UNIVERSITI PUTRA MALAYSIA

AUTOMATIC QUADRATURE SCHEME FOR EVALUATING SINGULAR INTEGRAL WITH CAUCHY KERNEL USING CHEBYSHEV POLYNOMIALS

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By

NUR AMALINA BINTI JAMALUDIN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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This thesis is dedicated to all my family members especially my parents Jamaludin Bin Abdul Rahim and Norizah Binti Isa.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of requirements for the degree of Master of Science.

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Chairman: Zainidin K. Eshkuvatov, PhD

Faculty : Science

In this thesis, an automatic quadrature scheme is presented for evaluating the product type indefinite integral

$$Q(f,x,y,c)=\int_x^y w(t)K(c;t)f(t)dt,\ -1\le x,y\le 1,\ -1<c<1$$

where $w(t)=1\big/\sqrt{1-t^2}$, $K(c,t)=1\big/(t-c)$ and $f(t)$ is assumed to be a smooth function. In constructing an automatic quadrature scheme for the case $-1<x<y<1$ the density function $f(t)$ is approximated by the truncated Chebyshev polynomial $p_N(t)$ of the first kind of degree N. The approximation $p_N(t)$ yields an integration rule $Q(p_N,x,y,c)$ to the integral $Q(f,x,y,c)$. An

automatic quadrature scheme for the case $x=-1,\ y=1$ can easily be constructed by replacing $f(t)$ with $p_N(t)$ and using the known formula

$$\int_{-1}^{1}\frac{T_k(t)}{\sqrt{1-t^2}\,(t-c)}dt=\pi U_{k-1}(c),\quad k=1,\dots,N\,.$$

In both cases the interpolation conditions are imposed to determine the unknown coefficients of the Chebyshev polynomials $p_N(t)$. The evaluations of $Q(f,x,y,c)\cong Q(p_N,x,y,c)$ for the set (x,y,c) can be efficiently computed by using backward direction method.

The estimation of errors for an automatic quadrature scheme are obtained and convergence problem are discussed in the classes of functions $C^{N+1,\alpha}[-1,1]$ and $L_p^w[-1,1]$.

The **C** code is developed to obtain the numerical results and they are presented and compared with the exact solution of SI for different functions $f(t)$. Numerical experiments are presented to show the efficiency and the accuracy of the method. It asserts the theoretical results.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Master Sains

SKEMA KUADRATUR AUTOMATIK UNTUK PENYELESAIAN KAMIRAN SINGULAR JENIS CAUCHY KERNEL MENGGUNAKAN POLINOMIAL CHEBYSHEV

Oleh

NUR AMALINA BINTI JAMALUDIN

April 2010

Pengerusi: Zainidin K. Eshkuvatov, Phd

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Dalam tesis ini, skema kuadratur automatik dipamer bagi menilai kamiran jenis pendaraban tak terhingga

$$Q(f,x,y,c)=\int_x^y w(t)K(c;t)f(t)dt,\ -1\le x,y\le 1,\ -1<c<1$$

dengan $w(t)=1\big/\sqrt{1-t^2}$, $K(c,t)=1\big/(t-c)$ dan $f(t)$ dianggap menjadi fungsi licin. Dalam membina skema kuadratur automatik untuk kes $-1<x<y<1$, fungsi ketumpatan $f(t)$ dihampirkan dengan polinomial Chebyshev tercantas, $p_N(t)$ jenis pertama berdarjah *N*. Penghampiran $p_N(t)$ menghasilkan formula kamiran $Q(p_N,x,y,c)$ kepada kamiran $Q(f,x,y,c)$. Skema kuadratur automatik bagi kes $x=-1,\ y=1$ boleh dibina

dengan mudah dengan menggantikan $f(t)$ kepada $p_N(t)$ menggunakan formula yang telah diketahui,

$$\int_{-1}^{1} \frac{T_k(t)}{\sqrt{1-t^2}\,(t-c)} dt = \pi U_{k-1}(c), \quad k = 1, \ldots, N$$

Dalam kedua-dua kes syarat interpolasi digunakan untuk menentukan pekali Chebyshev polinomial $p_N(t)$ yang tidak diketahui. Penghuraian $Q(f, x, y, c) \cong Q(p_N, x, y, c)$ untuk set (x, y, c) boleh dikira secara efisien dengan menggunakan Penjelmaan Fourier Pantas (PFP). Penganggaran ralat bagi skema kuadratur automatik yang dibina diperolehi dan masalah penumpuan untuk skema kuadratur automatik dibincangkan dalam kelas fungsi $C^{(N+1),\alpha}[-1,1]$ dan $L_p^w[-1,1]$.

Kod **C** dibangunkan bagi memperolehi keputusan berangka dan dipamerkan untuk dibandingkan dengan penyelesaian tepat kamiran singular untuk fungsi $f(t)$ yang berlainan. Eksperimen berangka dipamer bagi menunjukkan keberkesanan kaedah yang digunakan dan ia membuktikan keputusan teori.

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The Most Beneficent, The Most Merciful

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I certify that a Thesis Examination Committee has met on 21 April 2010 to conduct the final examination of Nur Amalina Binti Jamaludin on her thesis entitled "Automatic Quadrature Scheme for Evaluating Singular Integral with Cauchy Kernel using Chebyshev Polynomials" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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Date: 12 August 2010

DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledge. I also declare that it has not been previously or concurrently submitted for any other degree at Universiti Putra Malaysia or other institutions.

NUR AMALINA BINTI JAMALUDIN

Date: 21 April 2010

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LIST OF ABBREVIATIONS

SIE	: Singular integral equation
SI	: Singular integral
AQS	: Automatic Quadrature Scheme
EXACT	: Exact solution
ERR.	: Error
IQF	: Interpolatory Quadrature Formula
IVT	: Intermediate Value Theorem
MQF	: Modified Quadrature Formula
QF	: Quadrature Formula
QFs	: Quadrature Formulas

CHAPTER 1

INTRODUCTION

1.1 Historical introduction and types of the quadrature formulas

Numerical integration is the study of how the numerical value of an integral can be found. A fine example of ancient numerical integration is the Greek quadrature of the circle by means of inscribed and circumscribed regular polygons. This process led Archimedes to find an upper and lower bound for the value of π. Over the centuries, particularly since the sixteenth century, many methods of numerical integration have been derived. These include the use of the fundamental theorem of integral calculus, infinite series, differential equations and integral transforms. There is a method of approximate integration at which an integral is approximated by a linear combination of the values of the integrand, i.e.

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k), \ \infty \le a \le b \le \infty \qquad (1.1.1)$$

where $x_1, ..., x_n$, are points or abscissas and $w_1, w_2, ..., w_n$ are called weights accompanying these points.

One may properly ask why such primitive approximations as (1.1.1) should be developed and utilized. The answer is very simple: The sophisticated methods in mathematics do not always work, and even if they work it may not be advantageous to use them. For example, if the indefinite integral is an elementary function and antiderivatives can be obtained without difficulties, it may be complicated to use it.

For example:

$$\int_0^x \frac{dt}{1+t^4} = \frac{1}{4\sqrt{2}} \log \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} + \frac{1}{2\sqrt{2}} \left[\arctan \frac{x}{\sqrt{2-x}} + \arctan \frac{x}{\sqrt{2}+x} \right].$$

Whereas the integration

$$S = \int_a^b e^{-x^2} dx,$$

leads to a function that cannot be expressed in infinite terms by combinations of algebraic, logarithmic, or exponential operations.

Another reason for approximate integration occurs when we are solving a functional equation for the unknown function that appears in the integrand of some integral.

A final reason for developing rules of approximate integration is that in many instances, we are encounter with the problem of integrating experimental data. In such cases, theoretical devices may not be wholly applicable.

Numerical integration has been of the great interest to the pure mathematician. The history reveals that many great mathematicians have contributed to this field; Archimedes, Kepler, Huygens, Newton, Euler, Gauss, Jacobi, Chebyshev, Markoft, Feger, Polya, Cauchy, Schoenberg and Sobolev are among them.

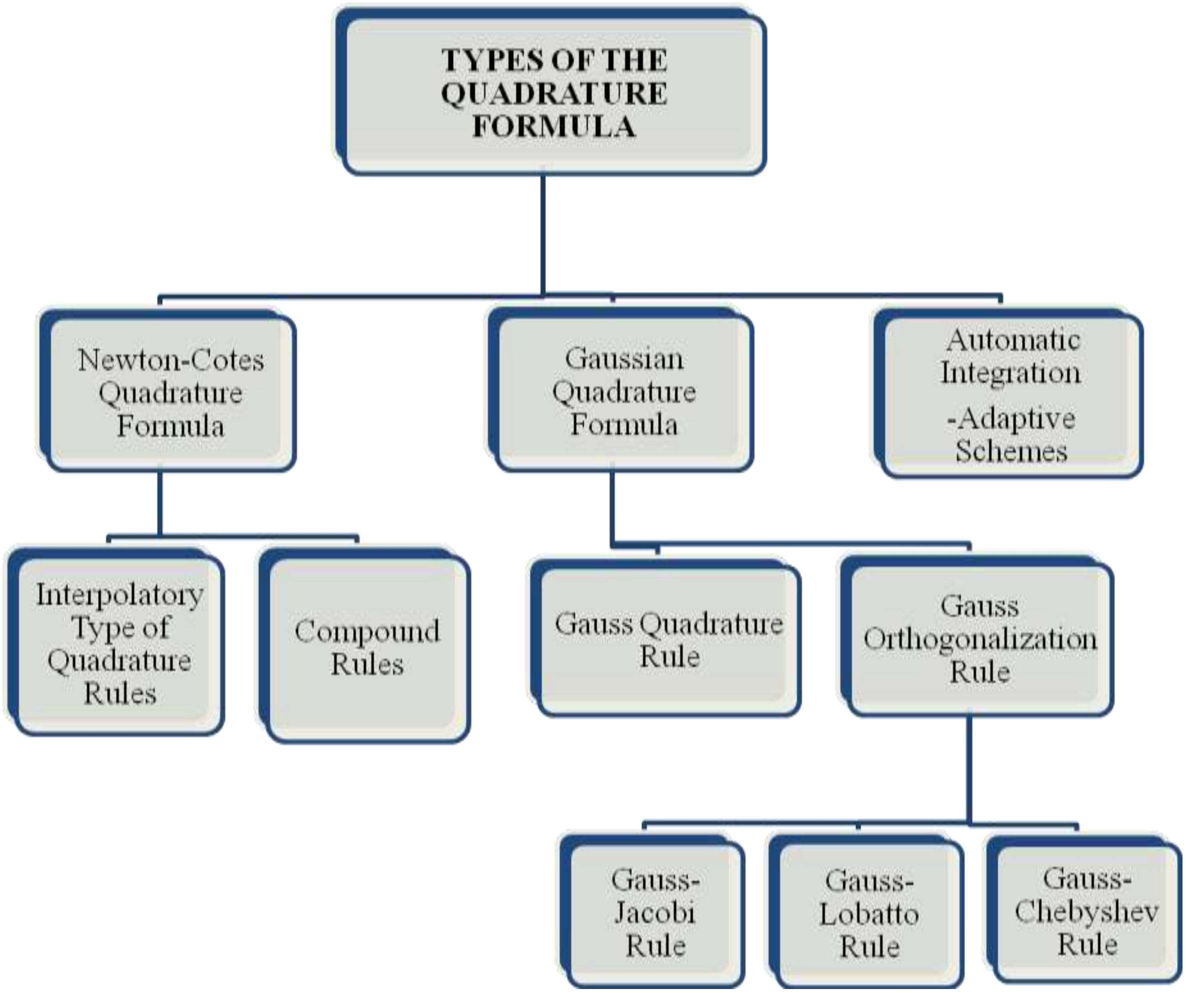

Figure 1.1: Scheme of the types of the quadrature formula

1.2 General integration rules.

Polynomial interpolation is the interpolation of a given data set by a polynomial. In other words, given some data points (obtained by sampling), the aim is to find a polynomial which goes exactly through these points.

Theorem 1.1: Let the nodes $x_i \in [a,b],\ 0 \le i \le n,$ be given and the node x_i be distinct, i.e., $x_i = x_j$ if and only if $i = j$ .Then there exists a unique polynomial p_n, of degree less than or equal to n , that satisfies either

$$p_n(x_i) = y_i, \quad i = 0,\ldots,n$$

for a given set of data values $\{y_i\}$, or

$$p_n(x_i) = f(x_i), \quad i = 0,\ldots,n$$

for a given function $f \in C[a,b]$.

Interpolatory polynomials are used to fit a function $f(x)$ over the interval $[a,b]$ and are applied in constructing the QFs for the integration problems. Consider the product type integral of the form

$$\begin{aligned} Q[f] = \int_a^b w(x) f(x) dx &= \sum_{k=1}^{n} w_k f(x_k) + E_n[f] \\ &= Q_n[f] + E_n[f], \qquad \infty \le a \le b \le \infty \end{aligned} \tag{1.2.1}$$

where $E_n[f]$ is an error term.

To construct a QF for evaluating the product integral (1.2.1) we usually divide the interval $[a,b]$ into subintervals $[x_k, x_{k+1}]$, $k=0,\ldots,n-1$, either of equal length $h=\dfrac{b-a}{n}$ or nodes not equal length $\Delta x_k = x_{k+1} - x_k$. Nodes $x_k, k=0,\ldots,n$ are chosen for not equal length as $x_k = x_{k-1} + \Delta x_k$, $x_0 = a$ and as $x_k = a + kh$ for equal length. If the function $f(x)$ is defined only over the interval $[a,b]$, then the node points $x_k, k=0,1,\ldots,n$ must belong to the interval $[a,b]$ entirely. If $f(x)$ is defined outside the interval of integrations, then all x_k need not belong to $[a,b]$. Some QFs are constructed to fit the integral based on some of the nodes outside the interval $[a,b]$. However, in many cases of integration problems the nodes x_k lies in the interval of integrations.

1.2.1 Interpolatory type quadrature rules

Definition 1.1: (Degree of precision)

The degree of precision of a QF is the positive integer n such that the error $E_n[p_i]=0$ for all polynomials $p_i(x)$ of degree $i \le n$, and $E_n[p_{n+1}] \ne 0$ for some polynomial $p_{n+1}(x)$ of degree n+1 , that is

$$\int_a^b w(x)\, p_i(x)\, dx = Q_n[p_i], \quad i=0,\ldots,n$$

and

$$\int_a^b w(x)\, p_{n+1}(x) dx \ne Q_n[p_{n+1}].$$

Let us illustrate interpolating quadrature formula which is exact for the polynomial of degree *n*. Let the interpolating polynomial $p_n(x)$ for $f(x)$ be such that

$$f(x) = p_n(x) + r(x), \tag{1.2.2}$$

where $r(x)$ is the remainder term, and

$$p_n(x) = \sum_{k=1}^{n} \frac{\pi(x)}{(x - x_k)\pi'(x_k)} f(x_k), \tag{1.2.3}$$

where $\pi(x) = (x - x_1)(x - x_2)...(x - x_n)$.

Polynomial of the form (1.2.3) is called Lagrange interpolating polynomial and it is known that the remainder term $r(x) = f(x) - p_n(x)$ in (1.2.2) is equal to zero, if $f(x)$ is a polynomial of degree $i \le n$. Then the exact value of the product integral (1.2.1) is

$$\int_a^b w(x) f(x) dx = \int_a^b w(x) p(x) dx + \int_a^b w(x) r(x) dx. \tag{1.2.4}$$

The behavior of the remainder $r(x)$ depends on the preciseness of the interpolating polynomial $p_n(x)$. Thus, if $r(x)$ is small throughout the interval $[a,b]$, then the second integral on the right-hand side (1.2.4) can be neglected. This situation leads to the approximate equation

$$\int_a^b w(x) f(x) dx \approx \sum_{k=1}^{n} A_k f(x_k), \tag{1.2.5}$$

where

$$A_k = \int_a^b w(x) \frac{\pi(x)}{(x - x_k)\pi'(x_k)} dx. \tag{1.2.6}$$

Formula of type (1.2.5) are called the IQF.

Remark: The necessary and sufficient condition for the QF (1.2.5) to be interpolatory is that it would be exact for all possible polynomials $p(x)$ of degree at most *n*-1.

1.2.2 Newton-Cotes formula (Basic QF)

Let the interval $[a,b]$ be divided into n equal subintervals of length $h=\frac{b-a}{n}$ and the points $x_k = a+kh,\ k=0,...,n.$

A classical quadrature rule has the form

$$\int_a^b w(x)\, f(x)\,dx = \sum_{k=1}^{n} w_k\, f(x_k) + E_n[f],$$

where $w(x)$ denotes a weight function, w_k are the weights, x_k are the nodes (quadrature points) and $E_n[f]$ the error term.

Let $w(x)=1$, the first basic rule is obtained from the left-hand Riemann sum

$$\int_a^b f(x)\,dx \approx h\sum_{k=0}^{n-1} f(a+kh) = R_n^{(1)}(f),$$

and the right-hand Riemann sum

$$\int_a^b f(x)\,dx \approx h\sum_{k=1}^{n} f(a+kh) = R_n^{(2)}(f).$$

These are known as the rectangular rules. The second basic rule is the midpoint rule