Why Teach Mathematical Problem Solving?

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Abstract

Problem solving is a basic skill needed by today's learners. Guided by recent research in mathematical problem solving, the changing professional standards, new workplace demands, technological changes and recent changes in learning theories, educators are to include integrated learning environments which encourage learners to use this higher order thinking skills, specifically problem solving skills. Teaching mathematical problem solving should take an approach of teaching in context where students are learning mathematics while solving realistic problems. Both holistic and componential opportunity modelings are approaches that have been emphasized in higher learning.

Introduction

The teaching of problem solving skills is currently receiving a great deal of attention. The main reason is that increasingly fast-paced changes in the society make it necessary for people to think for themselves in order to face challenges in life and to solve novel problems. In general, assessment of students' achievement suggests that today's students maybe failing to develop effective problem solving skills and thinking. Yet, problem-solving skill has been important to an individual's performance in everyday life and to an individual's economic and financial success.

The goal of teaching thinking and problem solving is not unique to the 21^{st} century but has been espoused for many centuries. It has stipulated that increasing thinking and problem solving is related to development of "mental discipline" and by subjecting learners to take subjects such as mathematics or philosophy learners can enhance mental discipline. Many have suggested that subjecting learners to the rigors of mathematics especially provide training for individuals to be active and retentive learners.

What is a problem?

Each and every one of us will have decisions to make and problems to solve throughout our lives. These problems come in various ways. To solve these problem specific strategies and skills are needed. Both specific strategies and diverse skills are equally important to be an effective problem-solver. For instance, solving problems in mathematics requires mathematical skills. Solving writing problems requires verbal skills. Solving every day lives problem require different kinds of skills such as budget-balancing for one's household would require accounting skills and good sense of monitoring cash-flow while solving problem to arrive in time for a meeting require one to be familiarize with the flow of traffic ensuring no related vehicle problem cropping up.

The following is a list of problems, some are familiar problems reflecting the concerns of most people in society while others will be problems with which you are unconcerned yourself, but which you would recognize as problems for other people.

- Explore polygonal numbers.
- A school population was predicted to increase by 50 students per year for the next 10 years. If the current enrolment is 700 students, what will the enrolment be after 10 years?
- How can I draw four straight lines through a three-by-three array of nine dots without taking your pen off the paper?

The problems listed above have two things in common. First, they have specific goal to be determined and secondly, for all the cases there is no immediate means to achieve the goal because the goal is blocked either through lack of resources or knowledge. These scenarios can be used as a basis for a definition of the concept of problem, problem solving and thinking. Whenever one have a goal which is blocked for any reason- lack of resources, lack of information, and so no – one indeed have a problem. Whatever one does in order to achieve the goal is problem solving. Thinking or problem solving calls for specific skills, strategies, and insights appropriate to the kind and specific details of the problem one is facing.

In other words, a problem is a situation, quantitative or otherwise, that confronts an individual or group of individuals that requires solution, and for which no path to the answer is known. Hence a problem is also situation encountered that requires thought and synthesis of previously learned knowledge to resolve.

Much of problem solving research involves George Polya's [7] heuristics for solving problems. In his book *How To Solve It,* he presents techniques for problem solving which are not only interesting but are also meant to ensure that principles learned in mathematics courses will transfer as widely as possible. His techniques are called heuristics, strategies that aid in solving problems. He outlines a four-step process or heuristics method:

1. Understanding the problem. *What is the unknown? What are the data? What is the condition? Draw a figure, introduce suitable symbols. Classify the various conditions given.*
2. Devising a plan. *Find the connection between the data and the unknown. Recall any previous but related problem.*
3. Carrying out the plan. *Check each step.*
4. Looking back. *Examine the solution obtained. Is the argument and the answer correct?*

Problem solving has also been discussed as far back as 1910 by John Dewey in his book titled *How We Think.* He outlined five steps for problem solving as in the following order:

1. Recognizing that a problem exists – being aware of a difficulty, a sense of frustration, wondering or doubt.
2. Identifying the problem – clarify and define the goal to be sought, as defined by the situation which poses the problem.
3. Employing previous experiences, such as relevant information, former solutions, or ideas to formulate hypotheses and problem solving propositions.
4. Testing the hypotheses or possible solutions or otherwise reformulate the problem.
5. Evaluate the solutions and draw conclusion based on the evidence. This involves ability to incorporate the successful solution into one's existing understanding.

Importance of problem solving skills

The goal of problem solving education is to develop tools for thinking that will contribute framework of actions that can be applied to our ever-changing knowledge base. Hence, problem-solving skills require an integrated use of thinking skills and an appropriate knowledge consisting of both declarative and procedural knowledge.

Students at all levels of education are expected to acquire information and recall it at some later time, discover and solve novel problems, evaluate the strength and nature of evidence, use reasons to support conclusions, recognize propaganda and other persuasive techniques, consider likely outcomes of actions, and question bogus claims. Yet, they are rarely taught how! Most postsecondary instruction involves transmitting facts to students with very little emphasis on how to discover the facts on their own or when to question them. But facts change, and they often change quickly in the natural sciences. The twin abilities of how to acquire information and how to use the information to think effectively will provide the best education for today's college student-the citizens of the 21st century. One way to encourage inquisitive and creative scientists, mathematicians, and engineers is by fostering a problem-solving attitude.

Problem solving skills can enrich our life as these skills are needed to be used at home, in our recreational pursuits, and especially in our relationships with other people. Through experiences in problem solving we generate new ideas and innovative solutions for a given need or problem. When people work together to find the best solutions to problems, goals are set, interconnections of ideas are built and the working environment is invigorated. Through problem solving we can generate new ideas and innovative solutions for a given need or problem. Ideas generated are often of higher quality than those we have always used.

The importance of teaching problem solving especially in mathematics is so that students will learn to read mathematics. Students will also learn to develop and use models for standard type problems. In mathematics the convenient arrangement of materials is as follow: First, develop a specific skill for instance *how to solve for angles in a triangle* or *how to solve simultaneous equations.* Secondly, assign set of practice problems which are straightforward or drill-like for instance ten pairs of simultaneous equations to be solved or five set of calculation of angles of any given triangles. Thirdly, students are asked to solve variety of verbal problems related to simultaneous equations or determination of angles of any given triangles.

Problem solving should be an integral part of all mathematics learning. To solve problems, students must draw upon their knowledge of the concepts and skills they have learned and apply them to a novel situation. Through this process, students develop new mathematical understanding. Problem solving should not be an isolated part of the teaching and learning program. Problem solving should cut across all content areas, and in mathematics: numbers and operation, algebra, geometry, measurement, and data analysis and probability. Furthermore, problem solving provides link between facts, algorithms, and real-life problems situations we all face, hence mathematics can be regarded as problem solving and problem solving is thinking. An emphasis on problem solving in classroom teaching and learning can also lessen the gap between the real world and the classroom world hence setting positive and conducive mood in the classroom especially in a mathematics lesson. Also in many mathematics classes, students do not see any connections between the various ideas taught during the year. Most students especially the young students see little connection between what happens in school and what happens in real life.

While students can hardly see connections between the various mathematical ideas taught during the year, problem solving can provide opportunities to see connections between mathematical ideas. Problem solving is more exciting, more challenging, and more interesting to students than simply mathematical drill and practice. Research has shown that students' performance increases upon carefully selected sequence of problem solving activities. These also yield success that stimulate and leads to positive attitude towards mathematics and problem solving.

Specifically teaching problem solving will develop students' abilities to use models for standard type problems and challenges students to more difficult or non-routine problems. Students will be taught to read verbal problems slowly and reflect on what has been read, must determine what it means and reread and reexamine the problem as often as necessary. During these processes, students will develop sense of pride seeing how their skills have grown over the years of mathematical training.

Finally, problem solving permits students to learn and to practice heuristic thinking. A careful selection of problems will enhance and provide development of problem solving and reasoning skills necessary in the mathematics classroom as well as in real life. Hence teaching problem solving should have its benefits and be emphasized in the elementary and secondary mathematics curriculum.

In recent research, Posamentier [6] taught students five strategies for use with mathematical problems. These strategies should generalize to scientific problems as well. Three of these strategies are useful in any context:

1. Draw a diagram, if at all possible.
   Even if you finally solve the problem by algebraic or other means, a diagram can help give you a "feel" for the problem. It may suggest ideas or plausible answers. You may even solve the problem graphically.

2. Consider a similar problem with fewer variables.
   If the problem has a large number of variables and is too confusing to deal with comfortably, construct and solve a similar problem with fewer variables. You may then be able to 1) adapt the method of solution to the more complex problem 2) take the result of the simpler problem and build up from there.

3. Try to establish sub-goals.
   Can you obtain part of the answer, and perhaps go on from there? Can you decompose the problem so that a number of easier results can be combined to give the total result you want?
   In addition he suggested two other strategies which are more specific to mathematical problems:

4. Look for patterns.
   Is there an n or other parameter in the problem that takes on an integer value? List the integer parameter in order, and look for a pattern. Go beyond n objects and see what happens when you pass from n to n + 1.

5. Use contradiction or contra-positive when solving proofs.
   In using contradiction, for the sake of argument assumed that the statement you want to make is false. Using this assumption, go on to prove that either one of the given conditions in the problem is false, or that what you wish to prove is true. On the other hand, in using contra-positive: Instead of proving the statement "If X is true, then Y is true," you can prove the equivalent statement, "If Y is false, then X is false."

Of course, a skills approach to solving problems is not a substitute for instruction in a content area. There is no substitute for knowledge in an academic domain. It is, however, a practical addition to every course because a skills approach requires students to become actively involved in the process of solving problems and because it makes implicit thinking strategies more explicit and, thus, easier to learn.

Teaching mathematical problem solving

Earlier it has been discussed that problem solving is the mental process that we use to arrive at a best answer to an unknown, the goal or some decision which require substantial cognitive or thinking demands due to sets of constraints. The problem situation is not one that has been encountered before, or recalled from memory a procedure or a solution from past experience. In this context, the skills needed in mathematical problem solving process include:

1. a knowledge base pertinent to the content of the problem.
2. the ability to identify, locate, obtain, and evaluate missing information.
3. the ability to learn on one's own.
4. the ability to think such as classify, check for consistency, reason; motivation and perseverance, identify relationship, creativity, and generalize.
5. the ability to simplify and broaden perspectives.
6. the ability to cope with ambiguity, fear, and anxiety.
7. interpersonal and group skills.
8. communication skills.
9. awareness of how one thinks one's preference or learning style.

There are three components of problem solving is 1) content knowledge and procedural knowledge in the particular domain 2) tacit knowledge or intuitive knowledge ("experience") and 3) skills at using problem solving strategies.

We need content and procedural knowledge in order to solve problems. Poor problem solvers have shown that they memorized an unstructured set of facts or ideas. Hence their knowledge are not connected by any sort of pointer words and their knowledge is not systematically arranged. In contrast research have shown that good problem solvers have carefully developed, organized, hierarchical knowledge structure centered around fundamental principles and abstractions. Their knowledge is complete with pointers which includes the applicability for all concepts. The second component of problem solving skills is tacit knowledge. Tacit information from the knowledge domain is essential in the creation of internal representations and checking and monitoring answers and decisions made during the process. The third component consist of all the skills defined at the beginning of this section such as classify, check for consistency, identify relationships, motivation and perseverance, interpersonal and group skills, awareness of how one thinks, learning style, preference in processing information, etc.

Procedural knowledge can also be described as the knowledge of operators and the conditions under which these can be used to reach certain goals (Byrnes & Wasik, [2]). This type of knowledge to some degree is said to be automated as it enables people to solve problems quickly and efficiently (Schneider & Stern, [8]; Hiebert & Carpenter, [4]). According to Johnson [5], automatization is accomplished through practice and allows for a quick activation and execution of procedural knowledge. In addition, as compared to the application of conceptual knowledge, its application involves minimal conscious attention and few cognitive resources. The automated nature of procedural knowledge implies that it is not or only partly open to conscious inspection and hence can be hardly verbalized or transformed by higher mental processes. As a consequence, it is tied to specific problem types (Baroody, [1]).

Recently, a review of some of the existing literature within the domain of mathematics learning by Star [9] indicates that very little useful theory has been developed to explain how conceptual knowledge and procedural knowledge are related. He postulates that the reason for this deficiency is that "knowledge of concepts is treated as a multi-faceted and rich construct, while procedural knowledge is very narrowly defined and operationalised" (p. 80). He also differentiates both types of knowledge based on the philosophical and historical foundations. Finally, he suggests that distinguishing between knowledge type and depth of knowledge will illuminate alternative ways in which procedural knowledge can be known and understood. Star [9] emphasized that re-conceptualisation of procedural knowledge will reminds us that "understanding in mathematics is the synthesis of knowing and doing, not the accomplishment of one in the absence of the other.

Soifer [10] categorize problem solving approaches into two overall categories: Some use a holistic approach and consider the process in the context of solving problems completely. For example, ask students to display how they solved a problem in the text and then reflect on how they did it. Other approaches break the problem-solving process into components, develop each component, and then give practice in applying the relevant skill to the complete problem-solving process. For example, we could focus on creativity, give practice in brainstorming, and, using triggers on the properties of red brick, embed it in the subject, such as in algebra, by brainstorming the possible use of algebra functions in mathematics, and then we could practice using that skill by solving ordinary homework problems.

Within each of these two broad categories there are subcategories consisting of giving the student opportunity, modeling the process, and facilitating the student's appreciation of the process. For the component approach, the subcategories consist of giving the students of opportunity, providing them with explicit practice on the component, enriching the explicit practice by embedding it in the subject domain, and then providing explicit practice in applying the skill in the holistic activity of solving problems.

The holistic opportunity approach gives the students many problems to solve, corrects their efforts based on the answers that they get, and hopes that they acquire skill at solving problems. This approach, although it is better than giving the students no problems at all to solve, does not develop their problem-solving skills. Yet, this approach is used extensively in teaching and learning mathematics as well as other subjects.

Holistic and component opportunity modeling

There are many variations on the holistic modeling approach. Teachers can show sample solutions on the board, hand out sample solutions, show through case studies how professionals have handled the problem, or have the students work at the board and display for others how they solved the problem. Most of the efforts fail because they focus on the knowledge components used in arriving at a solution, not on the mental process skills needed to solve the problem. For example, no explicit descriptions or given of how to draw a diagram. No one describes the five failed attempts to create a representation of the problem to be solved; the students only see the "correct" one. Indeed, the necessary amount of detail can rarely be included in sample solutions, given the limitations on the number of pages in a textbook. Even the much-discussed case studies usually fall short displaying the process. Only the technical details of a large, open-ended professional problem are displayed. Such a display serves many useful roles; rarely does it improve problem-solving skill.

Furthermore, teachers have great difficulty displaying the process because all the situations with which they work are "exercises" to them. They display how they solve exercises; the students need to see how someone solves problems. Even if the teacher could faithfully describe how they have solved the problems, their approach may not work for about two- thirds of the students because of personal differences in the processing of information. Teachers always come up with suggestions that work and that move toward a reasonable solution. Thus, students do not see the teacher's frustration, mistakes, or false moves.

Having students work problems on the board makes it more probable that"problem solving" instead of "exercise solving" will be displayed. Yet, unless students receive training in how to display the process, only the tip of the iceberg will be shown.

Under the component opportunity approach, we isolate the component, create appropriate "problems," and give students and opportunity to solve them. Some of the components needed to solve problems were identified earlier. However, they vary greatly in the degree to which they tell you how to do the task. Most give little or no guidance in that respect. It is up to the readers to analyze the mental skill, discover the process used, and enriches the opportunity by facilitating the student's acquisition of the skill.

Summary

Many options are available. The holistic approach has been used extensively in the past, but it suffers because it addresses such a host of skills simultaneously that it is hard for both student and teacher to monitor growth. The recent surge of interest in development of problem-solving skills has tended to isolate and explicitly develop skills, but they suffer unless sufficient care is taken to embed the application in the discipline and to transfer it to the whole task of solving problem.

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