Chromatic Equivalence Class of the Join of Certain Tripartite Graphs

G.C. Lau & Y.H. Peng

1Faculty of I. T. and Quantitative Science
Universiti Teknologi MARA (Johor Branch)
Segamat, Johor, Malaysia

2Department of Mathematics, and
3Institute for Mathematical Research
Universiti Putra Malaysia 43400 UPM Serdang, Malaysia
E-mail: yhpeng@fsas.upm.edu.my

ABSTRACT

For a simple graph $G$, let $P(G;\lambda)$ be the chromatic polynomial of $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent, denoted $G \sim H$ if $P(G;\lambda) = P(H;\lambda)$. A graph $G$ is said to be chromatically unique, if $H \sim G$ implies that $H \cong G$. Chia [4] determined the chromatic equivalence class of the graph consisting of the join of $p$ copies of the path each of length 3. In this paper, we determined the chromatic equivalence class of the graph consisting of the join of $p$ copies of the complete tripartite graph $K_{1,2,3}$.

MSC: 05C15; 05C60

Keywords: Tripartite graphs; Chromatic polynomial; Chromatic equivalence class

INTRODUCTION

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph $G$, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent, or $\chi$-equivalent, denoted $G \sim H$ if $P(G) = P(H)$. It is clear that the relation $\sim$ is an equivalence relation on the family of graphs. We denote by $[G]$ the equivalence class determined by $G$ under $\sim$. A graph $G$ is said to be chromatically unique, or $\chi$-unique, if $[G] = \{G\}$, i.e., $H \sim G$ implies that $H \cong G$. Many families of $\chi$-unique graphs are known (see [8, 9]), relatively fewer results concerning the chromatic equivalence class of graphs are known (see [2, 3, 4]). In this paper, our main purpose is to determine the chromatic equivalence class of the graph consisting of the join of $p$ copies of the complete tripartite graph $K_{1,2,3}$.

In what follows, we let $K_n$ denote the complete graph on $n$ vertices, $K_{p_1,p_2,...,p_t}$ the complete $t$-partite graph having $n_i$ vertices in the $i$-th partite set, $P_n$ and $C_n$ the path and cycle on $n$ vertices, respectively and $\chi(G)$ the chromatic number of $G$. Let $W_n$ denote the wheel of order $n$ and $\mathcal{U}_n$ the graph obtained from $W_n$ by deleting a spoke of $W_n$. Also let $n(A,G)$ denote the number of subgraph $A$ in $G$ and $i(A,G)$ the number of induced subgraph $A$ in $G$. 
The join of two graphs $G$ and $H$, denoted $G + H$, is the graph obtained from the union of $G$ and $H$ by joining every vertex of $G$ to every vertex of $H$.

Let $F$ be a graph and let $G = F + F + \ldots + F$ or $pF$ denote the join of $p$ ($\geq 2$) copies of $F$. We wish to determine $\left[ G \right]$. Let $J_p(F)$ denote the set of all graphs $H$ which are of the form $H = H_1 + H_2 + \ldots + H_p$, where $H_i \in [F]$, $i = 1, 2, \ldots, p$.

In [4], Chia posed the following problem

**Problem:** What are those graphs $F$ for which $J_p(F) = \left[ G \right]$?

and solve the problem for the case $F = P_4$. In this paper, by making very minor modification to the technique used in [4], we solve the above problem for the case $F = K_{1,2,3}$.

**PRELIMINARY RESULTS AND NOTATIONS**

A spanning subgraph is called a *clique cover* if its connected components are complete graphs. Let $G$ be a graph on $n$ vertices. Let $s_k(G)$ denote the number of clique cover of $G$ with $k$ connected components, $k = 1, 2, \ldots, n$. If the chromatic polynomial of $G$ is

$$P(G, \lambda) = \sum_{k=1}^{n} s_k(G)(\lambda)$$

where $(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$, then the polynomial

$$\sigma(G, k) = \sum_{k=1}^{n} s_k(G)x^k$$

is called the *$\sigma$*-polynomial of $G$ (see Brenti(1992)). It is easy to see that $\sigma(G, x) = x^n$ if and only if $G = K_n$ since $s_k(G) = 0$ for $k < \chi(G) = n$. Also note that $s_1(G) = 1$ and $s_{\chi(G)}(G) = m$ if $G$ has $m$ edges. Clearly, $P(G, \lambda) = P(H, \lambda)$ if and only if $\sigma(G, x) = \sigma(H, x)$ and $s_k(G) = s_k(H)$ for $k = 1, 2, \ldots$. If $\sigma(G, x) = xf(x)$ for some irreducible polynomial $f(x)$ over the rational number field, then $\sigma(G, x)$ is said to be irreducible.

**Lemma 2.1.** (Farrell (1980)) Let $G$ and $H$ be two graphs such that $G \sim H$. Then $G$ and $H$ have the same number of vertices, edges and triangles. If both $G$ and $H$ has no $K_4$ as subgraph, then $i(C_4, G) = i(C_4, H)$. Moreover,

$$-i(C_5, G) + i(K_{2,3}, G) + 2i(U_5, G) + 3i(W_5, G) = -i(C_5, H) + i(K_{2,3}, H) + 2i(U_5, H) + 3i(W_5, H).$$

**Lemma 2.2.** (Brenti (1992)) Let $G$ and $H$ be two disjoint graphs. Then

$$\sigma(G + H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K_{n_1,n_2,\ldots,n_r}, x) = \prod_{i=1}^{r} \sigma(K_{n_i}, x).$$
Chromatic Equivalence Class of the Join of Certain Tripartite Graphs

Lemma 2.3. (Liu (1992)) Let $G$ be a connected graph with $n$ vertices and $m$ edges. Assume that $G$ is not the complete graph $K_3$. Then

$$s_{m-2}(G) \leq \binom{m-1}{2}$$

and equality holds if and only if $G$ is the path $P_{m+1}$.

A CHROMATIC EQUIVALENCE CLASS

We first have the following lemma which follows readily from Lemma 2.1.

Lemma 3.1. $[K_{1,2,3}] = \{K_{1,3}, K_{2,2,2} \setminus \{e\}\}$ where $e$ is an edge of $K_{2,2,2}$.

We now have our main theorem as follow.

Theorem 3.1. Let $G = K_{1,2,3} + K_{1,2,3} + \ldots + K_{1,2,3}$ be the join of $p$ copies of $K_{1,2,3}$. Then $[G] = J_p(K_{1,2,3})$.

Proof. Let $H \sim G$, we will show that $H \in J_p(K_{1,2,3})$. Since $P(G) = P(H)$ implies that $\sigma(G) = \sigma(H)$, it is more convenient to look at $\sigma(G)$ and $\sigma(H)$. First note that $\sigma(K_{1,3}) = x(x^3 + 3x^2 + x) = \sigma(K_{2,2,2} - e)$ with $[K_{1,3}] = \{K_{1,3}, K_{2,2} \setminus \{e\}\}$, and $\sigma(K_{1,2,3}) = x(x^2 + x)(x^3 + 3x^2 + x) = P(K_{2,2,2} - e)$. So, $\sigma(G) = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = [x^4 + x^2 + x^3 + 3x^2 + x]^p$, having $p$ irreducible factors of $x, x^2 + x$ and $x^3 + 3x^2 + x$ respectively.

Let $n$ and $m$ denote the number of vertices and edges in $H$ respectively. Then $n = 6p$ and $m = 36\binom{p}{2} + 11p = 18p^2 - 7p$ so that $\sigma(H) = \sigma(G) = \sum_{i=1}^{6p} s_i(G)x^i$. Moreover, $H$ is uniquely $3p$-colorable as $G$ is so.

Let $V_1, V_2, \ldots, V_{3p}$ be the color classes of the unique $3p$-coloring of $H$. Let $V_i$ denote the subgraph induced by $V_i \cup V_j, i \neq j$. Call $V_i$ a 2-color subgraph of $H$.

Case (i): Every $V_i$ has exactly two vertices.

In this case, $V_i$ is either a path $P_4$ or else a cycle $C_4$ because, by Theorem 12.16 of [6], $V_i$ is connected for $i \neq j$. Note that the number of 2-color subgraphs in $H$ is $\binom{3p}{2} = \frac{1}{2}(9p^2 - 5p) + p$. By looking at the number of edges in $H$, we see that exactly $p$
of the 2-color subgraphs $V_g$ are $P_4$ and the rest of the 2-color subgraphs are $C_4$. This means that $\overline{H}$ has only $P_4$ and $K_2$ as subgraph so that $H = sP_4 + rK_2 (s, r \geq 0)$. Consequently, 

$$\sigma(H) = [x(x^2 + x^3 + x^2 + x)]^p = \sigma(G).$$

Obviously, $s, r \geq 1$ so that $\sigma(H) = (x^4 + 3x^3 + x^2)(x^3 + x^2 + x)\sigma(H_1)$ and that by Lemma 3.1, $H = (K_{2,2,2} - e) + H_1$ for some graph $H_1$. Since $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$, by induction on $p$, we have $H_1 \in J_p(K_{1,2,3})$. This implies that $H \in J_p(K_{1,2,3})$.

Case (ii): Not every $V_i$ has exactly two vertices.

Then there is a $j$ such that $|V_j| = 1$. Without loss of generality, let $|V_j| = 1$ for $j = 1, \ldots, r, r \geq 1$. Then $H = K_r + H_*$ for some graph $H_*$. Let $F_1, F_2, \ldots, F_t$ be the connected components of $H_*$. Then $H = K_r + F_1 + \ldots + F_t$ with $H_* = F_1 + \ldots + F_t$.

If for some $i$, $F_i = K_2$, then $\overline{H}$ contains a subgraph $K_1 \cup K_r$. This means that $H = K_1 + H'$ for some graph $H'$ and so 

$$\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H') = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = \sigma(G).$$

Clearly, $\sigma(H')$ must contain a factor $(x^2 + x)$ so that $\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H^*)\sigma(H_1)$ (where $\sigma(H^*) = x^2 + x$) for some graph $H_1$. Obviously, $\overline{H^n} = K_2$. Hence, $H' = K_{1,2,3} + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on $p$, we have $H \in J_p(K_{1,2,3})$.

If for some $i$, $F_i = K_3$ then $H = K_3 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^3 + 3x^2 + x)$ so that $H = K_{1,2,3} + H_1$ or $(K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on $p$, we have $H \in J_p(K_{1,2,3})$.

If for some $i$, $F_i = P_4 (=K_{2,2,2} - e)$, then $H = P_4 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^2 + x)$ so that $H = (K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on $p$, we have $H \in J_p(K_{1,2,3})$.

So, assume that $F_i$ is not $K_2, K_3$ or $P_4$ for any $i = 1, \ldots, t$. Let $n_i$ and $m_i$ denote the number of vertices and edges in $F_i$ respectively. Then $\sum_{i=1}^{t} m_i = 4p$, the number of edges in $\overline{H}$.

If $n_i \leq 3$, then $F_i = P_3$. However, this is impossible because $\sigma(G)$ does not contain
(x^3 + 2x^2) as a factor. Hence, \( n_i \geq 4 \). This implies that \( 6p = |V(G)| = r + \sum_{i=1}^{t} n_i \geq r + 4t \) so that \( t < 3p/2 \) because \( r \geq 1 \).

Since \( H = K_r + H_\ast \), we have \( \sigma(H) = x' \cdot \sigma(H_\ast) \). It follows that \( s_{n-2}(H) = s_{n-2}(H_\ast) \) where \( n_i \) is the number of vertices in \( H_i \). Note that

\[
\sigma(H_\ast) = \sum_{j=1}^{n_i} s_j(H_\ast)x^j = \prod_{i=1}^{t} \sigma(F_i)
\]

where

\[
\sigma(F_i) = \sum_{k=1}^{n_i} s_k(F_i)x^k = x^{n_i} + m_i x^{n_i-1} + s_{n_i-2}(F_i) x^{n_i-2} + \ldots,
\]

\( i = 1, \ldots, t \).

By multiplying all the terms in \( \prod_{i=1}^{t} \sigma(F_i) \) and by equating the coefficient of \( x^{n-2} \), we have by Lemma 2.3,

\[
s_{n-2}(H_\ast) = \sum_{1 \leq i < j \leq t} m_i m_j + \sum_{i=1}^{t} s_{n_i-2}(F_i)
\]

\[
\leq \sum_{1 \leq i < j \leq t} m_i m_j + \sum_{i=1}^{t} \left( \frac{m_i - 1}{2} \right).
\]

Consequently,

\[
s_{n-2}(H_\ast) \leq \sum_{1 \leq i < j \leq t} 2m_i m_j + \sum_{i=1}^{t} \left( \frac{m_i^2 - 3m_i + 2}{2} \right)
\]

\[
= \left( \sum_{i=1}^{t} m_i \right)^2 - 3 \sum_{i=1}^{t} m_i + 2t
\]

\[
= \frac{16p^2 - 12p + 2t}{2}
\]

\[
< \frac{16p^2 - 9p}{2}
\]

because \( t < 3p/2 \). However, this is a contradiction because \( s_{n-2}(H) = s_{6p-2}(G) \).

\( 4p + 16 \left( \frac{p}{2} \right) = (16p^2 - 8p) / 2 > s_{n-2}(H_\ast) \). This completes the proof.

**Remark:** Note that for even \( p \), our main result is a special case of Theorem 5.1 in (Ho, 2004).
ACKNOWLEDGEMENTS
The authors wish to thanks the referees for their valuable comments and suggestions.

REFERENCES


F. HARARY, Graph Theory (Addison-Wesley, Reading, MA, 1969).


