpha_k \leq \alpha$, $\alpha_k = 0$ (marginally stable origin), $\alpha_k > 0$ (unstable origin)	$\alpha_k < 0$ (asymptotically stable origin)
2.	Inherent System Behaviour	For $\alpha > 0$, the system drifts away from the target (origin is repelling)	The system converges toward the target (origin is attracting)
3.	Pursuer's Burden	Fights both the evader and the system's instability	Assisted by the system's stability while fighting only the evader
4.	Evader's Strategic Leverage	Can exploit instability to evade if $\varrho > \sigma$ but $\exists\, k$ such that $\alpha_k\lvert z_k^0\rvert \geq \varrho - \sigma$	Can only delay the inevitable pursuit completion if $\varrho > \sigma$
5.	Control Resource Advantage for Pursuit Completion	Requires strong advantage: $\varrho > \sigma + \epsilon$. For the strategy in Eq. (10), $\epsilon = \alpha\lVert z^0\rVert$	Requires simple advantage:$\varrho > \sigma$

Our results in Theorem 2 and Example 4 demonstrate that even when $\varrho > \sigma$, the guaranteed evasion time τ_e can be infinite, implying that no admissible pursuit strategy can complete the pursuit in finite time. This occurs precisely when there exists an index k with $\alpha_k > 0$ such that $\sigma < \varrho \leq \sigma + \alpha_k|z_k^0|$. Conversely, from Remark 5.2, τ_e is finite if and only if $\varrho > \sigma + \alpha_k|z_k^0|$ for all $k \in \mathbb{N}$ with $\alpha_k > 0$. Furthermore, as illustrated in Example 6.3, it is possible for the evading player to guarantee a certain delay (so τ_e is finite), while the pursuing player cannot complete the pursuit in finite time using the strategy in Eq. (10). This underscores the limitations of the pursuit strategy in Eq. (10) and motivates further investigation into construction of alternative strategies that can achieve finite-time pursuit completion under such conditions where τ_e is finite.

Established sufficient conditions for evasion in differential games with m pursuers and one evader, modelled by infinite systems of ODEs in Hilbert spaces, typically require either that the evading player's control resource bound exceeds those of individual pursuers under geometric constraints ($\sigma \geq \varrho_i$ for $i = 1, 2, \ldots, m$) (Kazimirova *et al.* 2024), or that the evading player's control resources (energies) exceeds the combined pursuing player's energies under integral constraints ($\sigma^2 \geq \sum_{i=1}^{m} \varrho_i^2$) (Ibragimov & Hasim 2010). Our findings demonstrates that these conditions can be relaxed a bit when the free dynamics exhibit instability since system instability provides an inherent advantage to the evader, effectively supplementing their control resources and enabling successful evasion under less restrictive conditions than previously established.

8. Conclusions

A two-player pursuit differential game modelled by an ISFODE in Eq. (6) in ℓ_2, with geometric constraints on the players' controls, has been analyzed for arbitrary bounded, nonnegative sequences $\alpha_1, \alpha_2, \ldots$. When $\alpha > 0$, the corresponding uncontrolled system is inherently unstable and drifts away from the origin. Consequently, the pursuing player faces greater challenges in completing the pursuit, while the evading player can more readily exploit this drift to achieve evasion.

We derived explicit formulas for guaranteed times for pursuit and evasion, given by Eqs. (9) and (17), respectively. As illustrated by Example 6.2, there exist initial states z^0 for which the value τ_p, as defined by Eq. (9), represents the optimal time for pursuit, and the strategies described in Eqs. (10) and (18) are optimal for the pursuing player and the evading player, respectively. Furthermore, Example 6.3 shows that while the condition $\varrho > \sigma$ is necessary for pursuit completion, it is not sufficient; in our setting, we found that $\varrho > \sigma + \alpha\|z^0\|$ ensures sufficiency with the aid of the strategy in Eq. (10).

Our findings suggest several directions for future work. First, we recommend investigating multi-pursuer single-evader evasion games in ℓ_2 governed by infinite systems of ODEs with unstable free dynamics. Specifically, it would be insightful to consider scenarios where the evader lacks a control resource advantage (in the sense of Ibragimov and Hasim (2010); Kazimirova *et al.* (2024)) but can still guarantee evasion by leveraging the system's inherent instability. Second, a discrete-time analogue of our differential game could be developed, following the approach in Samatov and Umaraliyeva (2025), which serves as a discrete counterpart to the continuous game in Samatov *et al.* (2021); the impact of instability in such discrete models merit exploration. Third, we suggest analyzing guaranteed pursuit and evasion times for differential games in Hilbert spaces modelled by two-coupled infinite systems of first-order ODEs where the uncontrolled system is unstable.

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