Malaysian Journal of Mathematical Sciences 1X(X): XX–XX (202X)

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Alternative Variational Iteration Method for Solving Time-Fractal-Fractional Partial Differential Equations

Ibrahim Onimisi Isah [1,3], Norazak Senu* [1,2], and Ali Ahmadian* [4,5,6]

[1] *Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia (UPM), 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia*
[2] *Department of Mathematics and Statistics, Universiti Putra Malaysia (UPM), 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia*
[3] *Department of Mathematical Sciences, Prince Abubakar Audu University, Anyigba, Nigeria*
[4] *Decisions Lab, Mediterranea University of Reggio Calabria, , Reggio Calabria, Italy*
[5] *Faculty of Engineering and Natural Sciences, Istanbul Okan University, Istanbul, Turkey*
[6] *Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon*

E-mail: norazak@upm.edu.my, ahmadian.hosseini@unirc.it
**Corresponding author*

Received:
Accepted:

Abstract

In this paper, the alternative variational iteration method (AVIM) is applied to solve time fractal-fractional partial differential equations. We investigate the one-dimensional time fractional diffusion-wave equation and a generalized time-fractional partial differential equation (PDE) with variable coefficients, modelling wave propagation in mathematical physics under the new generalized Caputo-type fractal-fractional derivative. The validity and efficiency of the method, as well as the effect of the fractal dimension, are demonstrated through numerical test examples. Graphical representations of numerical results for different parameter values are also provided. The acquired numerical results employing the proposed method are compared equally with those gotten using the traditional variational iteration method and AVIM with fractional derivatives only, as found in the literature. The results indicate that the proposed approach has the advantage of being able to solve the problem with fewer iterations due to the extra parameter and fractal dimension utilized in the fractal-fractional derivative.

Keywords: Fractal-fractional operator; new generalized Caputo derivative; alternative variational iterative method; Fornberg-Whitham equation with variable coefficients; diffusion equation.

1 Introduction

Recently, consideration for fractional order partial differential equations (FPDEs) has considerably increased due to their great importance and suitability in modelling the dynamical behaviours of problems and processes in different science and engineering fields Like time-fractional oscillation and relaxation phenomena, time-fractional wave and diffusive behaviours [37], dynamic processes within self-similar and porous frameworks, atom-field interactions within photonic crystals, viscoelastic phenomena, electrical network dynamics, signal processing applications, fractal geometric concepts, chemical processes and biological systems [11], among other disciplines within science and engineering [52, 18]. Several fractional derivative (FD) as well as integral operators that differ in terms of their properties have been defined and employed widely to model real-life problems in literature [43, 30]. One of the most widely used fractional derivatives is the Caputo and Caputo-type fractional derivatives due to their ability to handle conventional initial and boundary conditions (BCs) in the representation of physical problems. For instance, [45] applied the Caputo derivative to model the transmission of Tuberculosis. While the generalized Caputo derivative [36] shares the same characteristics as Caputo derivative [35, 48], the new generalized Caputo type fractional derivative employed in [21] is equipped with an extra parameter not found in other fractional derivatives, making it a more versatile version. The continuous emergence of new fractional operators may be attributed to the occurrence of new problems that reflect various real-world scenarios.

Worthy of note is that obtaining exact solutions to many FPDEs is often difficult, if not impossible. Therefore, approximate analytical and numerical methods are being developed to provide accurate approximations to FPDEs.The authors in [13] applied the analytical approach known as Kuratowski metric of non-compactness (MNC) to investigate the existence and stability of solutions to boundary value problem of Riemann-Liouville fractional differential equations of variable order. Furthermore, robust semi-analytic methods, such as the homotopy perturbation method (HPM)[16, 32], Adomian decomposition method (ADM) [34, 49], homotopy analysis method (HAM) [59, 23], reproducing kernel space method [24, 60], and generalized differential transform method (DTM)[44, 41], as well as numerical methods including finite difference method [42, 57], Adams-Bashforth-Moulton method [17], and operational matrix methods [28, 3], among others, have been devised to solve FPDEs. [25] proposed the variational iteration method (VIM) for solving non-linear differential equations. Subsequently, several modifications of the VIM have been suggested and applied to a variety of problems with the aim of improving convergence speed and suitability [25, 1].

In this study, we consider the fractional diffusion-wave equation [53] and the generalized non-linear time-fractional PDE with variable coefficients [5] as given in Equations (1) and (2) respectively below:

$$\frac{\partial^{\alpha}\varphi}{\partial\varsigma^{\alpha}} = c\frac{\partial^{2}\varphi}{\partial x^{2}} \tag{1}$$

with the initial and BCs:

$$\varphi(0,\varsigma) = \varphi(L,\varsigma) = 0,\ \varphi(x,0) = \phi(x),\ \frac{\partial\varphi(x,0)}{\partial x} = \sigma(x), \quad 0 \le x \le L,\ \varsigma > 0$$

where α is the fractional order and the coefficient c is a constant known as the diffusion constant, x and ς are the space and time variables respectively. For $\alpha = 1$, Equation (1) represents a diffusion equation; if $\alpha = 2$, it leads to a wave equation.

and

$$\begin{aligned}\mathfrak{D}_{\tau}^{\alpha}\phi(\kappa,\tau)-\mu(\kappa)\phi_{\kappa\kappa\tau}(\kappa,\tau)+2\varphi(\kappa)\phi_{\kappa}(\kappa,\tau)+\omega(\kappa)\phi(\kappa,\tau)\phi_{\kappa}(\kappa,\tau)\\-\theta(\kappa)\phi(\kappa,\tau)\phi_{\kappa\kappa\kappa}(\kappa,\tau)-\sigma(\kappa)\phi_{\kappa}(\kappa,\tau)\phi_{\kappa\kappa}(\kappa,\tau)=0,\quad \ell<\kappa<\mathcal{L},\quad \ell,\ \mathcal{L}\in\mathbb{R}\end{aligned}\tag{2}$$

with the initial and BCs:

$$\phi\left(\kappa,\ 0\right)=\psi\left(\kappa\right),\ \phi\left(\ell,\ \tau\right)\ =\ 0, \phi\left(\mathcal{L},\ \tau\right)=\ 0,\ \ \tau\geq 0,$$

where $\mu\left(\kappa\right)$, $\varphi\left(\kappa\right)$, $\omega\left(\kappa\right)$, $\theta\left(\kappa\right)$, and $\sigma\left(\kappa\right)$ are the variable coefficients.
Equation (2) represents a more general model for wave propagation, and for specific choices of the variable coefficients, we obtain a variety of interesting and important PDEs in literature. For instance, when $\alpha=1,\ \varphi=\frac{1}{2},\ \mu=\omega=\theta=1$ and $\sigma=3$, the very famous and well investigated Fornberg-Whitham equation [22] is obtained as:

$$\phi_{\tau}-\phi_{\kappa\kappa\tau}+\phi_{\kappa}+\phi\phi_{\kappa}-\phi\phi_{\kappa\kappa\kappa}-3\phi_{\kappa}\phi_{\kappa\kappa}=0\tag{3}$$

while $\alpha=1,\ \mu=\ \varphi=\ \sigma=0,\ \ \omega=6$ and $\theta=-1$ leads to the Korteweg-de Vries (KdV) equation [40]:

$$\phi_{\tau}+6\phi\phi_{\kappa}+\phi\phi_{\kappa\kappa\kappa}=0\tag{4}$$

and $\alpha=1,\mu=\sigma=1,\ \theta=-2$ and $\omega=3$ gives the Camassa–Holm Equation in [22]as:

$$\phi_{\tau}-\phi_{\kappa\kappa\tau}+2\varphi\phi_{\kappa}+3\phi\phi_{\kappa}-2\phi\phi_{\kappa\kappa\kappa}-\phi_{\kappa}\phi_{\kappa\kappa}=0\tag{5}$$

Over the past years, various methods have been employed to approximate solutions for different instances of Equations (1) and (2). [19] investigated the diffusion equation, which is prevalent in numerous applications and comprises of fractional derivatives and variable coefficients. Heydari et al [27] developed a wavelet method for solving time-fractional fractional diffusion equations while the authors in [29] studied the variable-order nonlinear fractional diffusion-wave equations. Additional approaches applied to diffusion equations can be found in [31, 53]. [50] introduced the alternative variational iterative method (AVIM) for the solution of linear and nonlinear differential equation models. [54] proposed another AVIM technique for obtaining solutions to the time-fractional Fornberg-Whitham equation, which involves the Caputo derivative. [56] utilized the AVIM method to derive solutions for time fractional partial differential equations with proportional delay. [14] explored three types of Fornberg-Whitham equations: original, modified, and time-fractional using the reproducing kernel Hilbert space method. [5] obtained approximate solutions for a third-order time-fractional Fornberg-Whitham equation involving variable coefficients by employing three numerical techniques: VIM, ADM and HAM. Heydari and Zhagharian [28] proposed a technique to address fractional Fornberg-Whitham equations with variable order. [47] proposed the VIM with the Elzaki transform to approximate solutions to fractional diffusion equation. The study demonstrated the effectiveness of combining the VIM and the Elzaki transform in handling fractional differential equations. [39] applied the Elzaki residual power series method to obtain semi-analytic approximation of time-fractional nonlinear equations occurring in magneto-acoustic waves. [58] combined Laplace transform and Picard's iterative scheme for obtaining optimal semi-analytical solutions of time-fractional evolution equations.[4] employed Yang homotopy perturbation method and the Yang transform decomposition method to solve fractional forced Korteweg-de Vries equation arising in fluids and plasmas. [46] proposed a traveling wave solution for solving non-linear space-time fractional Fornberg-Whitham equation. [33] proposed the optimal auxiliary function method for solving the fractional-order Fornberg-Whitham problem in the sense of the Caputo fractional operator. In [12], the Homotopy perturbation transform method was applied to solve the time-fractional FW equation with the Caputo-Fabrizio fractional derivative. The authors discussed the existence and uniqueness of this equation using fixed point theory. [20] developed a method called the q-Homotopy analysis conformable fractional Laplace transform method for analytical and numerical study of a space-time conformable fractional Fornberg-Whitham equation. [26] presented an extended separation method of semi-fixed variables for analytical solution of the time-fractional Fornberg-Whitham equation, the time-fractional porous medium equation and the time-fractional Hunter-Saxton equation, among other methods that have been depolyed to solving the model fractional PDEs under study.

While many dynamical behaviors of systems modeled by partial differential equations (PDEs) have been simulated using fractional PDEs, with several semi-analytical and numerical techniques in vogue for solving these fractional PDEs, a more general approach that has recently garnered significant interest is the utilization of fractal-fractional derivatives in place of fractional derivatives in partial differential equation models. This generalization of fractional derivatives with an additional fractal dimension provides a more general way to represent more complex systems and physical processes, addressing any potential limitations of the use of fractional derivatives in modelling problems that have complex patterns that exhibit self-similarity across different scales. [15, 38]. [6] proposed a novel derivative with fractional and fractal parameters, including power-law, exponential law, and Mittag-Leffler law. This derivative has been employed by several researchers to model various fractal-fractional problems and real-life systems including mathematical modelling of convective fluid motion in rotating cavity[2], groundwater dynamics [7], chaotic system dynamics based on circuit design [38] and modelling attractors of chaotic dynamical systems [8]. The new fractal-fractional derivative has been employed with different techniques such as reproducing kernel hilbert space method [9], deep learning via Hopfield neural networks [10]. [51] investigated a variant of the fractal-fractional diffusion equation using the operational matrix of Chebyshev cardinal functions. More recently, [55] introduced the new generalized Caputo-type fractal-fractional derivative (NGCFFD), incorporating another parameter (fractal dimension) in to the generalized Caputo fractional derivative. The authors applied this operator to solve fractal-fractional differential equations (FFDEs). Due to the limited research on fractal-fractional derivatives compared to traditional fractional derivatives, ongoing efforts are focused on developing and analyzing semi-analytical and numerical methods for solving fractal-fractional problems. As much as we are aware, the AVIM method has not been explored in conjunction with the NGCFFD for solving Equations (1) and (2) in the literature.

In light of the foregoing review, the present study expands upon the application of AVIM as introduced in [50] to address the solution of time-FPDEs. We establish the model FPDEs in terms of fractal-fractional derivatives and devise an iterative procedure based on the AVIM technique, employing the new generalized Caputo derivative, to solve FPDEs as outlined in [55]. The usefulness of the developed method is evaluated by comparing the numerical results got using the newly proposed approach with those obtained using methods discussed in the literature. The numerical solutions obtained via the suggested technique are found to align with those obtained using alternative approaches. The method offers good accuracy with fewer expansion terms, resulting in lower computational cost compared to conventional techniques. It effectively addresses non-linearities without requiring linearization and preserves analytical structure, which can give a deeper understanding of the behaviour of the solution.

We organize this article as thus: Section 2 enumerates key results and definitions from the literature. Section 3 introduces the procedure for the alternative variational iterative method (AVIM), followed by the derivation of the AVIM solution method for Equations (1) and (2) in Section 4. Section 5 provides numerical examples, and Section 6 analyzes the results and offers concluding remarks.

2 Preliminaries

Here, we offer fundamental definitions and properties from fractal and fractional calculus theory, essential for our current investigation.

Definition 2.1. ([48]). *The generalized fractional integral of a function* $u\,(x,t)$ *is denoted by* $\mathrm{I}_{\mathrm{a}+}^{\alpha,\beta}u\,(x,t)$ *and defined as:*

$$I_{\mathrm{a}+}^{\alpha,\beta}u(x,t)=\frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathrm{a}}^{t}\tau^{\beta-1}(t^{\beta}-\tau^{\beta})^{\alpha-1}\,d\tau,\quad \beta>0 \text{ and } t>a \tag{6}$$

Definition 2.2. ([48]). *Suppose* $u\,(x,t)$ *is continuous on* (a,b)*, then the generalized fractal-fractional integral,* $\mathrm{I}_{\mathrm{a}+}^{\alpha,\beta,\gamma}u\,(x,t)$ *of order* α *is defined as:*

$$u\,(x,t)=\frac{\gamma\beta^{1-\alpha}}{\Gamma\,(\alpha)}\int_{0}^{t}\tau^{\beta+\gamma-2}\left(t^{\beta}-\tau^{\beta}\right)^{\alpha-1}u\,(x,\tau)\,d\tau \tag{7}$$

Definition 2.3. ([55]). *Let* $\varsigma-1<\alpha\leq\varsigma$ *and* $\varsigma\ \epsilon\ \mathbb{N}$*. The Caputo fractional derivative of order* $\alpha>0$ *of the function* $u\,(x,t)$ *is defined as:*

$${}^{C}\mathfrak{D}_{\mathrm{a}+}^{\alpha}u\,(x,t)=\frac{1}{\Gamma\,(\varsigma-\alpha)}\int_{\mathrm{a}}^{t}(t-\tau)^{\varsigma-\alpha-1}u^{(\varsigma)}\,(x,\tau)\,d\tau,\ \ t>\mathrm{a} \tag{8}$$

Definition 2.4. ([48]). *Let* $u\,(x,t)\ \epsilon\ C^{\eta}\,([a,b])$ *. The new generalized Caputo-type FD having order* $\alpha>0$ *is defined as:*

$${}^{C}\mathfrak{D}_{\mathrm{a}+}^{\alpha,\beta}u\,(x,t)=\frac{\beta^{\alpha-\eta+1}}{\Gamma\,(\eta-\alpha)}\int_{\mathrm{a}}^{t}\tau^{\beta-1}\left(t^{\beta}-\tau^{\beta}\right)^{\eta-\alpha-1}\left(t^{1-\beta}\frac{d}{dt}\right)^{\eta}u\,(x,\tau)\,d\tau \tag{9}$$

where $t>a$*,* $\beta>0$ *and* $\eta-1<\alpha\leq\eta$*,* $\eta=\lceil\alpha\rceil$ *and*

the NGCD has these properties:

${}^{C}\mathfrak{D}_{\mathrm{a}+}^{\alpha,\beta}C=0$ *, C is a constant. And for* $\varsigma-1<\alpha\leq\varsigma$*,* $k>\varsigma-1$*,* $k\notin\mathbb{N}$*,*

$${}^{C}\mathfrak{D}_{\mathrm{a}+}^{\alpha,\beta}\left(t^{\beta}-\tau^{\beta}\right)=\begin{cases}\beta^{\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\left(t^{\beta}-\tau^{\beta}\right)^{k-\alpha}, & k\in\mathbb{N}_0 \text{ and } k\geq\lceil\alpha\rceil \text{ or k} >\lfloor\alpha\rfloor\\ 0, & k\in\mathbb{N}_0 \text{ and or k} <\lceil\alpha\rceil\end{cases} \tag{10}$$

Definition 2.5 ([55])**.** *Suppose* $u(x,t)$ *is continuous and fractal differentiable of order* γ *on* (a,b) *and* $0<\varsigma-1<\gamma$*,* $\alpha\leq\varsigma$*, then the new generalized Caputo-type fractal-fractional derivative (NGCFFD) of* $u(x,t)$ *of order* α *and* β *is denoted by* ${}^{C}_{\mathfrak{D}}{}^{\alpha,\beta,\gamma}u(x,t)$ *and defined as:*

$${}^{C}\mathfrak{D}^{\alpha,\beta,\gamma}u\,(x,t)=\frac{\beta^{\alpha-\varsigma+1}}{\Gamma\,(\varsigma-\alpha)}\int_{\mathrm{a}}^{t}\tau^{\beta-1}\left(t^{\beta}-\tau^{\beta}\right)^{\varsigma-\alpha-1}\left(t^{1-\beta}\frac{d}{dt^{\gamma}}\right)^{\varsigma}u\,(x,\tau)\,d\tau, \tag{11}$$

where $\frac{du}{dt^{\gamma}}=\lim_{t\to\tau}\frac{u(x,t)-v(x,\tau)}{t^{\beta}-\tau^{\beta}}$

Corollary 2.1. [55]). *Let* $\alpha,\gamma\ \epsilon\ (0,1)$ *and* $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ *and* $\beta\ \epsilon\ \mathbb{R}$*. Then, we have:*

$${}^{C}\mathfrak{D}^{\alpha,\beta,\gamma}t^{k}=\frac{\beta^{\alpha}\Gamma\left(\frac{k}{\beta}+1\right)}{\gamma\Gamma\left(\frac{k}{\beta}-\alpha+1\right)}t^{\frac{k}{\beta}-\alpha-\gamma+1} \tag{12}$$

Theorem 2.1. ([35]). *Let* $\varsigma-1<\alpha\leq\varsigma$, $\varsigma \in \mathbb{N}$ *and* $\alpha>0$. *Then the derivative operator* ${}^{C}\mathfrak{D}^{\alpha,\beta,\gamma}$ *satisfies the following property.*

$$\mathrm{I}_{\mathrm{a}+}^{\alpha,\beta,\gamma}\,{}^{C}\mathfrak{D}^{\alpha,\beta,\gamma}u\left(x,t\right)=\ u\left(x,t\right)-\sum_{k=0}^{\varsigma}\frac{\left(-1\right)^{k}\left(\gamma^{k}u\right)\left(a\right)}{k!}\left(\frac{a^{\beta}-t^{\beta}}{\beta}\right)^{k}.$$

Particularly, for $0<\alpha\leq1$, *we have that*

$$\mathrm{I}_{\mathrm{a}+}^{\alpha,\beta,\gamma}\,{}^{C}\mathfrak{D}^{\alpha,\beta,\gamma}u\left(x,t\right)=\ u\left(x,t\right)-u\left(x,a\right)$$

3 Alternative Variational Iteration Method (AVIM) Procedure

This section presents an overview of the overall framework of the alternative variational iterative technique for attaining approximate solutions to fractional PDEs, followed by a discussion of its usage in fractal-fractional PDEs.

Consider the non-linear FPDE:

$$ {}^{C}\mathfrak{D}_{\tau}^{\alpha}\phi\left(\kappa,\tau\right)+L\phi\left(\kappa,\tau\right)+N\phi\left(\kappa,\tau\right)=g\left(\kappa,\tau\right) \tag{13}$$

where $\varsigma-1<\alpha\leq\varsigma$; $\varsigma \in \mathbb{N}$. L and N refer to linear and non-linear operators, respectively., g is a specified analytic function called the source term and ${}^{C}\mathfrak{D}_{t}^{\alpha}$ is the Caputo fractional derivative of order α. The initial conditions for Equation (13) are given as:

$$\phi^{k}\left(\kappa,0\right)=f_{k}\left(\kappa\right),\qquad k=0,1,\ldots,\varsigma-1$$

The general framework of fractional VIM is aimed at obtaining a series solution to eq. (13) in the form:

$$\phi\left(\kappa,\tau\right)=\lim_{k\to\infty}\phi_{k}\left(\kappa,\tau\right),\qquad k=0,1,\ldots,\varsigma-1 \tag{14}$$

by constructing the correctional function:

$$\phi_{k+1}\left(\kappa,\tau\right)=\phi_{k}\left(\kappa,\tau\right)+J_{\tau}^{\alpha}\left[\lambda\left(s\right)\left({}^{C}\mathfrak{D}_{\tau}^{\alpha}\phi\left(\kappa,\tau\right)+L\phi\left(\kappa,\tau\right)+N\phi\left(\kappa,\tau\right)-g\left(\kappa,\tau\right)\right)\right] \tag{15}$$

to Equation (13), where J_{τ}^{α} denotes the Riemann-Liouville fractional integral and λ represents Lagrange multiplier that is determined by variational theory. Generally, λ is given by

$$\lambda=\frac{\left(-1\right)^{m}}{\left(m-1\right)!}\left(s-\tau\right)^{m-1},\qquad m\geq1 \tag{16}$$

This correctional function is formulated from a variational functional based on the residual of the given fractional PDE adapted to the specific type of fractional derivative, with a lagrange multiplier introduced in the formulation and an associated fractional integral.

Substituting λ into Equation (15) gives the general VIM iteration formula:

$$\phi_{k+1}\left(\kappa,\tau\right)=\phi_{k}\left(\kappa,\tau\right)+J_{\tau}^{\alpha}\left[\frac{\left(-1\right)^{m}}{\left(m-1\right)!}\left(s-\tau\right)^{m-1}\left({}^{C}\mathfrak{D}_{\tau}^{\alpha}\phi\left(\kappa,\tau\right)+L\phi\left(\kappa,\tau\right)+N\phi\left(\kappa,\tau\right)-g\left(\kappa,\tau\right)\right)\right] \tag{17}$$

Now, according to [50], the AVIM is deduced from the general VIM iteration formula given in Equation (17) with the following iterative procedure:
From Equation (17), we define the operator:

$$\mathcal{A}\left[\phi\left(\kappa,\tau\right)\right]=J_{\tau}^{\alpha}\left[\frac{\left(-1\right)^{m}}{\left(m-1\right)!}\left(s-\tau\right)^{m-1}\left({}^{C}\mathfrak{D}_{\tau}^{\alpha}\phi\left(\kappa,\tau\right)+L\phi\left(\kappa,\tau\right)+N\phi\left(\kappa,\tau\right)-g\left(\kappa,\tau\right)\right)\right] \tag{18}$$

and obtain the iterative components of Equation (18) as follows:

$$\begin{cases} \eta_0=\phi\left(\kappa,0\right) \\ \eta_1=\mathcal{A}\left[\eta_0\right] \\ \eta_2=\mathcal{A}\left[\eta_0+\eta_1\right] \\ \vdots \\ \eta_{k+1}=\mathcal{A}\left[\eta_0+\eta_1+\ldots+\eta_k\right] \end{cases} \tag{19}$$

where $\phi\left(\kappa,0\right)$ is the initial condition to the given FPDE.
Thus, we have $\phi\left(\kappa,\tau\right)=\lim_{k\to\infty}\phi_k\left(\kappa,\tau\right)\;=\sum_{k=0}^{\infty}\eta_k$
Consequently, the solution to the FPDE in Equation (13) using the AVIM is given by:

$$\phi\left(\kappa,\tau\right)=\sum_{k=0}^{\infty}\eta_k\left(\kappa,\tau\right) \tag{20}$$

4 AVIM for Fractal-fractional Partial Differential Equations (FFAVIM).

Here, we provide the contribution of this current research: the concept of the fractal-fractional AVIM method. To start with, we apply the NGCFFD to our model FPDEs. Subsequently, we derive the generalized Caputo-type fractal-fractional integral of a power function, which can be generalized to other types of functions.

Consider the non-linear fractal-fractional PDE for $m-1<\alpha\leq m;\ m\,\epsilon\,\mathbb{N}$:

$${}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi\left(\kappa,\tau\right)+L\phi\left(\kappa,\tau\right)+N\phi\left(\kappa,\tau\right)=g\left(\kappa,\tau\right) \tag{21}$$

where $\mathfrak{D}_{t}^{\alpha,\beta,\gamma}$ is the NGCFFD, L and N respectively represent linear and non-linear operators, g is a given function called the source term with initial conditions given as:

$$\phi^{k}\left(\kappa,0\right)=f_k\left(\kappa\right),\qquad k=0,1,\ldots,\alpha-1$$

To obtain the solution to Equation (21) by the proposed FFAVIM method, we define the FFAVIM iterative operator $\mathcal{A}\left[\phi\right]$, with the aid of Equation (18) as follows:

$$\mathcal{A}\left[\phi\right]=I_{\tau}^{\alpha,\beta,\gamma}\left[\frac{\left(-1\right)^{m}}{\left(m-1\right)!}\left(s-\tau\right)^{m-1}\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi\left(\kappa,\tau\right)+L\phi\left(\kappa,\tau\right)+N\phi\left(\kappa,\tau\right)-g\left(\kappa,\tau\right)\right)\right] \tag{22}$$

where $I_{t}^{\alpha,\beta,\gamma}$ is the new generalized fractal-fractional integral as presented in ***definition* 2.2**. This generalized fractal-fractional integral replaces the Riemann-Liouville fractional integral present in the AVIM iteration operator given in Equation (18).

Then, the iterative components to (22) are obtained in the following manner:

$$\begin{cases} \eta_0 = \phi\left(\kappa, 0\right) \\ \eta_1 = \mathcal{A}\left[\eta_0\right] \\ \eta_2 = \mathcal{A}\left[\eta_0 + \eta_1\right] \\ \vdots \\ \eta_{k+1} = \mathcal{A}\left[\eta_0 + \eta_1 + \ldots + \eta_k\right] \end{cases} \tag{23}$$

Consequently the required FFAVIM solution to Equation (21) is:

$$\phi\left(\kappa, \tau\right) = \sum_{k=0}^{\infty} \eta_k\left(\kappa, \tau\right) \tag{24}$$

In what follows, we derive the formula for obtaining the generalized fractal-fractional integral of a function that are needed to compute the iterative components of the FFAVIM iterative operator, $\mathcal{A}\left[\phi\right]$.

Lemma 4.1. *Given **definition** 2.2, then*

$$I_t^{\alpha,\beta,\gamma} t^k = \frac{\Gamma\left(1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)}{\beta^\alpha \Gamma\left(1 + \alpha + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)} t^{\alpha\beta + k + \gamma - 1} \tag{25}$$

Proof. Using the definition in definition 2.2, we have

$$I_t^{\alpha,\beta,\gamma} t^k = \frac{\gamma\beta^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_0^t \tau^{\beta+\gamma-2}\left(t^\beta - \tau^\beta\right)^{\alpha-1} \tau^k d\tau \tag{26}$$

After some mathematical simplifications involving change of variable, we obtain:

$$I_t^{\alpha,\beta,\gamma} t^k = \frac{\gamma\beta^{1-\alpha}}{\beta\,\Gamma\left(\alpha\right)} t^{\alpha\beta+\gamma+k-1} \int_0^1 v^{\frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}} \left(1 - v\right)^{\alpha-1} dv \tag{27}$$

$$= \frac{\gamma}{\beta^\alpha\,\Gamma\left(\alpha\right)} t^{\alpha\beta+\gamma+k-1} B\left(\left(1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right), \alpha\right) \tag{28}$$

$$= \frac{\gamma\,\Gamma\left(1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)}{\beta^\alpha\,\Gamma\left(1 + \alpha + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)} t^{\alpha\beta+\gamma+k-1} \tag{29}$$

which completes the proof. where $B\left(p, q\right), \quad p = 1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}, \quad q = \alpha$, represents the Beta function. □

Remark 4.1. *The generalized fractal-fractional integral of $u(x,t)$ in the case when $u\left(x,t\right)$ is a transcendental function can be obtained similarly by writing it in terms of series of t^k. For example, $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$.*

The FFAVIM procedure presented above is summarized in Algorithm 1 below:

Algorithm 1 Algorithm for FFAVIM Solver

Start algorithm

1: Define the FFPDE:

$$CD_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa,\tau) + L\varphi(\kappa,\tau) + N\varphi(\kappa,\tau) = g(\kappa,\tau)$$

where $CD_{\tau}^{\alpha,\beta,\gamma}$ is the NGCFFD operator, L and N are linear and nonlinear operators respectively, and $g(\kappa,\tau)$ is the source term.

2: Define the initial condition: $\varphi(\kappa, 0)$

3: Construct the FFAVIM iterative operator:

$$\mathcal{A}\left[\varphi\right] = I_{\tau}^{\alpha,\beta,\gamma}\left[\frac{(-1)^m}{(m-1)!}(s-\tau)^{m-1}\left(CD_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa,\tau) + L\varphi(\kappa,\tau) + N\varphi(\kappa,\tau) - g(\kappa,\tau)\right)\right]$$

where $I_{\tau}^{\alpha,\beta,\gamma}$ is the new generalized fractal-fractional integral.

4: Compute the iterative approximations:

- Set initial guess:

$$\eta_0 = \varphi(\kappa, 0)$$

- Compute iteratively:

$$\eta_1 = \mathcal{A}[\eta_0]$$
$$\eta_2 = \mathcal{A}[\eta_0 + \eta_1]$$
$$\eta_3 = \mathcal{A}[\eta_0 + \eta_1 + \eta_2]$$
$$\cdots$$
$$\eta_{k+1} = \mathcal{A}[\eta_0 + \eta_1 + \cdots + \eta_k]$$

5: Compute the final FFAVIM solution:

$$\varphi(\kappa,\tau) = \sum_{k=0}^{\infty}\eta_k(\kappa,\tau)$$

Output: Approximate solution $\varphi(\kappa,\tau)$ for the given FPDE.

4.1 Application of FFAVIM to Diffusion-wave Equation

This section presents the application of the proposed FFAVIM technique, using the NGCFFD to solve diffusion-wave equations.

Consider the time-fractal-fractional diffusion-wave (FFD-W) equation as:

$$ {}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\varphi\left(\kappa,\tau\right) = c\varphi_{\kappa\kappa}\left(\kappa,\tau\right) + f\left(\kappa\right) \tag{30} $$

with: $\varphi(0,\tau) = \varphi(L,\tau) = 0,\ \varphi(\kappa,0) = \phi(\kappa),\ \varphi_{\tau}(\kappa,0) = \sigma(\kappa),\ 0 \leq \kappa \leq L,\ \tau > 0$ and ${}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}$ is the NGCFFD.

To obtain the solution of Equation (30) by FFAVIM method, we define the iterative operator, $\mathcal{A}\left[\phi\right]$ as:

$$ \mathcal{A}\left[\phi\left(\kappa,\tau\right)\right] = I_{\tau}^{\alpha,\beta,\gamma}\left[\frac{(-1)^{m}}{(m-1)!}(s-\tau)^{m-1}\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\varphi\left(\kappa,\tau\right) - c\varphi_{\kappa\kappa}\left(\kappa,\tau\right) - f\left(\kappa\right)\right)\right] \tag{31} $$

The iterative components are then computed using Equation (23). The FFAVIM approximation would then be obtained in series form as in Equation (24)

4.2 Application of FFAVIM to Generalized Fractional PDE with Variable Coefficients

Here, we demonstrate how our proposed FFAVIM technique can be used to solve time-FFPDEs having variable coefficients in the sense of NGCFFD.

Consider the generalized non-linear time-fractal-fractional PDE with variable coefficients as:

$$ \begin{aligned} &{}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi(\kappa,\tau) - \mu(\kappa)\,\phi_{\kappa\kappa\tau}(\kappa,\tau) + 2\varphi(\kappa)\,\phi_{\kappa}(\kappa,\tau) + \omega(\kappa)\,\phi(\kappa,\tau)\,\phi_{\kappa}(\kappa,\tau) \\ &- \theta(\kappa)\,\phi(\kappa,\tau)\,\phi_{\kappa\kappa\kappa}(\kappa,\tau) - \sigma(\kappa)\,\phi_{\kappa}(\kappa,\tau)\,\phi_{\kappa\kappa}(\kappa,\tau) = 0, \quad \ell < \kappa < \mathcal{L}, \quad \ell,\mathcal{L} \in \mathbb{R} \end{aligned} \tag{32} $$

with the initial and BCs:

$$ \phi\left(\kappa,\ 0\right) = \phi\left(\kappa\right),\ \ \phi\left(\ell,\ \tau\right) = 0, \phi\left(\mathcal{L},\ \tau\right) = 0,\ \ \tau \geq 0, $$

where ${}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}$ is the NGCFFD.

To obtain the solution of Equation (32) using FFAVIM method, the iterative operator $\mathcal{A}\left[\phi\right]$ for Equation (32) is given as:

$$ \begin{aligned} \mathcal{A}[\phi(\kappa,\tau)] = I_{\tau}^{\alpha,\beta,\gamma}[&\frac{(-1)^{m}}{(m-1)!}(s-\tau)^{m-1}({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi(\kappa,\tau) - \mu(\kappa)\phi_{\kappa\kappa\tau}(\kappa,\tau) + 2\varphi(\kappa)\phi_{\kappa}(\kappa,\tau) \\ &+\omega(\kappa)\phi(\kappa,\tau)\phi_{\kappa}(\kappa,\tau) - \theta\left(\kappa\right)\phi(\kappa,\tau)\,\phi_{\kappa\kappa\kappa}\left(\kappa,\tau\right) - \sigma\left(\kappa\right)\ \phi_{\kappa}\left(\kappa,\tau\right)\phi_{\kappa\kappa}\left(\kappa,\tau\right))] \end{aligned} \tag{33} $$

The iterative components would be computed next with the help of Equation (23) to obtain the series solution of the form in Equation (24).

5 Numerical Examples

In this part, the efficiency of the proposed technique is demonstrated using some numerical examples.

Example 1

Consider the FFD-W equation as given in [53]:

$$
{}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa,\tau)=\varphi_{\kappa\kappa}(\kappa,\tau)\ ,\qquad 0<\alpha\leq 1 \tag{34}
$$
$$
\varphi(0,\tau)=\varphi(\pi,\tau)=0,\quad \varphi(\kappa,0)=sin\kappa,\quad \varphi_{\tau}(\kappa,0)=0,\ \ 0\leq\kappa\leq\pi,\ \ \tau>0
$$

The exact solution to the classical model (when $\alpha=\beta=\gamma=1$) is $\varphi(\kappa,\tau)=e^{-\tau}sin\kappa$.

To solve Equation (34) by our proposed FFAVIM, we construct the iteration formula and follow the FFAVIM algorithm as below:

$$
\mathcal{A}[\varphi]=I_{\tau}^{\alpha,\beta,\gamma}\left[(-1)\left(CD_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa,\tau)-\varphi_{\kappa\kappa}(\kappa,\tau)\right)\right]
$$

Then, the iterative components are given by:

$$
\begin{cases}
\eta_0=sin\kappa\\
\eta_1=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\eta_0\right)-\left(\eta_0\right)_{\kappa\kappa}\right]\\
\vdots\\
\eta_{k+1}=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\left[\eta_0+\eta_1+\ldots+\eta_k\right]\right)-\left(\left[\eta_0+\eta_1+\ldots+\eta_k\right]\right)_{\kappa\kappa}\right]
\end{cases} \tag{35}
$$

We compute each component of Equation (35) as follows:

$$
\begin{aligned}
\eta_1&=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\eta_0\right)-\left(\eta_0\right)_{\kappa\kappa}\right]\\
&=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\sin\kappa\right)-\left(\sin\kappa\right)_{\kappa\kappa}\right]\\
&=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\sin\kappa\right)+\sin\kappa\right]\\
&=-I_{\tau}^{\alpha,\beta,\gamma}\left[0+\sin\kappa\right]=-I_{\tau}^{\alpha,\beta,\gamma}\left[\sin\kappa\tau^{0}\right]=-\sin\kappa\ \ I_{\tau}^{\alpha,\beta,\gamma}\left[\tau^{0}\right]
\end{aligned}
$$

using Corollary 2.1, we get

$$
\eta_1=-\sin\kappa\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1}
$$

Similarly, for the second iterative component, we have

$$
\begin{aligned}
\eta_2&=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\left[\eta_0+\eta_1\right]\right)-\left(\left[\eta_0+\eta_1\right]\right)_{\kappa\kappa}\right]\\
&=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\left[\sin\kappa-\sin\kappa\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1}\right]\right)\right.\\
&\left.-\left(\left[\sin\kappa-\sin\kappa\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1}\right]\right)_{\kappa\kappa}\right]\\
&=-I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\left[-\sin\kappa\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1}\right]\right)\right.\\
&\left.-\left(\left[-\sin\kappa+\sin\kappa\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1}\right]\right)\right]
\end{aligned}
$$

and with the aid of Theorem 1 and Corollary 2.1, we obtain:

$$= -\sin\kappa \frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1} - I_{\tau}^{\alpha,\beta,\gamma}\left(\left[-\sin\kappa+\sin\kappa\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1}\right]\right)$$

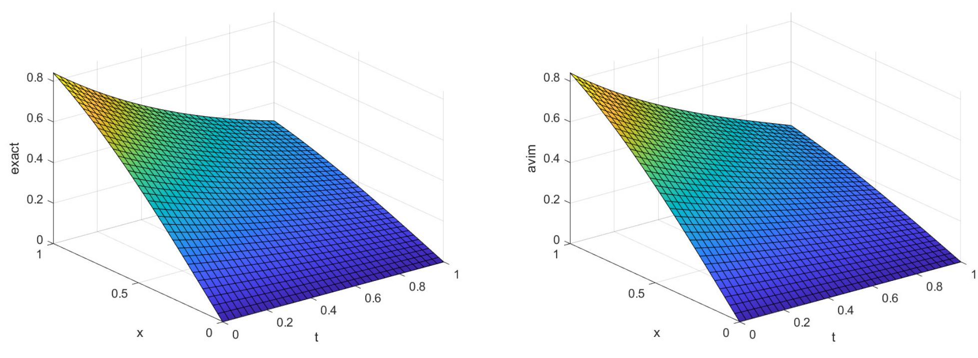
which simplifies to:

$$\eta_2 = \sin\kappa \frac{\gamma^2\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)}{\beta^{2\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)}\tau^{2\alpha\beta+2\gamma-2}$$

Higher-order iterative components can be computed by following a similar procedure. Thus, the following successive approximations are obtained for the iterative components for Example 1 up to the third order:

$$\begin{aligned}
\eta_0 &= sin\kappa \\
\eta_1 &= -\sin\kappa \frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\tau^{\alpha\beta+\gamma-1} \\
\eta_2 &= \sin\kappa \frac{\gamma^2\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)}{\beta^{2\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)}\tau^{2\alpha\beta+2\gamma-2} \\
\eta_3 &= \sin\kappa \frac{\gamma^3\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\Gamma\left(1+2\alpha+\frac{3\gamma}{\beta}-\frac{3}{\beta}\right)}{\beta^{3\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)}\tau^{3\alpha\beta+3\gamma-3} \\
&\vdots
\end{aligned} \tag{36}$$

Table(1) shows the result and absolute errors of example 1 using only three levels of FFAVIM iterations when $\alpha = 1,\ \beta = 1.014,\ \ \gamma = 0.98$. It is apparent that we do not require many iterations (only three) to obtain good approximation to the exact value. The outcome displayed in Table (2) is the results of using FFAVIM for example 1 with different α values: $\alpha = 0.5,\ 0.75$, and constant β and γ values.



(a) Exact solution (b) $\alpha = 1\ ,\ \ \beta = 1.014,\quad \gamma = 0.98$

Figure 1: Graphical simulation of third-order FFAVIM for $\alpha = 1$ and exact solutions of Example 1.

Table 1: Approx. solution of *Example 1* with third-order FFAVIM for $\alpha = 1,\ \beta = 1.014,\ \gamma = 0.98$.

κ	τ	Exact solution	FFAVIM Approx.	Absolute Error
	0.1	0.09033301095	0.09033593761	2.926660 E-6
	0.2	0.08173668840	0.08180429106	6.760266 E-5
	0.3	0.07395841409	0.07408157185	1.231577 E-4
	0.4	0.06692034044	0.06705706090	1.367205 E-4
0.1	0.5	0.06055202806	0.06062786565	7.583759 E-5
	0.6	0.05478974073	0.05469444280	9.529793 E-5
	0.7	0.04957580754	0.04915911386	4.166937 E-5
	0.8	0.04485804569	0.04392539906	9.326466 E-4
	0.9	0.04058923823	0.0388976630	1.691575 E-3
	1.0	0.03672666153	0.03398090423	2.745757 E-3
	0.1	0.17976344430	0.1797692684	1.582410 E-6
	0.2	0.16265669080	0.1627912206	1.345298 E-4
	0.3	0.14717786020	0.1474229451	2.450849 E-4
	0.4	0.13317203500	0.1334441099	2.720749 E-4
0.2	0.5	0.12049904030	0.1206499577	1.509174 E-4
	0.6	0.10903204050	0.1088423967	1.896438 E-4
	0.7	0.09865626990	0.09782704607	8.292239 E-4
	0.8	0.08926788460	0.08741190995	1.855975 E-3
	0.9	0.08077292220	0.07740667346	3.366249 E-3
	1.0	0.07308636240	0.06762228247	5.464079 E-3

Figure(1) is a comparison of the FFAVIM solution of example 1 with the exact solution of the classical model ($\alpha = 1$). The effectiveness of the method is clear from the two graphical simulations. While Figure(2)represents the surface plot of the FFAVIM solution of example 1 for two different α values. The effectiveness of the FFAVIM can be visualized from these plots. Figure(3) shows the effect of varying the fractional parameter α which defines memory effect. Although, as α decreases, convergence to the exact solution decreases, the FFAVIM is still able to provide suitable approximate solution to the problem.

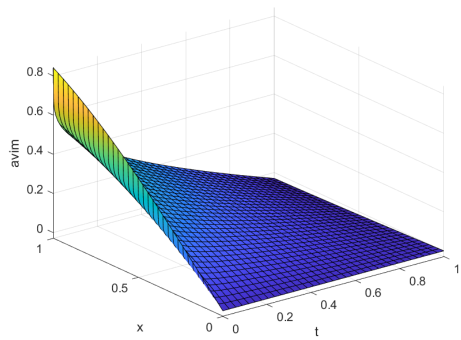
Example 2

Consider the fractal-fractional PDE with constant coefficients in the sense of NGCFFD [[54]]:

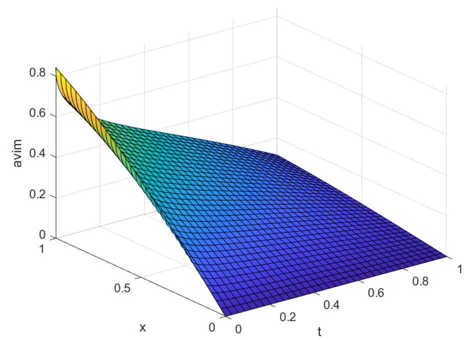
$$\begin{aligned} {}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi\left(\kappa,\tau\right) - \phi_{\kappa\kappa\tau}\left(\kappa,\tau\right) + \phi_{\kappa}\left(\kappa,\tau\right) + \phi\left(\kappa,\tau\right)\phi_{\kappa}\left(\kappa,\tau\right)\ - \phi(\kappa,\tau)\,\phi_{\kappa\kappa\kappa}\left(\kappa,\tau\right) \\ -3\phi_{\kappa}\left(\kappa,\tau\right)\phi_{\kappa\kappa}\left(\kappa,\tau\right) = 0\ ,\quad 0<\tau\leq 1\,,\quad 0<\alpha\ \leq 1 \end{aligned} \tag{37}$$

Table 2: Absolute errors of third-order FFAVIM solution of *Example 1* for $\kappa = 0.1$ and different values of α, $\beta = 1.014, \gamma = 0.98$.

κ	τ	Exact solution	FFAVIM $\alpha = 0.5$	Abs. Error $\alpha = 0.5$	FFAVIM $\alpha = 0.75$	Abs. Error $\alpha = 0.75$
	0.1	0.09033301095	0.07148829286	1.884471809E-2	0.08255739134	7.77561961E-3
	0.2	0.08173668840	0.06253119485	1.920549355E-2	0.07299952570	8.73716270E-3
0.1	0.3	0.07395841409	0.05571294351	1.824547058E-2	0.06561837954	8.34003455E-3
	0.4	0.06692034044	0.04964625123	1.727408921E-2	0.05939933823	7.52100221E-3
	0.5	0.06055202806	0.04384532392	1.670670414E-2	0.05385047621	6.70155185E-3
	0.6	0.05478974073	0.03808519375	1.670454698E-2	0.04867098277	6.11875796E-3
	0.7	0.04957580754	0.03224594660	1.732986094E-2	0.04365320398	5.92260356E-3
	0.8	0.04485804569	0.02625814293	1.859990276E-2	0.03864257334	6.21547235E-3
	0.9	0.04058923823	0.02007946475	2.050977348E-2	0.03351797086	7.07126737E-3
	1.0	0.03672666153	0.01368334036	2.304332117E-2	0.02818088578	8.54577575E-3



(a) $\alpha = 0.25$



(b) $\alpha = 0.5$

Figure 2: Graphical simulation of third-order FFAVIM solution of Example 1 for different α values , $\beta = 1.014, \quad \gamma = 0.98$

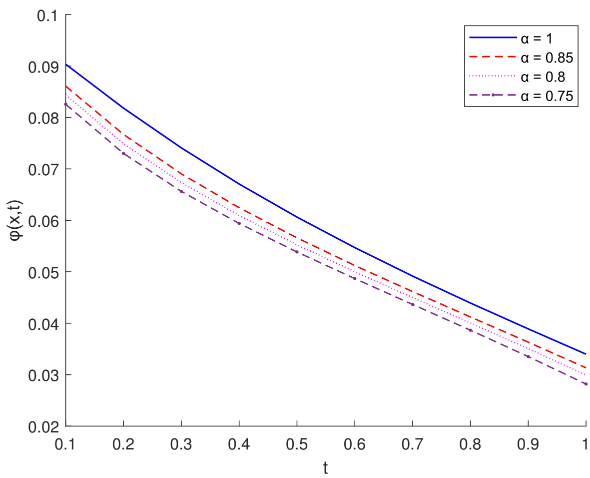


Figure 3: Plots of third-order FFAVIM solution of Example 1 for different α values with $\beta = 1.014, \ \gamma = 0.98$

with initial condition: $\phi(\kappa, 0) = \frac{4}{3}e^{\frac{1}{2}\kappa}$

Equation (37) is the well-known fractional order Fornberg-Whitham equation, an important equation modelling wave breakage in mathematical physics. The exact travelling wave solution to Equation (37) (classical model) is $\phi(\kappa, \tau) = \frac{4}{3}e^{\frac{1}{2}\kappa - \frac{2}{3}\tau}$.

To obtain its solution by our proposed FFAVIM, the iteration formula is equally constructed and the FFAVIM algorithm is utilized to complete the solution process as presented below:

$$\begin{aligned}\mathcal{A}[\phi] = I_{\tau}^{\alpha,\beta,\gamma}\Big[(-1)\Big({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi(\kappa,\tau) - \phi_{\kappa\kappa\tau}(\kappa,\tau) + \phi_{\kappa}(\kappa,\tau) + \phi(\kappa,\tau)\,\phi_{\kappa}(\kappa,\tau)\\ - \phi(\kappa,\tau)\,\phi_{\kappa\kappa\kappa}(\kappa,\tau) - 3\phi_{\kappa}(\kappa,\tau)\,\phi_{\kappa\kappa}(\kappa,\tau)\Big)\Big]\end{aligned}$$

with iterative components:

$$
\begin{cases}
\eta_0 = \frac{4}{3}e^{\frac{1}{2}\kappa} \\
\eta_1 = -I_\tau^{\alpha,\beta,\gamma}\left[\left({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}\eta_0\right) - (\eta_0)_{\kappa\kappa\tau} + (\eta_0)_\kappa + \eta_0(\eta_0)_\kappa - \eta_0(\eta_0)_{\kappa\kappa\kappa}\right] \\
\quad -3(\eta_0)_\kappa(\eta_0)_{\kappa\kappa} \\
\vdots \\
\eta_{k+1} = -I_\tau^{\alpha,\beta,\gamma}\left[\begin{array}{c} \left({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}[\eta_0+\eta_1+\ldots+\eta_k]\right) - ([\eta_0+\eta_1+\ldots+\eta_k])_{\kappa\kappa\tau} \\ +([\eta_0+\eta_1+\ldots+\eta_k])_\kappa + [\eta_0+\eta_1+\ldots+\eta_k]\left([\eta_0+\eta_1+\ldots+\eta_k]\right)_\kappa \\ -[\eta_0+\eta_1+\ldots+\eta_k]\left([\eta_0+\eta_1+\ldots+\eta_k]\right)_{\kappa\kappa\kappa} \\ -3\left([\eta_0+\eta_1+\ldots+\eta_k]\right)_\kappa\left([\eta_0+\eta_1+\ldots+\eta_k]\right)_{\kappa\kappa} \end{array}\right]
\end{cases}
\tag{38}
$$

We derive the components of Equation (38) as follows:

$$
\begin{aligned}
\eta_1 &= -I_\tau^{\alpha,\beta,\gamma}\left[\left({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}\eta_0\right) - (\eta_0)_{\kappa\kappa\tau} + (\eta_0)_\kappa + \eta_0(\eta_0)_\kappa - \eta_0(\eta_0)_{\kappa\kappa\kappa} - 3(\eta_0)_\kappa(\eta_0)_{\kappa\kappa}\right] \\
&= -I_\tau^{\alpha,\beta,\gamma}\left[\left({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}\frac{4}{3}e^{\frac{1}{2}\kappa}\right) - \left(\frac{4}{3}e^{\frac{1}{2}\kappa}\right)_{\kappa\kappa\tau} + \left(\frac{4}{3}e^{\frac{1}{2}\kappa}\right)_\kappa + \frac{4}{3}e^{\frac{1}{2}\kappa}\left(\frac{4}{3}e^{\frac{1}{2}\kappa}\right)_\kappa\right. \\
&\quad \left. -\frac{4}{3}e^{\frac{1}{2}\kappa}\left(\frac{4}{3}e^{\frac{1}{2}\kappa}\right)_{\kappa\kappa\kappa} - 3\left(\frac{4}{3}e^{\frac{1}{2}\kappa}\right)_\kappa\left(\frac{4}{3}e^{\frac{1}{2}\kappa}\right)_{\kappa\kappa}\right]
\end{aligned}
$$

On simplification, we get:

$$
\eta_1 = -\frac{2}{3}e^{\frac{1}{2}\kappa}\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\tau^{\alpha\beta+\gamma-1}}{\beta^\alpha\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}
$$

In a similar vein, we can obtain the third- and higher-order approximations by following the same procedure.

Consequently, successive approximations up to the third order are given as:

$$
\begin{aligned}
\eta_0 &= \frac{4}{3}e^{\frac{1}{2}\kappa}\\
\eta_1 &= -\frac{2}{3}e^{\frac{1}{2}\kappa}\frac{\gamma\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)\tau^{\alpha\beta+\gamma-1}}{\beta^{\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\\
\eta_2 &= \frac{1}{3}e^{\frac{1}{2}\kappa}\frac{\gamma^2\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{2\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\left(\frac{\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\tau^{2\alpha\beta+2\gamma-2}}{\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)}\right.\\
&\left.-\frac{1}{2}\frac{(\alpha\beta+\gamma-1)\ \Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)\ \tau^{2\alpha\beta+2\gamma-3}}{\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)}\right)\\
\eta_3 &= \frac{1}{6}e^{\frac{1}{2}\kappa}\frac{\gamma^2\Gamma\left(1+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}{\beta^{2\alpha}\Gamma\left(1+\alpha+\frac{\gamma}{\beta}-\frac{1}{\beta}\right)}\left(-\frac{\gamma\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\Gamma\left(1+2\alpha+\frac{3\gamma}{\beta}-\frac{3}{\beta}\right)\tau^{3\alpha\beta+3\gamma-3}}{\beta\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\Gamma\left(1+3\alpha+\frac{3\gamma}{\beta}-\frac{3}{\beta}\right)}\right.\\
&-\frac{1}{4}\frac{\gamma(\alpha\beta+\gamma-1)(2\alpha\beta+2\gamma-3)\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)\Gamma\left(1+2\alpha+\frac{3\gamma}{\beta}-\frac{5}{\beta}\right)\tau^{3\alpha\beta+3\gamma-5}}{\beta\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)\Gamma\left(1+3\alpha+\frac{3\gamma}{\beta}-\frac{5}{\beta}\right)}\\
&+\frac{1}{2}\frac{\gamma(2\alpha\beta+2\gamma-2)\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\Gamma\left(1+2\alpha+\frac{3\gamma}{\beta}-\frac{4}{\beta}\right)\tau^{3\alpha\beta+3\gamma-4}}{\beta\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{2}{\beta}\right)\Gamma\left(1+3\alpha+\frac{3\gamma}{\beta}-\frac{4}{\beta}\right)}\\
&\left.+\frac{1}{2}\frac{\gamma(\alpha\beta+\gamma-1)\Gamma\left(1+\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)\Gamma\left(1+2\alpha+\frac{3\gamma}{\beta}-\frac{4}{\beta}\right)\tau^{3\alpha\beta+3\gamma-4}}{\beta\Gamma\left(1+2\alpha+\frac{2\gamma}{\beta}-\frac{3}{\beta}\right)\Gamma\left(1+3\alpha+\frac{3\gamma}{\beta}-\frac{4}{\beta}\right)}\right)\\
&\ \ \vdots
\end{aligned}
\tag{39}
$$

Similar results as in Table(1) are observed in Table(3) where we show the result and absolute errors of example 2 when $\alpha = 1,\ \beta = 1.014,\ \ \gamma = 0.98$. The FFAVIM gives a good approximation in terms of absolute errors with fewer terms needed in the series solution.

In Table(4), we present the result of solving Example 2 with varying value of the fractal parameter, β taken between 1.00 and 1.01 with $\alpha = 1$. The results are also compared with those obtained using the method in [[54]]. It is observed that the accuracy of FFAVIM improves just by varying the parameter β, with time t, which shows that the extra parameter β affords us the chance to obtain better results without increasing the number of terms in the iterative process that would have led to more computations. In addition, our proposed method also outperforms the method in [[54]] utilizing more number of iterations (five) compared to that of our method (three).

Similar outcome is observed in Table(5) where we present the results of applying our proposed FFAVIM method in comparison with the AVIM method in [[54]] to solve Example 2 with $\alpha = 0.9,\ \beta = 1.08\ and\ \gamma = 0.98$. The FFAVIM method competes favourably with the AVIM despite being two iteration terms less. This further confirms the importance of the extra parameter present in NGCFFD together with the fractal dimension utilized in FFAVIM procedure.

Figure(4)$-$6 visualizes the behaviour of the FFAVIM solution of example 2 for different α values when contrasted with that of the exact solution. It is clear that the proposed method gives graphical simulations which mimics that of the exact solution, confirming its suitability.

Table 3: Approx. solution of *Example 2* with third-order FFAVIM for $\alpha = 1,\ \ \beta = 1.014,\ \ \gamma = 0.98.$

κ	τ	Exact solution	FFAVIM Approx.	Absolute Error
	0.02	0.1780570525	0.1780639651	6.91260 E-6
-4	0.04	0.1756987155	0.1757334374	3.47219 E-5
	0.06	0.1733716145	0.1734389308	6.73163 E-5
	0.08	0.1710753356	0.1710753356	1.01402 E-4
	0.02	0.4840092501	0.4840280409	1.87908 E-5
-2	0.04	0.4775986256	0.4776930098	9.43842 E-5
	0.06	0.4712729093	0.4714558942	1.82985 E-4
	0.08	0.4651219777	0.4653066161	2.75640 E-4
	0.02	3.576371501	3.576510347	1.38846 E-4
2	0.04	3.529003039	3.529700447	6.97408 E-4
	0.06	3.482261964	3.483614049	1.35209 E-3
	0.08	3.436139968	3.438176688	2.03672 E-3
	0.02	9.721585665	9.721963087	3.77422 E-4
4	0.04	9.592824829	9.593769902	9.45073 E-4
	0.06	9.465769420	9.594720585	1.89576 E-3
	0.08	9.344379956	9.345933217	5.53638 E-3

Table 4: Comparison of absolute errors between third-order FFAVIM and fifth-order AVIM in [54] for *Example 2*, $\alpha = 1,\ \beta \in [1.00,\ 1.01],\ \gamma = 0.98.$

κ	τ	Exact Solution	FFAVIM	AVIM in [54]	Abs. Error of FFAVIM	Abs. Error of AVIM in [54]
	0.2	1.49832638	1.498326756	1.49837370	3.7500E-7	4.7320E-5
	0.4	1.31129527	1.311295782	1.31123148	5.1000E-7	6.3790E-5
0.5	0.6	1.14761064	1.147614708	1.14746139	4.0732E-6	1.4924E-4
	0.8	1.00435820	1.004339344	1.00422572	1.8865E-5	1.3248E-4
	1.0	0.87898750	0.878993035	0.87897781	5.5279E-6	9.6900E-6
	0.2	1.92388915	1.923889637	1.92394991	4.8101E-7	6.0760E-5
	0.4	1.68373645	1.683737113	1.68365456	6.5553E-7	8.1890E-5
1.0	0.6	1.47356122	1.473566454	1.47336959	5.2301E-6	1.9163E-4
	0.8	1.28962146	1.289597245	1.28945134	2.4222E-5	1.7012E-4
	1.0	1.12864229	1.128649398	1.12862985	7.0980E-6	1.2440E-5

Table 5: Comparison of absolute errors between third-order FFAVIM and fifth-order AVIM in [54] for *Example 2*, $\alpha = 0.9,\ \beta = 1.08,\ \gamma = 0.98.$

κ	τ	Exact Solution	FFAVIM	AVIM in [54]	Abs. Error FFAVIM	Abs. Error of AVIM in [54]
	0.2	1.49832638	1.490158079	1.4405122780	8.168302E-3	5.7814102E-2
	0.4	1.31129527	1.313205682	1.2556273427	1.910410E-3	5.5667927E-2
0.5	0.6	1.14761064	1.163917708	1.1079229036	1.630707E-3	3.9687736E-2
	0.8	1.00435820	1.036950706	0.9856303461	3.259250E-2	1.8727850E-2
	1.0	0.87898750	0.9287880425	0.8823294271	4.980054E-2	3.3419270E-3
	0.2	1.92388915	1.913400850	1.8496543780	1.048831E-2	7.4234772E-2
	0.4	1.68373645	1.686189471	1.6122574219	2.453014E-2	7.1479028E-2
1.0	0.6	1.47356122	1.494499919	1.4226011679	2.093870E-2	5.0960052E-2
	0.8	1.28962146	1.331471064	1.2655744159	4.184960E-2	2.4047044E-2
	1.0	1.12864229	1.192587454	1.1329334103	6.394515E-2	4.2911200E-3

Table 6: Comparison of second-order FFAVIM solution of *Example 3* with VIM in [5] for $\alpha = 1, \ \beta = 0.98, \ \gamma = 0.98$.

κ	τ	FFAVIM	VIM in [5]	|FFAVIM-VIM|
	0.1	0.2895464737	0.3414716667	5.1925193 $E-2$
	0.2	0.9047899428	0.9577533333	5.2963391 $E-2$
	0.3	1.8354881748	1.8768450000	4.1356825 $E-2$
0.1	0.4	3.0696448141	3.0967466667	2.7101853 $E-2$
	0.5	4.5984369297	4.6154583333	1.7021404 $E-2$
	0.6	6.4147424850	6.4309800000	1.6237515 $E-2$
	0.7	8.5125052282	8.5413116667	2.8806438 $E-2$
	0.8	10.886400921	10.944453333	5.8052412 $E-2$
	0.1	0.3722695731	0.5605466667	1.8827709 $E-1$
	0.2	0.9924681385	1.2664533333	2.7398519 $E-1$
	0.3	1.8854063834	2.2357200000	3.5031362 $E-1$
0.2	0.4	3.0409321041	3.4663466667	4.2541456 $E-1$
	0.5	4.4513724066	4.9563333333	5.0496093 $E-1$
	0.6	6.1104349749	6.7036800000	5.9324503 $E-1$
	0.7	8.0127093689	8.7063866667	6.9367730 $E-1$
	0.8	10.153398232	10.962453333	8.0905510 $E-1$

Table 7: Comparison of second-order FFAVIM solution of *Example 3* with VIM in [5] for $\alpha = 0.75, \ \beta = 0.98, \ \gamma = 0.98$.

κ	τ	FFAVIM	VIM in [5]	|FFAVIM-VIM|
	0.1	1.049456121	1.060903465	1.1447344 $E-2$
	0.2	2.631479963	2.591106901	4.0373062 $E-2$
	0.3	4.558209979	4.477801402	8.0408577 $E-2$
0.1	0.4	6.755030059	6.655117794	9.9912265 $E-2$
	0.5	9.178970821	9.083168720	9.5802101 $E-2$
	0.6	11.80119343	11.73432536	6.6868070 $E-2$
	0.7	14.60063609	14.58793515	1.2700940 $E-2$
	0.8	17.56105735	17.62777490	6.6717550 $E-2$
	0.1	1.096826786	1.331249022	2.3442224 $E-1$
	0.2	2.581554361	2.884825181	3.0327082 $E-1$
	0.3	4.351547692	4.721597056	3.7004936 $E-1$
0.2	0.4	6.347247222	6.794716853	4.4746963 $E-1$
	0.5	8.533567918	9.073961079	5.4039316 $E-1$
	0.6	10.88667099	11.53773303	6.5106204 $E-1$
	0.7	13.38900518	14.16957486	7.8056968 $E-1$
	0.8	16.02695688	16.95638418	9.2942730 $E-1$

Example 3

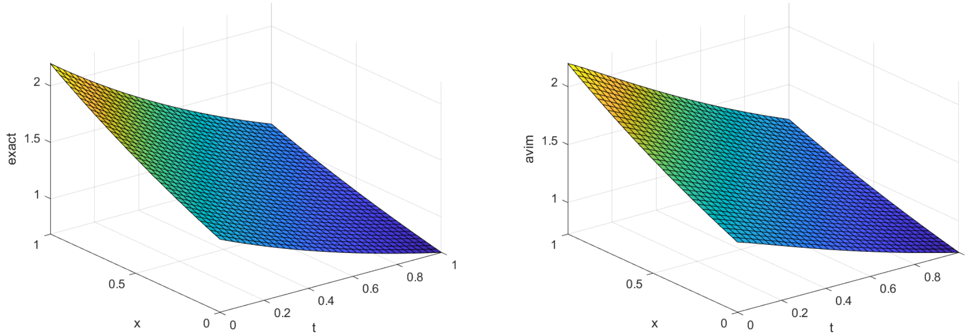
Consider the fractal-fractional PDE with variable coefficients [5]:

$$ {}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi-\kappa^{2}\phi_{\kappa\kappa\tau}+2\phi_{\kappa}+\phi\phi-\phi\phi_{\kappa\kappa\kappa}-3\phi_{\kappa}\phi_{\kappa\kappa}=0 \ , \quad 0<\tau\leq 1 \ , \quad 0<\alpha\ \leq 1 \tag{40} $$

with initial condition: $\phi\left(\kappa,0\right)=\kappa^{2}$

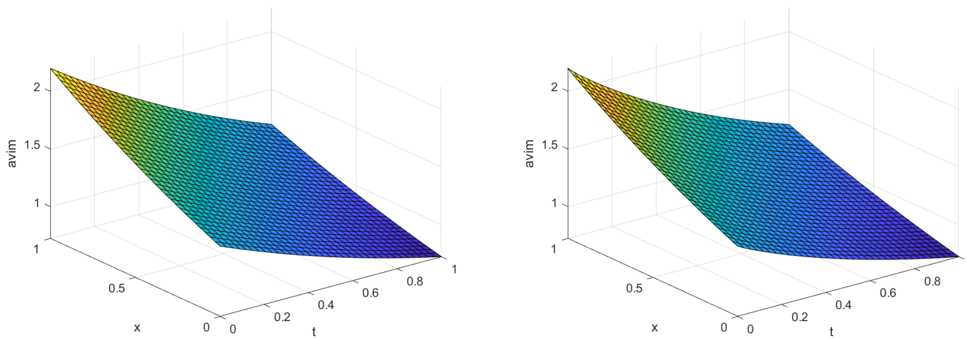
The exact solution to the classical model of the given FFPDE is not presented in [5].

The iteration formula to Equation (40) using the FFAVIM method is equally obtained by means of Equations



(a) Exact solution (b) $\alpha = 1$, $\beta = 1.08$, $\gamma = 0.98$

Figure 4: Graphical simulation of third-order FFAVIM and exact solutions of Example 2.



(a) $\alpha = 0.95$ (b) $\alpha = 0.9$

Figure 5: Graphical simulation of third-order FFAVIM solution of Example 2 for different values of α, $\beta = 1.08$, $\gamma = 0.98$

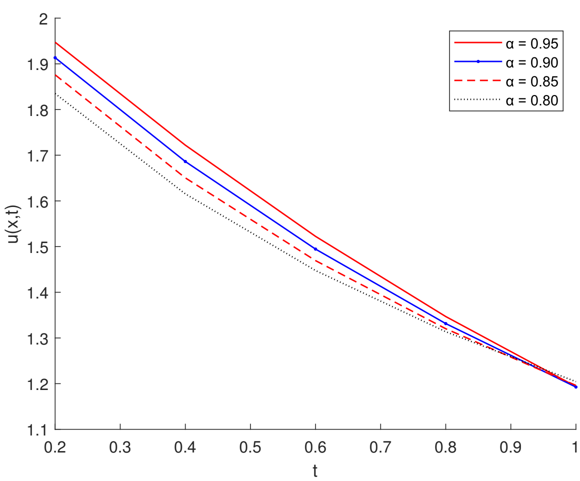


Figure 6: Plots of third-order FFAVIM solution of Example 2 for different values of α with $\beta = 1.08$, $\gamma = 0.98$

(33) and its iterative components also computed as below:

$$\mathcal{A}\left[\phi\right] = I_{\tau}^{\alpha,\beta,\gamma}\left[(-)\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\phi - \kappa^{2}\phi_{\kappa\kappa\tau} + 2\phi_{\kappa} + \phi\phi\ {}_{-}\phi\phi_{\kappa\kappa\kappa} - 3\phi_{\kappa}\phi_{\kappa\kappa}\right)\right]$$

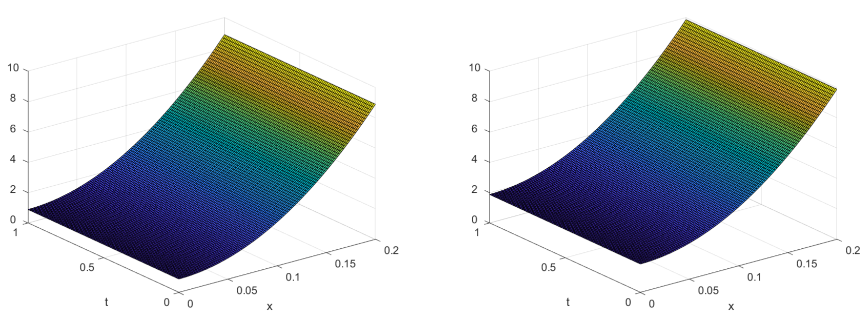
whose iterative components are:

$$\begin{cases} \eta_0 = \kappa^2 \\ \eta_1 = -I_{\tau}^{\alpha,\beta,\gamma}\left[\left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\eta_0\right) - \kappa^2(\eta_0)_{\kappa\kappa\tau} + 2(\eta_0)_{\kappa} + \eta_0(\eta_0)_{\kappa} - \eta_0(\eta_0)_{\kappa\kappa\kappa} - (\eta_0)_{\kappa}(\eta_0)_{\kappa\kappa}\right] \\ \vdots \\ \eta_{k+1} = -I_{\tau}^{\alpha,\beta,\gamma}\left[\begin{array}{c} \left({}^{C}\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right) - \kappa^2\left(\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right)_{\kappa\kappa\tau} \\ +2\left(\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right)_{\kappa} + \left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\left(\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right)_{\kappa} \\ -\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\left(\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right)_{\kappa\kappa\kappa} - 3\left(\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right)_{\kappa}\left(\left[\eta_0 + \eta_1 + \ldots + \eta_k\right]\right)_{\kappa\kappa} \end{array}\right] \end{cases} \tag{41}$$

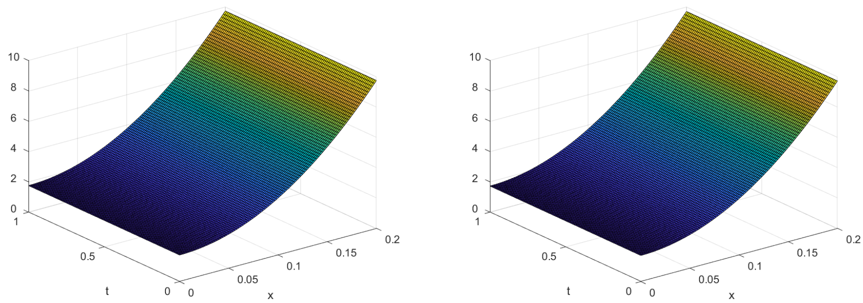
Again, applying the FFAVIM algorithm in a manner analogous to the previous examples, we obtain the following successive approximations up to the second order for this example:

$$\begin{aligned} \eta_0 &= \kappa^2 \\ \eta_1 &= -(2\kappa^3 - 8\kappa)\frac{\gamma\Gamma\left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\tau^{\alpha\beta+\gamma-1}}{\beta^{\alpha}\Gamma\left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)} \\ \eta_2 &= \frac{-12\gamma^2\Gamma\left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)}{\beta^{2\alpha}\Gamma\left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)}\left(\frac{\kappa^3\left(\alpha\beta + \gamma - 1\right)\Gamma\left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta}\right)\tau^{2\alpha\beta+2\gamma-3}}{\Gamma\left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta}\right)}\right. \\ &-\frac{1}{6}\frac{\left(5\kappa^4 - 66\kappa^2 + 16\right)\Gamma\left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta}\right)\tau^{2\alpha\beta+2\gamma-2}}{\Gamma\left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta}\right)} \\ &\left.+\frac{1}{6}\frac{\left(6\kappa^5 - 152\kappa^3 + 224\kappa\right)\gamma\Gamma\left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\Gamma\left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta}\right)\tau^{3\alpha\beta+3\gamma-3}}{\beta^{\alpha}\Gamma\left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\Gamma\left(1 + 3\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta}\right)}\right) \\ &\vdots \end{aligned} \tag{42}$$

In Table(6) and Table(7), we display the results of approximating the variable-coefficient FFPDE in Example 3 with a second-order FFAVIM at different α values. The numerical values are tallied alongside those of the traditional VIM in [[5]] proposed for the same problem. The exact solution isn't available, however, the correctness of the proposed FFAVIM is affirmed via the comparison of the difference between the FFAVIM solution and that of VIM.



(a) $\alpha = 1$ (b) $\alpha = 0.9$



(c) $\alpha = 0.85$ (d) $\alpha = 0.80$

Figure 7: Graphical simulation of second-order FFAVIM solution of Example 3 for different α values, $\beta = 1.08, \quad \gamma = 0.98$

Figure(7) visualizes the behavior of the FFAVIM solution of Example 3 in $3D$ graphical plot for different values of fractional parameter, α. Again, the suitability of the proposed approach is clear from the graphical simulations in Figure(7)(a)$-$ Figure(7)(d) for $\alpha = 1,\ 0.9,\ 0.85$ and 0.80 respectively.

In Figure(8), further comparison is made between the plot of FFAVIM solution of Example 3 for different α values. It is apparent that the method is suitable for FFPDEs whose coefficients are not necessarily constants.

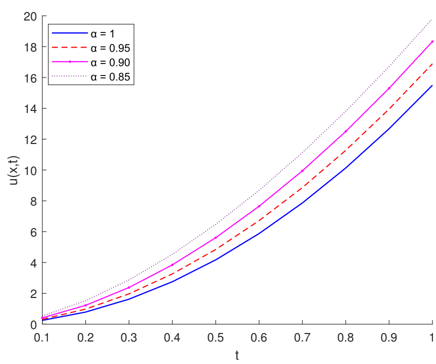


Figure 8: Plots of third-order FFAVIM solution of Example 3 for different α values with $\beta = 1.01, \quad \gamma = 0.98$

6 Conclusions

In this study, we introduced and successfully applied a modified Alternative Variational Iteration Method (FFAVIM) to approximate solutions of generalized time-fractal-fractional partial differential equations (FFPDEs) with both constant and variable coefficients. The fractal-fractional derivative was considered in the sense of the generalized Caputo-type fractal-fractional derivative (NGCFFD). The proposed method was implemented directly, without requiring linearization or perturbation, and provided analytical solutions in series form. To demonstrate the effectiveness of this technique, numerical examples are solved and results are presented in tables and graphs. The results indicate that accurate approximations can be achieved with fewer iterations of the FFAVIM, aided by the additional parameter and fractal dimension incorporated in the NGCFFD. This advantage distinguishes FFAVIM from other existing methods. Furthermore, the FFAVIM compares favorably with the classical AVIM in [[54]] and the VIM in [5], both of which require more iterative steps to attain similar accuracy. The findings suggest that the proposed method has strong potential for solving a broader class of real-world mathematical models beyond those explored in this study. Additionally, future research may focus on extending this approach to handle integro-differential equations and systems of fractal-fractional partial differential equations.

Author contributions Isah, I.O.: Idea Conceptualization, Methodology, Coding, original draft. Senu,N.: Supervision, reviewing and validation of original draft. Ahmadian, A.: Supervision, reviewing and editing, validation of original draft.

Acknowledgement The authors would like to thank...

Conflicts of Interest The authors declare that they have no conflict of interest.

References

[1] T. A. Abassy, M. A. El-Tawil & H. El Zoheiry (2007). Solving nonlinear partial differential equations using the modified variational iteration padé technique. *Journal of Computational and Applied Mathematics*, *207*(1), 73–91. https://doi.org/10.1016/j.cam.2006.07.024.

[2] K. A. Abro & A. Atangana (2020). A comparative study of convective fluid motion in rotating cavity via atangana–baleanu and caputo–fabrizio fractal–fractional differentiations. *The European Physical Journal Plus*, *135*(2), 1–16. https://doi.org/10.1140/epjp/s13360-020-00136-x.

[3] A. Ahmadian, S. Salahshour & C. S. Chan (2016). Fractional differential systems: a fuzzy solution based on operational matrix of shifted chebyshev polynomials and its applications. *IEEE Transactions on Fuzzy Systems*, *25*(1), 218–236. https://doi.org/10.1109/TFUZZ.2016.2554156.

[4] W. Alhejaili, E. A. Az-Zo'bi, R. Shah & S. El-Tantawy (2024). On the analytical soliton approximations to fractional forced korteweg–de vries equation arising in fluids and plasmas using two novel techniques. *Communications in Theoretical Physics*, *76*(8), 085001. https://doi.org/10.1088/1572-9494/ad53bc.

[5] F. Alsidrani, A. Kılıçman & N. Senu (2023). Approximate solutions for time-fractional fornberg–whitham equation with variable coefficients. *Fractal and Fractional*, *7*(3), 260. https://doi.org/10.3390/fractalfract7030260.

[6] A. Atangana (2017). Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. *Chaos, solitons & fractals*, *102*, 396–406. https://doi.org/10.1016/j.chaos.2017.04.027.

[7] A. Atangana & E. F. D. Goufo (2020). Cauchy problems with fractal–fractional operators and applications to groundwater dynamics. *Fractals*, *28*(08), 2040043. https://dx.doi.org/10.1142/S0218348X20400435.

[8] A. Atangana & S. Qureshi (2019). Modeling attractors of chaotic dynamical systems with fractal–fractional operators. *Chaos, solitons & fractals*, *123*, 320–337. https://doi.org/10.1016/j.chaos.2019.04.020.

[9] N. Attia, A. Akgül, D. Seba, A. Nour & M. B. Riaz (2022). Reproducing kernel hilbert space method for solving fractal fractional differential equations. *Results in physics*, *35*, 105225. https://doi.org/10.1016/j.rinp.2022.105225.

[10] İ. Avcı, H. Lort & B. E. Tatlıcıoğlu (2023). Numerical investigation and deep learning approach for fractal–fractional order dynamics of hopfield neural network model. *Chaos, Solitons & Fractals*, *177*, 114302. https://doi.org/10.1016/j.chaos.2023.114302.

[11] J. Azevedo, C. Cuevas & E. Henriquez (2019). Existence and asymptotic behaviour for the time-fractional keller–segel model for chemotaxis. *Mathematische Nachrichten*, *292*(3), 462–480. https://doi.org/10.1002/mana.201700237.

[12] A. K. Badsara & V. S. Chouhan (2024). A fixed point theorem for time-fractional fornberg-whitham equation arising in atmospheric science. *Contemporary Mathematics*, pp. 1265–1281. https://doi.org/10.37256/cm.5220243290.

[13] Z. Bouazza, M. Souid, C. Hussin, A. Mandangan & S. Sabit (2023). Variable-order implicit fractional differential equations based on the kuratowski mnc technique. *Malaysian Journal of Mathematical Sciences*, *17*(3). https://doi.org/10.47836/mjms.17.3.05.

[14] B. Boutarfa, A. Akgül & M. Inc (2017). New approach for the fornberg–whitham type equations. *Journal of Computational and Applied Mathematics*, *312*, 13–26. https://doi.org/10.1016/j.cam.2015.09.016.

[15] W. Chen, H. Sun, X. Zhang & D. Korošak (2010). Anomalous diffusion modeling by fractal and fractional derivatives. *Computers & Mathematics with Applications*, *59*(5), 1754–1758. https://doi.org/10.1016/j.camwa.2009.08.020.

[16] S. Das, P. K. Gupta & P. Ghosh (2011). An approximate solution of nonlinear fractional reaction–diffusion equation. *Applied mathematical modelling*, *35*(8), 4071–4076. https://doi.org/10.1016/j.apm.2011.02.004.

[17] K. Diethelm, N. J. Ford & A. D. Freed (2004). Detailed error analysis for a fractional adams method. *Numerical algorithms*, *36*, 31–52. https://doi.org/10.1023/B:NUMA.0000027736.85078.be.

[18] K. Diethelm & N. Ford (2010). The analysis of fractional differential equations. *Lecture notes in mathematics*, *2004*. https://doi.org/10.1006/jmaa.2000.7194.

[19] M. Donatelli, M. Mazza & S. Serra-Capizzano (2016). Spectral analysis and structure preserving preconditioners for fractional diffusion equations. *Journal of Computational Physics*, *307*, 262–279. https://doi.org/10.1016/j.jcp.2015.11.061.

[20] C. D. Enyi, E. R. Nwaeze & M. E. Omaba (2024). Numerical and analytical solution to a conformable fractional fornberg–whitham equation. *Mathematical Sciences*, *18*(2), 167–180. https://doi.org/10.1007/s40096-022-00491-z.

[21] V. S. Erturk & P. Kumar (2020). Solution of a covid-19 model via new generalized caputo-type fractional derivatives. *Chaos, Solitons & Fractals*, *139*, 110280. https://doi.org/10.1016/j.chaos.2020.110280.

[22] B. Fornberg & G. B. Whitham (1978). A numerical and theoretical study of certain nonlinear wave phenomena. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, *289*(1361), 373–404. https://doi.org/10.1098/rsta.1978.0064.

[23] M. Ganjiani (2010). Solution of nonlinear fractional differential equations using homotopy analysis method. *Applied Mathematical Modelling*, *34*(6), 1634–1641. https://doi.org/10.1016/j.apm.2009.09.011.

[24] F. Geng, M. Cui & B. Zhang (2010). Method for solving nonlinear initial value problems by combining homotopy perturbation and reproducing kernel hilbert space methods. *Nonlinear Analysis: Real World Applications*, *11*(2), 637–644. https://doi.org/10.1016/j.nonrwa.2008.10.033.

[25] J. He (1997). A new approach to nonlinear partial differential equations. *Communications in Nonlinear Science and Numerical Simulation*, *2*(4), 230–235. https://doi.org/10.1016/S1007-5704(97)90007-1.

[26] Y. He & W. Rui (2025). Extended separation method of semi-fixed variables together with analytical method for solving time fractional equation. *Mathematical Methods in the Applied Sciences*,. https://doi.org/10.1002/mma.10856.

[27] M. Heydari, M. Hooshmandasl, F. M. Ghaini & C. Cattani (2015). Wavelets method for the time fractional diffusion-wave equation. *Physics Letters A*, *379*(3), 71–76. https://doi.org/10.1016/j.physleta.2014.11.012.

[28] M. Heydari & S. Zhagharian (2023). Genocchi polynomials for variable-order time fractional fornberg–whitham type equations. *Partial Differential Equations in Applied Mathematics*, *8*, 100554. https://doi.org/10.1016/j.padiff.2023.100554.

[29] M. H. Heydari, Z. Avazzadeh & Y. Yang (2019). A computational method for solving variable-order fractional nonlinear diffusion-wave equation. *Applied Mathematics and Computation*, *352*, 235–248. https://doi.org/10.1016/j.amc.2019.01.075.

[30] R. Hilfer (2000). *Applications of fractional calculus in physics*. World scientific,.

[31] J. Huang, Y. Tang, L. Vázquez & J. Yang (2013). Two finite difference schemes for time fractional diffusion-wave equation. *Numerical Algorithms*, *64*, 707–720. https://doi.org/10.1007/s11075-012-9689-0.

[32] R. W. Ibrahim (2012). Numerical solution for complex systems of fractional order. *Journal of Applied Mathematics*, *2012*. https://doi.org/10.1155/2012/678174.

[33] N. Iqbal, S. Hussain, M. N. Tufail & W. W. Mohammed (2024). Solving the fractional fornberg-whitham equation within caputo framework using the optimal auxiliary function method. *Physica Scripta*, *99*(5), 055234. https://doi.org/10.1088/1402-4896/ad3488.

[34] H. Jafari & V. Daftardar-Gejji (2006). Solving a system of nonlinear fractional differential equations using adomian decomposition. *Journal of Computational and Applied Mathematics*, *196*(2), 644–651. https://doi.org/10.1016/j.cam.2005.10.017.

[35] F. Jarad, T. Abdeljawad & D. Baleanu. On the generalized fractional derivatives and their caputo modification 2017. http://hdl.handle.net/20.500.12416/2171.

[36] U. N. Katugampola (2014). Existence and uniqueness results for a class of generalized fractional differential equations. *arXiv preprint arXiv:1411.5229*,. https://doi.org/10.48550/arXiv.1411.5229.

[37] N. Kedia, A. A. Alikhanov & V. K. Singh (2022). Stable numerical schemes for time-fractional diffusion equation with generalized memory kernel. *Applied Numerical Mathematics*, *172*, 546–565. https://doi.org/10.1016/j.apnum.2021.11.006.

[38] N. Khan, Z. Ahmad, J. Shah, S. Murtaza, M. D. Albalwi, H. Ahmad, J. Baili & S.-W. Yao (2023). Dynamics of chaotic system based on circuit design with ulam stability through fractal-fractional derivative with power law kernel. *Scientific Reports*, *13*(1), 5043. https://doi.org/10.1038/s41598-023-32099-1.

[39] S. Kumar, V. Joshi & M. Kapoor (2024). Elzaki residual power series approach for semi-analytic approximation of time-fractional nonlinear equations occurring in magneto-acousticwaves. *Physica Scripta*, *100*(1), 015274. https://doi.org/10.1088/1402-4896/ad9d8e.

[40] J. Lenells (2004). Traveling wave solutions of the camassa-holm and korteweg-de vries equations. *Journal of Nonlinear Mathematical Physics*, *11*(4), 508–520. https://doi.org/10.2991/jnmp.2004.11.4.7.

[41] J. Liu & G. Hou (2011). Numerical solutions of the space-and time-fractional coupled burgers equations by generalized differential transform method. *Applied Mathematics and Computation*, *217*(16), 7001–7008. https://doi.org/10.1016/j.amc.2011.01.111.

[42] V. E. Lynch, B. A. Carreras, D. del Castillo-Negrete, K. Ferreira-Mejias & H. Hicks (2003). Numerical methods for the solution of partial differential equations of fractional order. *Journal of Computational Physics*, *192*(2), 406–421. https://doi.org/10.1016/j.jcp.2003.07.008.

[43] J. T. Machado, V. Kiryakova & F. Mainardi (2011). Recent history of fractional calculus. *Communications in nonlinear science and numerical simulation*, *16*(3), 1140–1153. https://doi.org/10.1016/j.cnsns.2010.05.027.

[44] S. Momani & Z. Odibat (2008). A novel method for nonlinear fractional partial differential equations: combination of dtm and generalized taylor's formula. *Journal of Computational and Applied Mathematics*, *220*(1-2), 85–95. https://doi.org/10.1016/j.cam.2007.07.033.

[45] R. Nawaz, N. Nik Long & S. Shohaimi (2024). Caputo fractional differential equations for low–risk individuals of the tuberculosis transmission disease. *Malaysian Journal of Mathematical Sciences*, *18*(4). https://doi.org/10.47836/mjms.18.4.14.

[46] A. Nazari-Golshan (2024). Exploring new traveling wave solutions by solving the nonlinear space–time fractal fornberg- whitham equation. *Scientific Reports*, *14*(1), 18642. https://doi.org/10.1038/s41598-024-66298-1.

[47] S. Noor, A. W. Alrowaily, M. Alqudah, R. Shah & S. A. El-Tantawy (2024). Innovative solutions to the fractional diffusion equation using the elzaki transform. *Mathematical and Computational Applications*, *29*(5), 75. https://doi.org/10.3390/mca29050075.

[48] Z. Odibat & D. Baleanu (2020). Numerical simulation of initial value problems with generalized caputo-type fractional derivatives. *Applied Numerical Mathematics*, *156*, 94–105. https://doi.org/10.1016/j.apnum.2020.04.015.

[49] Z. Odibat & S. Momani (2008). Numerical methods for nonlinear partial differential equations of fractional order. *Applied Mathematical Modelling*, *32*(1), 28–39. https://doi.org/10.1016/j.apm.2006.10.025.

[50] Z. M. Odibat (2010). A study on the convergence of variational iteration method. *Mathematical and Computer Modelling*, *51*(9-10), 1181–1192. https://doi.org/10.1016/j.mcm.2009.12.034.

[51] M. Partohaghighi, M. Mortezaee, A. Akgül, A. M. Hassan & N. Sakar (2024). Numerical analysis of the fractal-fractional diffusion model of ignition in the combustion process. *Alexandria Engineering Journal*, *86*, 1–8. https://doi.org/10.1016/j.aej.2023.11.038.

[52] I. Podlubny (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. elsevier,.

[53] S. S. Ray (2006). Exact solutions for time-fractional diffusion-wave equations by decomposition method. *Physica Scripta*, *75*(1), 53. https://doi.org/10.1088/0031-8949/75/1/008.

[54] M. G. Sakar & H. Ergören (2015). Alternative variational iteration method for solving the time-fractional fornberg–whitham equation. *Applied Mathematical Modelling*, *39*(14), 3972–3979. https://doi.org/10.1016/j.apm.2014.11.048.

[55] A. Shloof, N. Senu, A. Ahmadian & S. Salahshour (2021). An efficient operation matrix method for solving fractal–fractional differential equations with generalized caputo-type fractional–fractal derivative. *Mathematics and Computers in Simulation*, *188*, 415–435. https://doi.org/10.1016/j.matcom.2021.04.019.

[56] B. K. Singh & P. Kumar (2017). Fractional variational iteration method for solving fractional partial differential equations with proportional delay. *International journal of differential equations*, *2017*. https://doi.org/10.1155/2017/5206380.

[57] C. Tadjeran & M. M. Meerschaert (2007). A second-order accurate numerical method for the two-dimensional fractional diffusion equation. *Journal of Computational Physics*, *220*(2), 813–823. https://doi.org/10.1016/j.jcp.2006.05.030.

[58] S. Ullah, F. Yaqub, T. Munir & H. Zeb (2024). Optimal semi-analytical solutions of time-fractional evolution equations. *Journal of Applied Mathematics and Computing*, pp. 1–21. https://doi.org/10.1007/s12190-024-02310-1.

[59] K. Vishal, S. Kumar & S. Das (2012). Application of homotopy analysis method for fractional swift hohenberg equation–revisited. *Applied Mathematical Modelling*, *36*(8), 3630–3637. https://doi.org/10.1016/j.apm.2011.10.001.

[60] Y. Wang, L. Su, X. Cao & X. Li (2011). Using reproducing kernel for solving a class of singularly perturbed problems. *Computers & Mathematics with Applications*, *61*(2), 421–430. https://doi.org/10.1016/j.camwa.2010.11.019.