UNIVERSITI PUTRA MALAYSIA

CENTRAL EXTENSIONS OF NILPOTENT LIE AND LEIBNIZ ALGEBRAS

MOUNA BIBI LANGARI

IPM 2010 2

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By

MOUNA BIBI LANGARI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

June 2010

DEDICATION

To

My Parents

For their encouragement

My husband, my son and my daughter

Jalal, Amir Ali and Moattar

For their great patience

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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By

MOUNA BIBI LANGARI

June 2010

Chair: Isamiddin Rakhimov, PhD

Faculty: Institute for Mathematical Research

This thesis is concerned with the central extensions of nilpotent Lie and Leibniz algebras. The researcher has used the Skjelbred and Sund's method to construct all the $6-$dimensional non-isomorphic Lie algebras over complex numbers.

Skjelbred and Sund have published in 1977 their method of constructing all nilpotent Lie algebras of dimension n introducing those algebras of dimension $< n$, and their automorphism groups. The deployed method in classifying nilpotent Lie algebras for dimension n is necessarily comprised of two main steps. For the first step, a plausibly redundant list of Lie algebras was confirmed which included all $n-$dimensional nilpotent Lie algebras. The next step is to discard the isomorphic copies generated from the list of Lie algebras.

By then, the $n-$dimensional nilpotent Lie algebras are confirmed as central extension of nilpotent Lie algebras for smaller dimensions. As the result, the

action of the automorphism group is deployed to drastically decrease the number of isomorphic Lie algebras which may appear in the final list. The maple program is by then deployed to discard the final isomorphic copies from the calculation. Based on the elaborated method, the researcher must find center of Lie algebras then drived algebras.

In order to locate a basis for second cohomology group of Lie algebras, the researcher must locate cocycles as well as coboundary. Further, there is the requirement to locate the most appropriate action for finding orbits. By then, every orbit will submit a representative that is illustrated by θ.

The following description illustrates the most significant part of the deployed method: Suppose g is an $n-$dimensional Lie algebra on complex number $\mathbb{C}$, where for every $2-$cocycle in the space of second cohomology group, there is a central extension $g(\theta) = g \oplus V$ of θ by V constituted the way it is explained. On the vector space $g \oplus V$, define a Lie product $[\cdot, \cdot]_\theta$ by $[(x, v), (y, w)]_\theta = ([x, y]_g, \theta(x, y))$ for all $x, y \in g$ and $v, w \in V$. Thus, for every located, constitute $g(\theta)$. Once done, the isomorphic constitutes must be removed.

The researcher has deployed this method for classification of nilpotent Leibniz algebras with minor modification. A detailed illustration of this method will be given in Chapter 2. The researcher has presented the most comprehensive description comprised of necessary details for the prospective researchers to pursue.

In the third Chapter, by using this method the researcher was able to classify nilpotent Lie algebras of dimensions 3, 4, and 5 over complex numbers which were all required to calculate the $6-$dimensional nilpotent Lie algebras.

In the fourth Chapter the same procedure has been utilized to the classification of $6-$dimensional nilpotent Lie algebras over complex numbers.

In the Chapter 5, the modified of Skjelbred and Sund's method has been used for classifying filiform Leibniz algebras of dimension 5 over complex numbers. In particular, some examples of classification of $5-$dimensional nilpotent Leibniz algebras are provided.

For the final chapter, Chapter six, the researcher has presented a comparison of his constituted list with the list of Beck and Kolman. The researcher has provided recommendations and suggestions for further research at the end of chapter six for interested researchers.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENGEMBANGAN BERPUSAT ALJABAR LIE DAN LEIBNIZ NILPOTEN

Oleh

MOUNA BIBI LANGARI

Jun 2010

Pengerusi: Isamiddin Rakhimov, PhD

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Tesis ini berkaitan dengan pengembangan berpusat aljabar Lie dan Leibniz nilpoten. Penyelidik menggunakan kaedah Skejelbred dan Sund untuk membina kesemua aljabar Lie berdimensi 6 dan tidak berisomorfisma terhadap nombor kompleks.

Skjelbred dan Sund telah menerbitkan pada 1977 kaedah mereka di dalam membina kesemua aljabar Lie nilpoten berdimensi n, memperkenalkan aljabar tersebut untuk dimensi $< n$, dan kumpulan automorfisma mereka. Kaedah yang digunapakai di dalam mengkelaskan aljabar Lie nilpoten berdimensi n perlu mempunyai 2 langkah penting. Langkah pertama, satu senarai aljabar Lie nilpoten perlu disahkan yang mana ia mempunyai kesemua aljabar Lie nilpoten berdimensi n. Langkah seterusnya adalah untuk menyingkirkan salinan isomorfik yang dihasilkan daripada senarai aljabar Lie.

Dengan itu, aljabar Lie nilpoten berdimensi n akan disahkan sebagai pengembangan berpusat daripada aljabar Lie nilpoten berdimensi yang lebih kecil. Kesannya, gerak kerja kumpulan automorfisma digunapakai untuk mengurangkan jumlah aljabar Lie dengan cepat yang mana akan ditunjukkan di dalam bab terakhir. Di sini, pengaturcaraan MAPLE kemudiannya digunapakai untuk menyingkirkan salinan isomorfik terakhir daripada pengiraan. Berdasarkan kaedah yang panjang lebar, penyelidik perlu mencari pusat aljabar Lie dan kemudiannya menemui aljabar.

Untuk mengesan asas kumpulan kohomoloji kedua bagi aljabar Lie, penyelidik perlu mengesan cocycles (samakitaran) dan juga samasempadan. Seterusnya, terdapat keperluan untuk mengesan tindakan yang paling sesuai untuk mencari orbit. Selanjutnya, setiap orbit akan menyediakan sebuah wakil yang akan diterangkan oleh θ.

Keterangan seterusnya menerangkan bahagian yang paling penting di dalam kaedah yang digunapakai. Jika g adalah satu aljabar Lie berdimensi n terhadap nombor kompleks, di mana untuk setiap samakitaran 2 di dalam ruang kumpulan kohomoloji kedua, terdapat satu pengembangan berpusat $g(\theta) = g \oplus V$ bagi θ oleh V yang dibentuk sebagaimana ia diterangkan. Untuk ruang vektor $g \oplus V$, ditakrifkan hasildarab Lie $[\cdot,\cdot]_\theta$ sebagai $[(x,v),(y,w)]_\theta = ([x,y]_g, \theta(x,y))$ bagi semua $x, y \in g$ dan $v, w \in V$. Oleh itu, untuk sebarang $g(\theta)$ yang dikesan dan dibentuk. Apabila selesa, isomorfik perlu dibuang.

Penyelidik menggunapakai kaedah ini untuk pengkelasan aljabat Lie nilpoten dengan sedikit pengubahsuaian. Penerangan terperinci bagi kaedah ini akan diberi di dalam Bab 2. Penyelidik mengemukakan gambaran yang paling lengkap yang menpunyai perincian yang perlu untuk penyelidik berpotensi mendapatkannya.

Di dalam Bab ketiga, dengan menggunakan kaedah ini, penyelidik telah mampu mengkelaskan aljabar Lie nilpoten berdimensi $3, 4$, dan 5 terhadap nombor kompleks yang diperlukan kesemuanya untuk mengira aljabar Lie nilpoten berdimensi 6.

Di dalam Bab keempat, kaedah dan langkah yang sama telah digunapakai untuk pengkelasan aljabar Lie nilpoten berdimensi 6 terhadap nombor kompleks.

Di dalam bab 5, kaedah Skjelbred dan Sund yang telah diubahsuah telah digunakan untuk mengkelaskan aljabar Leibniz filiform berdimensi 5 terhadap nombor kompleks. Secara khususnya, beberapa contoh pengkelasan aljabar Leibniz nilpoten berdimensi 5 akan disediakan.

Bagi bab yang terakhir, Bab 6, penyelidik telah mengemukakan suatu perbandingan daripada senarai yang dihasilkan dengan senarai Beck dan Kolman. Penyelidik telah mengemukakan cadangan dan saranan untuk kajian seterusnya di akhir bab 6 untuk mana-mana penyelidk yang berminat.

ACKNOWLEDGEMENTS

First and foremost, praise to Allah for giving me strength, courage and patience in completing this research.

I would like to thank my supervisor Associate Professor Dr. Isamiddin Rakhimov for his support and patience throughout this process. This work could not have been carried out without both direct and indirect help and support from him. Besides learning algebra I also learned a lot of things from him in last two years.

I would like to thank Professor Dato Dr. Kamel Ariffin M. Atan as the director of the Institute for Mathematical Research (INSPEM) for his encouragement. I am highly grateful to the Institute for Mathematical Research and Universiti Putra Malaysia (U.P.M.) for all the fruitful years of my study that left an enduring positive impression on my life and professional development.

I also would like to express my sincere and deepest gratitude to my supervisory committee members, Associate Professor Dr. Mohamad Rushdan Md. Said and Dr. Siti Hasana Bt Sapar for their advice and helpful comments. I want to thank my husband Jalal for his cooperation and encouragement. It is to my husband and my son Amir Ali and my daughter Moattar Sadat that I dedicate this work, with all my love and heartfelt gratitude.

I am indebted to my Mom and Dad for all their support and sacrifice. They have always been my source of inspiration and have encouraged me to learn throughout my life. Thanks to my brothers Matin, Moein, Mobin and Milad for their encouragements. I am grateful to Mr. Saeed Vahdati for his invaluable help with regard to any computer-related problems, and Gholamreza Zareian for all their help.

I owe my big gratitude to my teachers Mr. Abbasnejad, Khojasteh and Solaymani for all their encouragements and help. I would like to express my sincere gratitude to Mr. Reza Gholinejad and his wife Hapipah Khalid for their encouragements and support in Malaysia. My deepest appreciation goes to all my friends especially Soghra, Kobra, Masoumeh Barghchi and Hamideh, for their encouragements and everyone in the Mathematics Department, Universiti Putra Malaysia who has helped in one way or another.

There is some supporting people that I might have forgotten: exhaustion plays havoc on memory. My humblest apologies to those who consider having earned for themselves being cited amongst these lines.

I certify that a Thesis Examination Committee has met on **23 June 2010** to conduct the final examination of Mouna Bibi Langari on her thesis entitled "**Central Extensions of Nilpotent Lie and Leibniz Algebras**" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

Members of the Thesis Examination Committee were as follows:

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science.

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Date: 6 September 2010

DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

Mouna Bibi Langari

Date: 23 June 2010

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CHAPTER 1

INTRODUCTION

1.1 Chapter Outline

In this chapter we introduce some basic definitions and notations from the theory of Lie algebras that are used throughout the thesis. Most of them can be found in any standard books on Lie algebras for example [12], [20], [22] and [31].

1.2 Basic Definitions and Knowledge

We begin with the definitions of vector space V, algebra and derivation of algebra.

Definition 1.2.1. *A vector space is a set V on which two operations $+$ and $\cdot$ are defined, called vector addition and scalar multiplication. The operation $+$ (vector addition) must satisfy the following conditions:*

Closure: *If u and v are any vectors in V, then the sum $u + v \in V$.*

Commutative law: *For all vectors u and v in V, $u + v = v + u$.*

Associative law: *For all vectors u, v, w, in V, $u+(v+w) = (u+v)+w$.*

Additive identity: *The set V contains an additive identity elements, denoted by 0, such that for any vector v in V, $0 + v = v$ and $v + 0 = v$.*

Additive inverses: *For each vector v in V, the equations $v + x = 0$*

and $x + v = 0$ have a solution x in V, called an additive inverse of v, and denoted by $-v$.

The operation (scalar multiplication) is defined between complex numbers(or scalar) and vectors, must satisfy the following conditions:

Closure: *If v is any vector in V , and c is any complex number, then the product $c \cdot v$ belong to V.*

Distributive law: *For all complex numbers c and vectors u,v in V,*

$$c \cdot (u + v) = c \cdot u + c \cdot v.$$

Distributive law: *For all complex numbers c, d and vector v in V,*

$$(c + d) \cdot v = c \cdot v + d \cdot v.$$

Associative law: *For all complex number c, d and all vectors v in V,*

$$c \cdot (d \cdot v) = (c \cdot d) \cdot v.$$

Unitary law: *For all vectors v in V, $1 \cdot v = v$.*

Definition 1.2.2. *A is called an algebra over the field $\mathbb{C}$ if A is a vector space over $\mathbb{C}$, such that $f : A \times A \longrightarrow A$ and*

$$f(\alpha a + \beta b, c) = \alpha f(a, c) + \beta f(b, c)$$

$$f(a, \gamma b + \delta c) = \gamma f(a, b) + \delta f(a, c)$$

for $a, b, c \in A$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Definition 1.2.3. *Let A be any algebra over a field F ($F = \mathbb{C}$), and define a derivation of A as a linear operator D on A satisfying*

$$(xy)D = (xD)y + x(yD), \forall x, y \in A.$$

Definition 1.2.4. *Let G be a group and let X be a set. A left group action is a function $\cdot : G \times X \longrightarrow X$ such that:*

$1_G \cdot x = x$ *for all* $x \in X$.

$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ *for* $g_1, g_2 \in G$ *and* $x \in X$.

A right group action is a function $\cdot : X \times G \longrightarrow X$ such that

$x \cdot 1_G = x$ *for all* $x \in X$,

$x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$ *for* $g_1, g_2 \in G$ *and* $x \in X$.

1.3 Introduction and Methodology

This thesis is concerned with the classification of 6−dimensional nilpotent Lie algebras. Some examples of classification of 5−dimensional nilpotent Leibniz algebras are also provided.

Skjelbred and Sund [45] published in 1977 their method of constructing all nilpotent Lie algebras of dimension n given those algebras of dimension $< n$, and their automorphism group.

For the purpose of classification and comparison across our lists, Maple version 12 is an efficient instrument. Using Maple provides the researcher with a reliable instrument in order to compute algebraic systems. In the current research, the software was used to compute: (1) the Jacobi identities; (2) the cocycles; (3) the isomorphism between two algebras, particularly, as a special case, the automorphism groups; and (4) solving all kinds of equations. Further, the researcher has used Maple 12 for various calculations and computations across the current research.

Here we give a brief outline of the method that we use for classifying nilpotent Lie and Leibniz algebras of dimension n. For more details, the readers are refered to Chapter 2. It essentially consists of two steps. In the first step a possibly redundant list of Lie and Leibniz algebras is constructed for all $n-$dimensional nilpotent Lie and Leibniz algebras. Secondly we remove the isomorphic copies from the list.

For the first step we follow the method of Skjelbred and Sund [46] for Lie case and for Leibniz algebras with adjustments in some steps of method, that is, we construct nilpotent Lie and Leibniz algebras as central extensions of nilpotent Lie and Leibniz algebras of smaller dimension.

Let g be a nilpotent Lie or Leibniz algebra of dimension $m < n$, and V a vector space of dimension $n-m$. Let θ be a cocycle in $H^2(g, V)$ (or $HL^2(g, V)$). Then θ defines a Lie and Leibniz algebras structure on $g \oplus V$, which is called the central extension of g by θ (for Lie case θ is Lie cocycle and for Leibniz algebra θ is Leibniz cocycle).

By this construction we get all nilpotent Lie and Leibniz algebras of dimension n. The main problem is that different cocycles may yield isomorphic Lie or Leibniz algebras. Now the automorphism group of g, denoted $\mathrm{Aut}(g)$, acts on $H^2(g, V)$. The orbits of this group correspond exactly to the isomorphism classes of the central extensions of g. So all that is left is to compute the orbits of $\mathrm{Aut}(g)$ on $H^2(g,V)$ (or $HL^2(g,V)$).

However, if the dimension of the field is infinite then this last space is an infinite set. Literature review reveals that there is no algorithm devised for computing the orbits in this method. Therefore we use the action of the automorphism group in hand calculations to get a list of orbit representatives that is as small as we can possibly make it. We then construct the corresponding Lie and Leibniz algebras, and obtain a list that usually contains some isomorphisms. In order to get rid of the isomorphisms, we use a Maple 12 program.

We can illustrate the inclusion of various types of Lie and Leibniz algebras in the following figures:

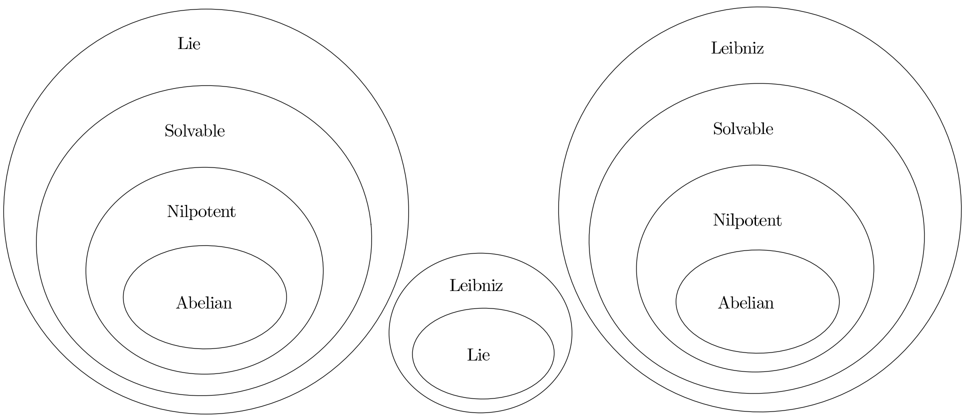

Figure 1.1. The Inclusions of Various Types of Lie and Leibniz Algebras

To any Lie algebras we can associate the following sequence:

$$D^0g = g \supset D^1g = [g,g] \supset ... \supset D^kg = [D^{k-1}g, D^{k-1}g] \supset ...,$$

and called a derived sequence of g. We say that g is solvable if there exists an integer $k \geq 1$ such that $D^kg = \{0\}$.

1.4 Literature Review

Let us consider Lie case first:

The classification of finite-dimensional Lie algebras are divided into three parts: (1) classification of nilpotent Lie algebras; (2) description of solvable Lie algebras with given nilradical; (3) description of Lie algebras with given radical.

The third problem reduces to the description of semisimple subalgebras in the algebra of derivations of a given solvable algebra [28].

The second problem reduces to the description of orbits of certain unipotent linear groups [29]. According to a theorem of Levi [22], in characteristic zero a finite -dimensional Lie algebra can be written as the direct sum of the a semisimple subalgebra and its unique maximal solvable ideal. If the field is algebraically closed, all semisimple Lie algebras and their modules are classified [20]. In 1945, Malcev [29] reduced the classification of complex solvable Lie algebras to several invariants plus the classification of nilpotent Lie algebras. Quite recently, W. de Graaf [9] listed at most all 4-dimensional solvable Lie algebras over arbitrary fields.

The first problem is the most complicated and many attempts have been made on this topic, and a number of lists have been published. To mention just a few: the first non-trivial classification of some classes of low-dimensional nilpotent Lie algebras are due to Umlauf [47] in dimensions ≤ 6 over complex field, Umlauf present the list of nilpotent Lie algebras of dimension $m \leq 6$. Umlauf