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A Differential Game of Evasion from Many Pursuers

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ABSTRACT

This paper considers a game problem with many pursuers described by infinite systems of differential equations of second order. On the controls of players geometric constraints are imposed. The aim of the pursuers is to capture the evader, while the aim of the evader is the opposite. The theorem on evasion is proved in this paper.

Key words: differential game, pursuer, evader, control, evasion.
Subject classifications: 49N70, 49N75, 49N90

INTRODUCTION

Many investigations are devoted to the differential games, see for example [2]-[12]. Fundamental results were obtained by [8]-[10].

Of special interests are differential games with many pursuers [3], [6]-[7], [11]. Pshenichniy [11] obtained the necessary and sufficient conditions for the capture of a single evader by a group of pursuers where the players have equal resources.

Evasion differential games attract the attention of many authors, see for example [10], [12]. Some control problems and game problems in system with distributed parameters can be reduced to the ones described by infinite system of differential equations (for example, see [2], [4]). Chernous'ko [2] considered control problems in which geometric constraints are imposed on controls.

It was found optimal pursuit time [4] for the game problem with one pursuer described by parabolic equation. Guaranteed pursuit time was obtained [5] for the game problem with one pursuer and described by infinite system of differential equations of second order.

In the present paper we investigate a game problem of m pursuers and one evader described by infinite systems of differential equations of second order. It is the reformulation of the game problem for hyperbolic partial differential equations [2]. Geometric constraints are imposed on the controls of players. The theorem on evasion is proved in this paper.

STATEMENT OF PROBLEM

Let $\lambda_1, \lambda_2, \ldots$ be a sequence of positive numbers and r be any given number. We introduce the space

$$l_r^2 = \{\alpha = (\alpha_1, \alpha_2, \ldots) : \sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 < \infty\}$$

with inner product and the norm

$$(\alpha, \beta)_r = \sum_{i=1}^{\infty} \lambda_i^r \alpha_i \beta_i, \quad \alpha, \beta \in l_r^2, \ \| \alpha \|_r = \left(\sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 \right)^{1/2}.$$

Consider the differential game described by countable many differential equations

$$\ddot{z}_{ik} = -\lambda_k z_{ik} - u_{ik} + v_k, \quad z_{ik}(0) = z_{ik}^0, \quad \dot{z}_{ik}(0) = z_{ik}^1, \quad k = 1, 2, \ldots, \tag{1}$$

where $\quad z_{ik}, u_{ik}, v_k, z_{ik}^0, z_{ik}^1 \in R^1, \quad z_i^0 = (z_{i1}^0, z_{i2}^0, \ldots) \in l_{r+1}^2,$

$$z_i^1 = (z_{i1}^1, z_{i2}^1, \ldots) \in l_r^2, \ \| z_i^0 \|_{r+1} + \| z_i^1 \|_r \neq 0,$$

$u_i = (u_{i1}, u_{i2}, \ldots)$ is control parameter of the ith pursuer, $i = 1, 2, \ldots, m,$ and $v = (v_1, v_2, \ldots)$ is control parameter of the evader. Denote

$$L_2(0, T; l_r^2) = \{f(\cdot) = (f_1(\cdot), f_2(\cdot), \ldots) : f_i(\cdot) \in L_2(0, T), \ \sum_{k=1}^{\infty} \lambda_k^r f_k^2(t) < \infty\}$$

Definition 1. A function $u_i(\cdot) = (u_{i1}(\cdot), u_{i2}(\cdot), \ldots)$ satisfying the condition

$$\| u_i(t) \| = \left(\sum_{k=1}^{\infty} \lambda_k^r u_{ik}^2(t) \right)^{1/2} \le \rho_i, \quad u_{ik}(\cdot) \in L_2(0,T),$$

where $\rho_1, \rho_2, ..., \rho_m$ and T are given positive numbers, is called the admissible control of the i th pursuer.

Definition 2. A function $v(\cdot) = (v_1(\cdot), v_2(\cdot), ...)$ satisfying the constraint

$$\| v(t) \| = \left(\sum_{k=1}^{\infty} \lambda_k^r v_k^2(t) \right)^{1/2} \le \sigma, \ \ v_k(\cdot) \in L_2(0,T),$$

where σ is given positive number is called the admissible control of the evader.

Definition 3. Assume that $w(\cdot) = (w_1(\cdot), w_2(\cdot), ...) \in L_2(0, T, l_r^2)$. The function $z(t) = (z_1(t), z_2(t), ...), 0 \le t \le T$, where each of the coordinate $z_k(t)$

1) is continuously differentiable on $(0;T)$ and satisfies initial conditions $z_k(0) = z_k^0, \ \ \dot{z}_k(0) = z_k^1;$
2) has the second derivative $\ddot{z}_k(t)$ almost everywhere on $(0;T)$ satisfying almost everywhere equation

$$\ddot{z}_k(t) = -\lambda_k z_k(t) + w_k(t),$$

is called the solution of the system

$$\ddot{z}_k = -\lambda_k z_k + w_k(t), z_k(0) = z_k^0, \dot{z}_k(0) = z_k^1, k = 1, 2, ... \tag{2}$$

Equation (2) has the solution

$$z_k(t) = z_k^0 \cos(\alpha_k t) + z_k^1 \frac{\sin(\alpha_k t)}{\alpha_k} + \int_0^t \frac{\sin(\alpha_k (t-s))}{\alpha_k} w_k(s) ds, \tag{3}$$

where $\alpha_k = \sqrt{\lambda_k}$.

Its derivative is

$$\dot{z}_k(t) = -z_k^0\alpha_k \sin(\alpha_k t) + z_k^1 \cos(\alpha_k t) + \int_0^t \cos(\alpha_k (t-s))w_k(s)ds\,. \tag{4}$$

Let $C(0,T;l_r^2)$ be the space of functions $z(t)$, $z:[0,T]\to l_r^2$, being continuous in the norm of l_r^2. If

$$w(t) = (w_1(t), w_2(t), ...) \in L_2(0,T;l_r^2),\quad z^0 = (z_1^0, z_2^0, ...) \in l_{r+1}^2,\quad z^1 = (z_1^1, z_2^1, ...) \in l_r^2,$$

then the following assertion can be proven analogous with [1] and it is not studied in this paper.

Assertion. The functions $z(t) = (z_1(t), z_2(t), ...)$, and $\dot{z}(t) = (\dot{z}_1(t), \dot{z}_2(t), ...)$, where $z_k(t)$ and $\dot{z}_k(t)$ are defined by the formulas (3)-(4), belong to $C(0,T;l_{r+1}^2)$ and $C(0,T;l_r^2)$ respectively.

Definition 4. We say that evasion is possible in the game (1) from an initial position $\{z^0, z^1\}$, where

$$z^0 = \{z_1^0, z_2^0, ..., z_m^0\},\ \ z_i^0 \in l_{r+1}^2,\ \ z^1 = \{z_1^1, z_2^1, ..., z_m^1\},\ \ z_i^1 \in l_r^2,$$

if it can be chosen an admissible control $v(\cdot)$ of the evader such that for any admissible controls of pursuers $u_i(\cdot)$, $i = 1, 2, ..., m,$ and arbitrary number $\vartheta, \vartheta > 0,$ the relation $\| z_i(t) \|_{r+1} + \| \dot{z}_i(t) \|_r \neq 0$ is true for all $t \in [0, \vartheta]$ and $i \in \{1, 2, ..., m\}.$

MAIN RESULT

In this sections we present the theorem on evasion.

Theorem 1. If $\rho_i \le \sigma$ for all $i = 1, 2, ..., m,$ then from any initial position $\{z^0, z^1\}$

$$z^0 = \{z_1^0, z_2^0, ..., z_m^0\},\ \ z_i^0 \in l_{r+1}^2,\ \ z^1 = \{z_1^1, z_2^1, ..., z_m^1\},\ \ z_i^1 \in l_r^2,$$

$$\| z_i^0 \|_{r+1} + \| z_i^1 \|_r \neq 0,\ \ i = 1, 2, ..., m,$$

evasion is possible in the game (1).

Proof. Let ϑ be an arbitrary positive number. We consider the system

$$\begin{aligned} z_{ik}(t) &= z_{ik}^0 cos(\alpha_k t) + z_{ik}^1 \frac{sin(\alpha_k t)}{\alpha_k} + \int_0^t \frac{sin(\alpha_k (t-s))}{\alpha_n} w_{ik}(s)ds \\ \dot{z}_{ik}(t) &= -\alpha_k z_{ik}^0 sin(\alpha_k t) + z_{ik}^1 cos(\alpha_k t) + \int_0^t cos(\alpha_k (t-s)) w_{ik}(s)ds, \end{aligned} \tag{5}$$

where $\alpha_k = \sqrt{\lambda_k}$, on the time-interval $[0, \vartheta]$. In order to obtain more simple a system we transform the system (5) setting

$$\begin{bmatrix} x_{ik}(t) \\ y_{ik}(t) \end{bmatrix} = \begin{bmatrix} \cos(\alpha_k t) & -\sin(\alpha_k t) \\ \sin(\alpha_k t) & \cos(\alpha_k t) \end{bmatrix} \begin{bmatrix} \alpha_k z_{ik}(t) \\ \dot{z}_{ik}(t) \end{bmatrix}, \quad \begin{bmatrix} x_{ik}^0 \\ y_{ik}^0 \end{bmatrix} = \begin{bmatrix} \alpha_k z_{ik}^0 \\ \dot{z}_{ik}^1 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} x_{ik}(t) \\ y_{ik}(t) \end{bmatrix} = \begin{bmatrix} x_{ik}^0 - \int_0^t sin(\alpha_k s) w_{ik}(s)ds \\ y_{ik}^0 + \int_0^t cos(\alpha_k s) w_{ik}(s)ds \end{bmatrix}, \quad i = 1,...,m; k = 1,2,...,$$

where

$$x_i^0 = (x_{i1}^0, x_{i2}^0, ...) \in l_r^2, \quad y_i^0 = (y_{i1}^0, y_{i2}^0, ...) \in l_r^2.$$

Indeed,

$$\| x_i^0 \|_r^2 = \sum_{k=1}^{\infty} \lambda_k^r \, | x_{ik}^0 |^2 = \sum_{k=1}^{\infty} \lambda_k^r \alpha_k^2 \, | z_{ik}^0 |^2 = \| z_i^0 \|_{r+1}^2,$$

$$\| y_i^0 \|_r^2 = \sum_{k=1}^{\infty} \lambda_k^r \, | y_{ik}^0 |^2 = \sum_{k=1}^{\infty} \lambda_k^r \, | z_{ik}^1 |^2 = \| z_i^1 \|_r^2 .$$

Further we consider the system (6). As $\| x_i^0 \|_r + \| y_i^0 \|_r \neq 0, \; i = 1,2,...,m,$ then we can pick a natural number M for the M -dimensional vectors

$$X_i^0 = (x_{i1}^0, x_{i2}^0, ..., x_{iM}^0) \in R^M, \quad Y_i^0 = (y_{i1}^0, y_{i2}^0, ..., y_{iM}^0) \in R^M$$

to have relations $|X_i^0| + |Y_i^0| \neq 0, \quad i = 1, 2, ..., m,$ that is vectors X_i^0 and Y_i^0 as elements of the space R^M are not equal to zero simultaneously.

Without any loss of generality, let $M \geq m.$ Then there exist unit vectors $p, q \in R^M$ such that $pX_i^0 \geq 0, \; qY_i^0 \geq 0,$ for all $i = 1, 2, ..., m.$

Instead of considering the differential equations (1) we consider the following finite systems of equations

$$X_{ik}(t) = x_{ik}^0 + \int_0^t u_{ik}(s) sin(\alpha_k s) ds - \int_0^t v_k(s) sin(\alpha_k s) ds, \tag{7}$$

$$Y_{ik}(t) = y_{ik}^0 - \int_0^t u_{ik}(s) cos(\alpha_k s) ds + \int_0^t v_k(s) cos(\alpha_k s) ds, \tag{8}$$

where

$$X_i^0 = (x_{i1}^0, x_{i2}^0, ..., x_{iM}^0), \quad Y_i^0 = (y_{i1}^0, y_{i2}^0, ..., y_{iM}^0), \quad k = 1, 2, ..., M, \quad i = 1, ..., m,$$

with control parameters $u_i = (u_{i1}, ..., u_{iM})$ and $v = (v_1, ..., v_M)$ satisfying conditions

$$\sum_{k=1}^{M} \lambda_k^r u_{ik}^2(t) \leq \rho_i^2, \quad \sum_{k=1}^{M} \lambda_k^r v_k^2(t) \leq \sigma^2.$$

To prove the theorem it is sufficient to show that all $2M$-dimensional vector-functions

$$(X_i(t), Y_i(t)) = (x_{i1}(t), ..., x_{iM}(t), y_{i1}(t), ..., y_{iM}(t)),$$

are not equal to zero, that is

$$|X_i(t)| + |Y_i(t)| \neq 0, \quad 0 \leq t \leq \vartheta, \; i = 1, 2, ..., m.$$

Define

$$v_k(t) = \begin{cases} \dfrac{\sigma}{A}(-p_k sin(\alpha_k t) + q_k cos(\alpha_k t)), & 1 \le k \le M, \\ 0, & k > M, \end{cases} \tag{9}$$

where

$$A = \left(\sum_{k=1}^{M} \lambda_k^r (-p_k \sin(\alpha_k t) + q_k \cos(\alpha_k t))^2 \right)^{1/2}.$$

Using Cauchy-Schwartz inequality, substituting (9) into (7) and (8) we have

$$(X_i(t), p) + (Y_i(t), q) = (X_i^0, p) + (Y_i^0, q) +$$

$$+\int_0^t \sum_{k=1}^{M} \lambda_k^r u_{ik}(s)(-p_k \sin(\alpha_k s) + q_k \cos(\alpha_k s))ds +$$

$$+\int_0^t \sum_{k=1}^{M} \lambda_k^r v_k(s)(-p_k \sin(\alpha_k s) + q_k \cos(\alpha_k s))ds$$

$$= (X_i^0, p) + (Y_i^0, q) + \int_0^t \sum_{k=1}^{M} \lambda_k^r u_{ik}(s)(-p_k \sin(\alpha_k s) + q_k \cos(\alpha_k s))ds + \sigma \int_0^t A(s)ds$$

$$\ge (X_i^0, p) + (Y_i^0, q) - \rho_i \int_0^t A(s)ds + \sigma \int_0^t A(s)ds.$$

As $(X_i^0, p) \ge 0,\ (Y_i^0, q) \ge 0$ and $\rho_i \le \sigma,$ we have

$$(X_i(t), p) + (Y_i(t), q) \ge 0. \tag{10}$$

We assume the contrary, i.e. evasion is not possible in the game (1).
Let

$$X_i(\tau) = 0,\ Y_i(\tau) = 0 \tag{11}$$

at some $\tau \in [0, \vartheta]$ and $i \in \{1,..,m\}$. Then

$$(X_i(\tau), p) + (Y_i(\tau), q) = 0. \tag{12}$$

Using (10) and (12), we find that

(i) $\quad u_{ik}(s) = \dfrac{\sigma}{A(s)}(-p_k \sin(\alpha_k s) + q_k \cos(\alpha_k s)),\ i = 1,...,m;\ \ k = 1,2,...,M,$

almost everywhere on $[0,\tau]$ that is, $u_{ik}(s) = v_k(s)$ almost everywhere on $[0,\tau]$, in contrast, Cauchy-Schwartz inequality gives the inequality $(X_i(\tau), p) + (Y_i(\tau), q) > 0$ contradicting (12);

(ii) $\quad (X_i^0, p) = 0,\ \ (Y_i^0, q) = 0,\ \ \rho_i = \sigma, i = 1,2,...,m.$

Setting $u_{ik}(s) = v_k(s), 0 \le s \le \tau, i = 1,...,m,$ in (7) and (10), we get

$$X_{ik}(t) = x_{ik}^0 + \int_0^t v_k(s) sin(\alpha_k s) ds - \int_0^t v_k(s) sin(\alpha_k s) ds = x_{ik}^0,$$

$$Y_{ik}(t) = y_{ik}^0 - \int_0^t v_{ik}(s) cos(\alpha_k s) ds + \int_0^t v_k(s) cos(\alpha_k s) ds = y_{ik}^0,$$

so

$$X_i(t) = X_i^0,\ \ Y_i(t) = Y_i^0,\ \ 0 \le t \le \tau,\ \ i = 1,2,...,m.$$

In accordance with the choice $|X_i^0| + |Y_i^0| \ne 0,\ \ i = 1,...,m,$ and we have

$$|X_i(\tau)| + |Y_i(\tau)| \ne 0,\ \ 0 \le t \le \tau,\ \ i = 1,...,m,$$

that contradicts (11). This completes the proof of the theorem.

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