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Chromatic Equivalence Class of the Join of Certain Tripartite Graphs

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ABSTRACT

For a simple graph G, let $P(G;\lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$ if $P(G;\lambda) = P(H;\lambda)$. A graph G is said to be chromatically unique, if $H \sim G$ implies that $H \cong G$. Chia [4] determined the chromatic equivalence class of the graph consisting of the join of p copies of the path each of length 3. In this paper, we determined the chromatic equivalence class of the graph consisting of the join of p copies of the complete tripartite graph $K_{1,2,3}$.
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INTRODUCTION

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph G, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent*, or *χ-equivalent*, denoted $G \sim H$ if $P(G) = P(H)$. It is clear that the relation " $\sim$ " is an equivalence relation on the family of graphs. We denote by $[G]$ the equivalence class determined by G under " $\sim$ ". A graph G is said to be *chromatically unique*, or *χ-unique*, if $[G] = \{G\}$, i.e., $H \sim G$ implies that $H \cong G$. Many families of *χ-unique* graphs are known (see [8, 9]), relatively fewer results concerning the chromatic equivalence class of graphs are known (see [2, 3, 4]). In this paper, our main purpose is to determine the chromatic equivalence class of the graph consisting of the join of p copies of the complete tripartite graph $K_{1,2,3}$.

In what follows, we let K_n denote the complete graph on n vertices, $K_{p1,p2,...,pt}$ the complete t-partite graph having n_i vertices in the i-th partite set, P_n and C_n the path and cycle on n vertices, respectively and $\chi(G)$ the chromatic number of G. Let W_n denote the wheel of order n and U_n the graph obtained from W_n by deleting a spoke of W_n. Also let $n(A,G)$ denote the number of subgraph A in G and $i(A,G)$ the number of induced subgraph A in G.

The join of two graphs G and H, denoted $G + H$, is the graph obtained from the union of G and H by joining every vertex of G to every vertex of H.
Let F be a graph and let $G = F + F + \ldots + F$ or pF denote the join of p (≥ 2) copies of F. We wish to determine $[G]$. Let $J_p(F)$ denote the set of all graphs H which are of the form $H = H_1 + H_2 + \ldots + H_p$, where $H_i \in [F]$, $i = 1, 2, \ldots, p$.
In [4], Chia posed the following problem

Problem: *What are those graphs F for which* $J_p(F) = [G]$*?.*

and solve the problem for the case $F = P_4$. In this paper, by making very minor modify-cation to the technique used in [4], we solve the above problem for the case $F = K_{1,2,3}$.

PRELIMINARY RESULTS AND NOTATIONS

A spanning subgraph is called a *clique cover* if its connected components are complete graphs. Let G be a graph on n vertices. Let $s_k(G)$ denote the number of clique cover of G with k connected components, $k = 1, 2, \ldots, n$. If the chromatic polynomial of G is $P(G,\lambda) = \sum_{k=1}^{n} s_k(\overline{G})(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1)\cdots(\lambda - k + 1)$, then the polynomial $\sigma(G, k) = \sum_{k=1}^{n} s_k(\overline{G})x^k$ is called the σ-polynomial of G (see Brenti(1992)). It is easy to see that $\sigma(G, x) = x^n$ if and only if $G = K_n$ since $s_k(G) = 0$ for $k < \chi(G) = n$. Also note that $s_n(G) = 1$ and $s_{n-1}(G) = m$ *if* G has m edges. Clearly, $P(G,\lambda) = P(H,\lambda)$ if and only if $\sigma(G, x) = \sigma(H, x)$ and $s_k(G) = s_k(H)$ for $k = 1, 2, \ldots$.
If $\sigma(G, x) = xf(x)$ for some irreducible polynomial $f(x)$ over the rational number field, then $\sigma(G, x)$ is said to be irreducible.

Lemma 2.1. (Farrell (1980)) Let G and H be two graphs such that $G \sim H$. Then G and H have the same number of vertices, edges and triangles. If both G and H has no K_4 as subgraph, then $i(C_4, G) = i(C_4, H)$. Moreover,

$$
\begin{aligned}
&-i(C_5,G) + i(K_{2,3},G) + 2i(U_5,G) + 3i(W_5,G) \\
&\qquad = -i(C_5,H) + i(K_{2,3},H) + 2i(U_5,H) + 3i(W_5,H).
\end{aligned}
$$

Lemma 2.2. (Brenti (1992)) Let G and H be two disjoint graphs. Then

$$\sigma(G + H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K_{n_1,n_2,\ldots,n_t}, x) = \Pi_{i=1}^{t}\sigma(O_{n_i}, x).$$

Lemma 2.3. (Liu (1992)) Let G be a connected graph with n vertices and m edges. Assume that G is not the complete graph K_3. Then

$$s_{n-2}(G) \leq \binom{m-1}{2}$$

and equality holds if and only if G is the path P_{m+1}.

A CHROMATIC EQUIVALENCE CLASS

We first have the following lemma which follows readily from Lemma 2.1.

Lemma 3.1. $[K_{1,2,3}] = \{K_{1,2,3}, K_{2,2,2} - e\}$ where e is an edge of $K_{2,2,2}$.

We now have our main theorem as follow.

Theorem 3.1. Let $G = K_{1,2,3} + K_{1,2,3} + \ldots + K_{1,2,3}$ be the join of p copies of $K_{1,2,3}$. Then $[G] = J_p(K_{1,2,3})$.

Proof. Let $H \sim G$, we will show that $H \in J_p(K_{1,2,3})$. Since $P(G) = P(H)$ implies that $\sigma(G) = \sigma(H)$, it is more convenient to look at $\sigma(G)$ and $\sigma(H)$. First note that $\sigma(K_{1,3}) = x(x^3 + 3x^2 + x) = \sigma(K_{2,2} - e)$ with $[K_{1,3}] = \{K_{1,3}, K_{2,2} - e\}$, and $\sigma(K_{1,2,3}) = x(x^2 + x)(x^3 + 3x^2 + x) = P(K_{2,2,2} - e)$. So, $\sigma(G) = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = [(x^2 + x)(x^4 + 3x^3 + x^2)]^p$, having p irreducible factors of x, $x^2 + x$ and $x^3 + 3x^2 + x$ respectively.

Let n and m denote the number of vertices and edges in H respectively. Then $n = 6p$ and $m = 36\binom{p}{2} + 11p = 18p^2 - 7p$ so that $\sigma(H) = \sigma(G) = \sum_{i=1}^{6p} s_i(\overline{G})x^i$. Moreover, H is uniquely $3p$-colorable as G is so.

Let $V_1, V_2, \ldots, V_{3p}$ be the color classes of the unique $3p$-coloring of H. Let V_{ij} denote the subgraph induced by $V_1 \cup V_j, i \neq j$. Call V_{ij} a 2-color subgraph of H.

Case (i): Every V_i has exactly two vertices.

In this case, V_{ij} is either a path P_4 or else a cycle C_4 because, by Theorem 12.16 of [6], V_{ij} is connected for $i \neq j$. Note that the number of 2-color subgraphs in H is $\binom{3p}{2} = \frac{1}{2}(9p^2 - 5p) + p$. By looking at the number of edges in H, we see that exactly p

of the 2-color subgraphs V_{ij} are P_4 and the rest of the 2-color subgraphs are C_4. This means that $\overline{H}$ has only P_4 and K_2 as subgraph so that $H = s\overline{P_4} + r\overline{K_2}$ $(s,r \geq 0)$. Consequently,

$$\sigma(H) = [(x^4+3x^3+x^2)^s(x^2+x)^r] = [x(x^2+x)(x^3+3x^2+x)]^p = \sigma(G).$$

Obviously, $s, r \geq 1$ so that σ (H) = $(x^4 + 3x^3 + x^2)(x^2 + x)\sigma(H_1)$ and that by Lemma 3.1, $H = (K_{2,2,2} - e) + H_1$ for some graph H_1. Since $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$, by induction on p, we have $H_1 \in J_{p-1}(K_{1,2,3})$. This implies that H $\in J_p(K_{1,2,3})$.

Case (ii): Not every V_i has exactly two vertices.

Then there is a j such that $|V_j| = 1$. Without loss of generality, let $|V_j| = i$ for $j = 1, \ldots, r$, $r \geq 1$. Then $H = K_r + H_*$ for some graph H_*. Let $F_1, F_2, \ldots, F_t$ be the connected components of $\overline{H_*}$. Then $H = K_r + \overline{F_1} + \ldots + \overline{F_t}$ with $H_* = \overline{F_1} + \ldots + \overline{F_t}$.

If for some i, $F_i = K_3$, then $\overline{H}$ contains a subgraph $K_1 \cup K_3$. This means that $H = K_{1,3} + H'$ for some graph H' and so

$$\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H') = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = \sigma(G).$$

Clearly, $\sigma(H')$ must contain a factor $(x^2 + x)$ so that $\sigma(H) = (x^4+3x^3+x^2)\sigma(H'')\sigma(H_1)$ (where $\sigma(H'') = x^2 + x$) for some graph H_1. Obviously, $\overline{H''} = K_2$. Hence, $H = K_{1,2,3} + H_1$ with $\sigma(H_1) = [x(x^2+x)(x^3+3x^2+x)]^{p-1}$. Again, by induction on p, we have $H \in J_p(K_{1,2,3})$.

If for some i, $F_i = K_2$, then $H = K_2 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^3 + 3x^2 + x)$ so that $H = K_{1,2,3} + H_1$ or $(K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on p, we have $H \in J_p(K_{1,2,3})$.

If for some i, $F_i = P_4$ $(=K_{2,2} - e)$, then $H = P_4 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor (x^2+x) so that $H = (K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on p, we have $H \in J_p(K_{1,2,3})$.

So, assume that F_i is not K_2, K_3 or P_4 for any $i = 1, \ldots, t$. Let n_i and m_i denote the number of vertices and edges in F_i respectively. Then $\sum_{i=1}^{t} m_i = 4p$, the number of edges in $\overline{H}$.

If $n_i \leq 3$, then $F_i = P_3$. However, this is impossible because $\sigma(G)$ does not contain

$(x^3 + 2x^2)$ as a factor. Hence, $n_i \geq 4$. This implies that $6p = |V(G)| = r + \sum_{i=1}^{t} n_i \geq r + 4t$ so that $t < 3p/2$ because $r \geq 1$.

Since $H = K_r + H_*$, we have $\sigma(H) = x^r . \sigma(H_*)$ It follows that $s_{n-2}(\overline{H}) = s_{n_*-2}(\overline{H_*})$ where n_* is the number of vertices in H_*. Note that

$$\sigma(H_*) = \sum_{j=1}^{n_*} s_j(\overline{H_*})x^j = \prod_{i=1}^{t} \sigma(\overline{F_i})$$

where

$$\sigma(\overline{F_i}) = \sum_{k=1}^{n_i} s_k(F_i)x^k = x^{n_1} + m_i x^{n_i-1} + s_{n_i-2}(F_i)x^{n_i-2} + \ldots,$$

$i = 1, \ldots, t.$

By multiplying all the terms in $\Pi_{i=1}^{t}\sigma(\overline{F_i})$ and by equating the coefficient of x^{n_*-2}, we have by Lemma 2.3,

$$s_{n_*-2}(\overset{\rho}{H_*}) = \sum_{1\leq i\leq j\leq t} m_i m_j + \sum_{i=1}^{t} s_{n_i-2}(F_i)$$

$$\leq \sum_{1\leq i\leq j} m_i m_j + \sum_{i=1}^{t} \binom{m_i - 1}{2}.$$

Consequently, $$s_{n_*-2}(\overline{H_*}) \leq \frac{\sum_{1\leq i\leq j\leq t} 2m_i m_j + \sum_{i=1}^{t}(m_i^2 - 3m_i + 2)}{2}$$

$$= \frac{\left(\sum_{i=1}^{t} m_i\right)^2 - 3\sum_{i=1}^{t} m_i + 2t}{2}$$

$$= \frac{16p^2 - 12p + 2t}{2}$$

$$< \frac{16p^2 - 9p}{2}$$

because t < $3p/2$. However, this is a contradiction because $s_{n-2}(\overline{H}) = s_{6p-2}(\overline{G}) =$

$4p + 16\binom{p}{2} = (16p^2 - 8p)/2 > s_{n_*-2}(\overline{H}_*)$. This completes the proof.

Remark: Note that for even p, our main result is a special case of Theorem 5.1 in (Ho, (2004)).

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REFERENCES

F. Brenti, Expansions of chromatic polynomial and log-concavity, Trans. Amer. Math. Soc. 332 (1992) 729-756.

C.Y. Chao, On tree of polygons, Arch. Math. 45 (1985) 180-185.

G.L. Chia, On the chromatic equivalence class of a family of graphs, Discrete Math. 162 (1996) 285-289.

G.L. Chia, On the chromatic equivalence class of graphs, Discrete Math. 178 (1998) 15-23.

E.J. Farrell, On chromatic coefficients, Discrete Math. 29 (1980) 257-264.

F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).

C.K. Ho, On graphs determined by their chromatic polynomials, Ph.D. thesis (2004) University Malaya, Malaysia.

K.M. Koh and K.L. Teo, The search for chromatically unique graphs, Graphs Combin. 6 (1990) 259-285.

K.M. Koh and K.L. Teo, The search for chromatically unique graphs - II, Discrete Math. 172 (1997) 59-78.

R.Y. Liu, Chromatic uniqueness of Kn - E(kPs rPt), J. System Sci. Math. Sci. 12 (1992) 207-214 (Chinese, English Summary).