UNIVERSITI PUTRA MALAYSIA

CLASSIFICATION OF LOW DIMENSIONAL NILPOTENT LEIBNIZ ALGEBRAS USING CENTRAL EXTENSIONS

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CLASSIFICATION OF LOW DIMENSIONAL NILPOTENT LEIBNIZ ALGEBRAS USING CENTRAL EXTENSIONS

By

SEYED JALAL LANGARI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia in Fulfilment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

July 2010

DEDICATION

To

My Parents

For their encouragement

My wife, my son and my daughter

Mouna, Amir Ali and Moattar

For their great patience

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chair: Isamiddin Rakhimov, PhD

Faculty: Institute for Mathematical Research (INSPEM)

This thesis is concerned with the classification of low dimensional nilpotent Leibniz algebras by central extensions over complex numbers. Leibniz algebras introduced by J.-L. Loday (1993) are non-antisymmetric generalizations of Lie algebras. There is a cohomology theory for these algebraic objects whose properties are similar to those of the classical Chevalley-Eilenberg cohomology theory for Lie algebras. The central extensions of Lie algebras play a central role in the classification theory of Lie algebras.

We know that if a Leibniz algebra L satisfies the additional identity $[x, x] = 0,\ x \in L$, then the Leibniz identity is equivalent to the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \forall x, y, z \in L.$$

Hence, Lie algebras are particular cases of Leibniz algebras.

In 1978 Skjelbred and Sund reduced the classification of nilpotent Lie algebras in a given dimension to the study of orbits under the action of a group on the space of second degree cohomology of a smaller Lie algebra with coefficients in a trivial module. The main purpose of this thesis is to establish elementary properties of central extensions of nilpotent Leibniz algebras and apply the Skjelbred-Sund's method to classify them in low dimensional cases. A complete classification of three and four dimensional nilpotent Leibniz algebras is provided in chapters 3 and 4. In particular, Leibniz central extensions of Heisenberg algebras H_n is provided in chapter 4.

Chapter 5 concerns with application of the Skjelbred and Sund's method to the classification of filiform Leibniz algebras in dimension 5. Chapter 6 contains the conclusion and some proposed future directions.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGKELASAN ALJABAR LEIBNIZ NILPOTEN BERDIMENSI RENDAH DENGAN MENGGUNAKAN PENGEMBANGAN BERPUSAT

Oleh

SEYED JALAL LANGARI

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Tesis ini melibatkan pengkelasan aljabar Leibniz nilpoten berdimensi rendah. Aljabar Leibniz telah diperkenalkan oleh J.-L. Loday (1993) sebagai pengembangan aljabar Lie yang bukan anti simetrik. Terdapat teori kohomoloji terhadap objek - objek aljabar tersebut yang mana ciri - cirinya adalah sama dengan teori kohomoloji Chevalley-Eilenberg untuk aljabar Lie. Peluasan berpusat aljabar Lie memainkan peranan utama di dalam teori pengkelasan aljabar Lie.

Diketahui bahawa jika suatu aljabar Leibniz L memenuhi identiti penambahan $[x, x] = 0,\ x$ dalam L, maka identiti Leibniz adalah setara dengan identiti Jacobi

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \text{untuk semua} \quad x, y, z \quad \text{di dalam} \quad L.$$

Oleh sebab itu, aljabar Lie merupakan kes yang khusus bagi aljabar Leibniz.

Pada 1978 Skjelbred dan Sund telah mengurangkan pengkelasan aljabar Lie nilpoten di dalam dimensi yang diberi untuk mengkaji orbit di bawah tindakan suatu kumpulan di dalam ruangan kohomoloji berdarjah kedua daripada aljabar Lie yang lebih kecil bersama pekali-pekali dalam modul remeh. Tujuan utama tesis ini adalah untuk menetapkan ciri - ciri asas peluasan berpusat aljabar Leibniz nilpoten dan menggunapakai kaedah Skjelbred-Sund untuk mengkelaskan mereka di dalam kes berdimensi rendah. Suatu pengkelasan yang lengkap untuk aljabar Leibniz nilpotent berdimensi 3 dan 4 disediakan dalam bab 3 dan 4. Secara khususya, peluasan berpusat Leibniz untuk aljabar Heisenberg H_n dibentangkan dalam bab 4.

Bab 5 berkaitan dengan penggunaan kaedah Skjelbred dan Sund' kepada pengkelasan aljabar Leibniz filiform dalam dimensi5. Bab 6 mengandungi kesimpulan dan beberapa cadangan arahan masa akan datang.

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I certify that a Thesis Examination Committee has met on **5 July 2010** to conduct the final examination of Seyed Jalal Langari on his thesis entitled "**Classification of Low Dimensional Nilpotent Leibniz Algebras Using Central Extensions**" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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Date: 6 September 2010

DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

Seyed Jalal Langari

Date: 5 July 2010

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SOME CONCEPTS OF LEIBNIZ ALGEBRAS

1.1 Chapter outline

In this chapter we introduce some basic definitions and notations from the theory of Lie and Leibniz algebras that are used throughout the thesis. For Lie algebras, most of them can be found in any standard books [24].

1.2 Basic definitions and knowledge

We begin with the definition of Lie algebras.

1.2.1 Lie Algebras

Definition 1.2.1. *A Lie algebra L is a vector space over a field F equipped with a bilinear map, $[\cdot,\cdot] : L \times L \to L$ which has the following properties :*

1. $[x, x] = 0 \qquad \forall x \in L$

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \forall x, y, z \in L.$

Condition (2) is known as the Jacobi identity. As the Lie bracket $[\cdot,\cdot]$ is bilinear, we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Hence condition (1) implies

$$(1') \ [x, y] = -[y, x] \text{ for all } x, y \in L \quad (anti - symmetry).$$

If the field F does not have characteristic 2, then putting $x = y$ in $(1')$ shows that $(1')$ implies condition (1).

Example 1.2.2. *Any vector space V has a Lie bracket defined by $[x, y] = 0$ for all $x, y \in V$. This is the abelian Lie algebra structure on V. In particular, the field F may be regarded as a $1-$dimensional abelian Lie algebra.*

Example 1.2.3. *Let F=$\mathbb{R}$. The vector product $(x, y) \longmapsto x \wedge y$ defines the structure of a Lie algebra on $\mathbb{R}^3$. We denote this Lie algebra by $\mathbb{R}^3_\wedge$. Explicitly, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ then $x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.*

Example 1.2.4. *Let L be an associative algebra over F, then we define a bilinear operation $[\cdot, \cdot]$ on L by*

$$[x, y] := xy - yx \qquad \textit{for all } x, y \in L.$$

Consequently L together with $[\cdot, \cdot]$ is a Lie algebra.

Let $(L_1, [\cdot, \cdot]_1)$ and $(L_2, [\cdot, \cdot]_2)$ be Lie algebras over a field F. Then we say that a map $\varphi : L_1 \longmapsto L_2$ is a homomorphism if φ is a linear map and satisfies the condition:

$$\varphi([x, y]_1) = [\varphi(x), \varphi(y)]_2 \quad \text{for all } x, y \in L_1.$$

We say that φ is an isomorphism if φ is bijective. The map φ is an automorphism if φ is an isomorphism and $L_1 = L_2$.

An important homomorphism is the adjoint homomorphism. Let denote by $gl(L)$ the set of all linear maps from L to L. If L is a Lie algebra, we define

$$\begin{aligned} ad : L &\rightarrow gl(L) \\ x &\longmapsto ad_x \end{aligned},$$

where $(ad_x)(y) := [x, y]$ for $x, y \in L$.

Definition 1.2.5. *Let L be a Lie algebra. The center $C(L)$ of L is defined as*

$$C(L) = \{x \in L\ ; [x, L] = 0\} = \{x \in L\ ; [x, y] = 0,\ \forall\ y \in L\}.$$

Let n be the dimension of algebra L. By taking a basis $\{e_1, e_2, ..., e_n\}$ in L, all elements x and y of L can be represented as follows:

$$x = \sum_{j=1}^{n} x^j e_j \text{ and } y = \sum_{k=1}^{n} y^k e_k.$$

Notice that the product $x \cdot y$ of any two elements x and y in L, is completely determined by the product $e_j \cdot e_k$ of pairs of basis elements, that is:

$$x \cdot y = \sum_{j,k=1}^{n} x^j y^k e_j \cdot e_k.$$

Since $e_j \cdot e_k \in L$, the product $e_j \cdot e_k$ can also be expanded with respect to the basis $\{e_1, e_2, ..., e_n\}$:

$$e_j \cdot e_k = \sum_{i=1}^{n} C_{jk}^i e_i \quad (j, k = 1, ..., n).$$

The numbers C_{jk}^i are called the structure constants with respect to the basis $\{e_1, e_2, ..., e_n\}$. Therefore any algebra L is completely determined by its structure constants.

Let L be a Lie algebra. The structure constants of L are the numbers C_{ij}^k which are given by

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k.$$

As the basis is fixed, we can identify the anti-symmetry of commutator and the Jacobi identity with its structure constants. These constants satisfy the following conditions:

$$\begin{cases} C_{ij}^k = -C_{ji}^k \quad (1 \le i < j \le n, \ \ 1 \le k \le n) \\ \\ \sum_{l=1}^{n} (C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{lj}^s) = 0 \quad (1 \le i < j < k \le n), \quad 1 \le s \le n. \end{cases}$$

1.3 Introduction and methods

Usually one studies algebras where the product satisfies some further properties. In particular, Leibniz algebras are the algebras satisfying the identity

$$[x,[y,z]] - [[x,y],z] + [[x,z],y] = 0 \quad \text{for all } x,y,z \in L.$$

The following figure illustrates the inclusion of various types of algebras.

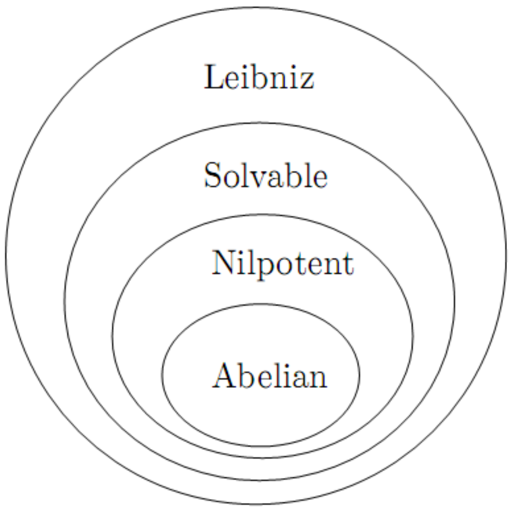

Figure 1.1. The inclusions of various types of algebras

To any Leibniz algebra we can associate the following sequence:

$$D^0L = L \supset D^1L = [L,L] \supset ... \supset D^kL = [D^{k-1}L, D^{k-1}L] \supset ....$$

Latter is called derived sequence of L. We say that L is solvable if there exists an integer $k \geq 1$ such that $D^k L = \{0\}$.

We are interested in the following question. How many essentially different (that is, non-isomorphic) nilpotent Leibniz algebras over complex numbers there are and what kind of approaches can be use to classify them.

Abelian Leibniz algebras are easily understood. For any natural number n, there is an abelian Leibniz algebra of dimension n (where for any two elements, the Leibniz bracket is zero).

It is well known (see [16]) that any two abelian Leibniz algebras of the same dimension over the same field are isomorphic. It is not hard to see that, Leibniz algebras of different dimensions can not be isomorphic.

This thesis is concerned with the classification of low-dimensional nilpotent Leibniz algebras. In 1997 Skjelbred and Sund published [52], where they have introduced a method of constructing all nilpotent Lie algebras of dimension n of given those algebras of dimension $< n$, and their automorphism group.

By minor but important adjustments, we apply the Skjelbred-Sund method to classify nilpotent Leibniz algebras in low dimensional cases. We have compared our list with that which obtained by S. Albeverio, Omirov, I. S. Rakhimov (see [4]).

Here the researcher presents a ketch of the method deployed in order to classify the nilpotent Leibniz algebras for dimension n. The following two steps are intended to present the details of the methodology.

There is the possibility to observed a list of redundant Leibniz algebras constituted in the analysis. This list represents all $n-$dimensional nilpotent Leibniz algebras.

The second step includes removing the isomorphic copies which appear in the list. The first step comprised of following the Skjelbred and Sund's method [53] which needs some methodological adjustments. The adjustment must include constructing the nilpotent Leibniz algebras into central extensions of nilpotent Leibniz algebras of a smaller dimension.

Suppose L is a nilpotent Leibniz algebra of dimension $m < n$, and V is a vector space of dimension $n - m$. By this assumption, then suppose θ be a Leibniz cocycle in $HL^2(L, V)$.

By then, θ will define a Leibniz algebra construct on $L \oplus V$. This is then called the central extension of L by θ. This construction will help us sketch all of the nilpotent Leibniz algebras for dimension n.

However, there is the problem that deferent cocycles are apt to submit isomorphic Leibniz algebras. The automorphism group of L, is denoted by $\mathrm{Aut}(L)$, and acts on $HL^2(L, V)$.

The orbits of the composed group resembles precisely the isomorphism classes of central extensions of L. Therefore, the finishing job is to calculate the orbits of $\mathrm{Aut}(L)$ on $HL^2(L, V)$.

Literature review reveals that there is no algorithm devised for computing the orbits in this method. The next step is the handy section of the methodology in which the researcher must use the action of the automorphism group to achieve the list of orbits representatives at the end.

At this stage, it is the time to use the Maple program to avoid the remaining isomorphism after constructing the resembling Leibniz algebras.

1.4 Literature Review

In this thesis we investigate the algebras, which are a generalization of Lie algebras. Therefore we give some short literature review for Lie case first.

The classification of finite-dimensional Lie algebras divides to three parts: (1) classification of nilpotent Lie algebras; (2) description of solvable Lie algebras with given nilradical; (3) description of Lie algebras with given radical.

The third problem reduces to the description of semisimple subalgebras in the algebra of derivations of a given solvable algebra [35].

The second problem reduces to the description of orbits of certain unipotent linear groups [36]. According to a theorem of Levi, in characteristic zero a finite -dimensional Lie algebra can be written as the direct sum of the a semisimple subalgebra and its unique maximal solvable ideal. If the field is algebraically closed, all semisimple Lie algebras and their modules are classified [27]. Around 1945, Malcev [36] reduced the classification of complex solvable Lie algebras to several invariants plus the classification of nilpotent Lie algebras. Quite recently, W. de Graaf [13] listed at most all 4-dimensional solvable Lie algebras over arbitrary fields.

The first problem is the most complicated and many attempts have been made on this topic, and a number of lists have been published. To mention just a few: the first non-trivial classification of some classes of low-dimensional nilpotent Lie algebras are due to Umlauf [54] in dimensions ≤ 6 over complex field, in this thesis, he presented the list of nilpotent Lie algebras of dimension $m \leq 6$. He gave also the list of nilpotent Lie algebras of dimension $m \leq 9$ admitting a basis $(e_0, e_1, ..., e_{m-1})$ with $[e_0, e_i] = e_{i+1}$ for $i = 1, ..., m-2$ (now, the nilpotent

Lie algebras with this property are called filiform Lie algebras). Umlauf's list of filiform Lie algebras is exact only for dimensions $m \leq 7$; in dimensions 8 and 9 this list contains errors and it is incomplete. Later on Dixmier [15] gives a complete list in dimension ≤ 5 over a commutative field.

In dimension 6, there are various lists obtained by Morozov over a field of characteristic 0 [37], Shedler over any field [51], Skjelbred and Sund over $\mathbb{R}$ [53], also Beck and Kolman [9] over $\mathbb{R}$. We remark that the first exact classification of the nilpotent complex Lie algebras in dimension $m \leq 6$ is probably due to Vergne [55]. She also showed the important role of filiform Lie algebras, terminology introduced by herself, in the study of variety of nilpotent Lie algebras laws. Nielsen [38] compares the tables of Morozov, Vergne, Skjelbred and Sund, and Umlauf and gives for the first time a complete and nonredundant list for nilpotent Lie algebras of dimension 6 over the real field.

In dimension 7, there are also several lists avilable: Safiulina [46], [47] over $\mathbb{C}$, Ramdhani [44], [45] over $\mathbb{R}$ and $\mathbb{C}$, Seeley [49] over $\mathbb{C}$, Ancochea and Goze [6] over $\mathbb{C}$, Ming-Peng Gong [23] over algebraically closed fields and $\mathbb{R}$. The lists above are obtained using different invariants. By introducing a new invariant-the weight system, Carles [12] compares the lists of Safiullina, Romdhani and Seeley, and has identified omissions and some mistakes in all of them. Later on in 1993, basing on his own thesis, by incorporating all the previous results, Seeley [50] published his list over $\mathbb{C}$.

There are also other partial classifications concerning some particular properties of nilpotent Lie algebras. Among them are: Favre [17] for nilpotent Lie algebras of maximal rank, Scheuneman [48], Gauger [20] and Revoy [43] for