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PAPER

Chaotic Bohmian trajectories and the role of quantum potential in rational frequencies system

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Abstract

Bohmian mechanics is an alternative formulation of quantum mechanics that maintains the notion of a well-defined trajectory governed by the wave function. The extra potential in this framework called quantum potential plays an important role in determining the dynamics of the Bohmian trajectory. In this study, we examine numerically the characteristics of a rational frequency system near a moving nodal point and how quantum potential plays an important role in generating chaotic motion for a specific mixture of wavefunction amplitudes. We show the transition from a regular to a chaotic trajectory when there is an abrupt change of quantum potential near the nodal point. This phenomenon occurs in the centre where the value of quantum potential is much higher than classical potential.

1. Introduction

The reason behind conventional quantum mechanics, which is still a probability theory, is that measurement outcomes are unpredictable. The development of a deterministic quantum theory has been the subject of considerable work; one of the more significant attempts was Bohmian mechanics [1]. Bohmian mechanics is primarily concerned with keeping a well-defined position while obeying the wavefunction that satisfies the Bohmian equations of motion

$$m\frac{d\mathbf{r}}{dt} = \hbar Im\left(\frac{\nabla\psi}{\psi}\right) \tag{1}$$

where $\mathbf{r}$ is the position vector, $\hbar$ is reduced Planck's constant, m is a mass and ψ is the wavefunction. At first sight, the concept of trajectory in Bohmian mechanics appears comparable to the classical one. However, the specifics show that this theory does not entirely reproduce the classical concept since it comprises a new possibility known as the quantum potential

$$Q = -\frac{\hbar^2\nabla^2 \ R}{2mR} \tag{2}$$

where $R = |\psi|^2$. Unlike the classical potential, the quantum potential varies according to the evolution of the wavefunction, and its characteristics depend on the form of the wavefunction, not the intensity [2]. It is the fundamental reason for the non-locality of Bohmian trajectories and is crucial to understanding their evolution. In this setting, Bohmian mechanics efficiently provides intuitive thoughts and notions that establish causal relationships between events in physical space and time. As a result, this framework can serve as an alternative strategy to solving some quantum mechanical problems such as quantum chaos.

Quantum chaos has been extensively studied in physics since it attempts to comprehend the relationship between chaos and quantum theories. It should be emphasized that classical chaotic notions like periodic orbits, repellors, phase space maps, and more are based on the classical concept of trajectory. However, fundamental incompatibilities between quantum and classical mechanics demonstrate themselves in microscopic systems. According to the Heisenberg uncertainty principle, a particle's position and momentum cannot be measured simultaneously, making position and trajectory which are two fundamental ideas in classical mechanics inapplicable. Furthermore, in conventional quantum mechanics, the wave-particle duality influences the concept of a physical system's state. The wavefunction's linearity makes it less sensitive to initial conditions, and its evolution must follow the Schrödinger equation. For these reasons, chaotic trajectories cannot be used directly in conventional quantum mechanics. To solve this challenge, physicists have studied quantum mechanical criteria for detecting chaos in the quantum system. A variety of methods such as Random Matrix Theory [3] and Out-of-Time-Ordered Correlators [4] have been discovered, but they still cannot be directly compared.

It has been shown that researchers have expressed a strong interest in Bohm's trajectory-based formulation of quantum mechanics. In Bohmian mechanics, chaos has been discovered even when the classical analogue of the system is not chaotic [5]. Chaotic Bohmian trajectories may also emerge in the absence of chaos in conventional quantum mechanics [6]. Several research studies have investigated chaos in Bohmian mechanics for various systems, including billiard systems [6], double-well potential [7], hydrogen atoms in external fields [8] and harmonic oscillators [9].

Studies by Parmenter and Valentine [5] reveal that some superpositions of harmonic oscillator states can lead to chaos and trajectories of these states show sensitive dependency on initial conditions. This chaotic behavior results from the complex structure of the quantum potential producing regions of rapid acceleration and deceleration. Dembiński *et al* work on time-dependent harmonic oscillators shows that periodic changes in an oscillator's frequency can lead to chaos with fractal characteristics in the phase space [10]. These results demonstrate the interpretative richness of Bohmian mechanics, since chaos arises even where one would expect the structure of the Schrödinger equation to be linear. In addition, the study of commensurate and incommensurate frequencies has proven useful. The analysis of chaos in the Bohmian mechanics of commensurate harmonic oscillators suggests that these systems have parts governed by the frequency ratio. Here, periodic or chaotic motion depends on the commensurability of the frequencies [9]. Chaos is also caused by multiple nodes which are randomly located in the configuration space, as has been found in the study of chaotic orbits in complex Bohmian systems with incommensurate frequencies [11]. In addition, Born's studies of distribution, order and chaos in Bohmian mechanics provided a complete understanding of two-dimensional quantum harmonic oscillators [12] and shed new light on the role of superpositions in chaotic dynamics.

However, there is still much to learn about the specific mechanism of chaos for Bohmian orbits. The interaction of nodal points and X-points, which together constitute the nodal point X-point complex (NPXPC), is one of the factors driving the creation of chaos. The places where the Bohmian equation of motion (1) becomes singular $\psi = 0$ are known as nodal points. The nodal point singularity has a strong effect on the nearby particles, which tend to move in a spiral or divergent pattern depending on the local features of the quantum velocity field. These interactions are crucial in causing trajectories to become unstable, which in turn leads to chaotic behaviour [13]. On the other hand, X-points are unstable saddle points in the velocity field that facilitate trajectory splitting and enhance divergence. They are situated close to nodal points. The NPXPC is a dynamic structure that creates extremely sensitive and chaotic dynamics in the Bohmian flow by combining the instability caused by X-points with the singularity influence of nodal points [14]. For the harmonic oscillator, moving nodal points create intricate trajectory flow patterns, particularly in superposed states, while their interplay with X-points induces sensitive dependence on initial conditions, a defining characteristic of chaos [15]. The global phase-space structure and dynamical complexity can be changed by the formation and destruction of nodal points as the system is driven. These findings illuminate the role of NPXPC as bridges between quantum coherence and classical-like behavior [15, 16]. Furthermore, various investigations have demonstrated that quantum potential, which is absent in classical physics, reaches its maximum value extremely close to the X-point, especially when the X-point is very close to the nodal point, i.e. the NPXPC is tiny in size [17].

According to the literature review, several authors have determined the quantum potential and provided graphics illustrating its shape in various situations [18–20]. However, to the best of our knowledge, no focus has been placed on the form of the quantum potential near nodal points of the wavefunction in the case of a rational frequencies system. This is because nearly all research indicates that Bohmian trajectories in such a system always exhibit regular motion [21]. In particular, the work of [13] stated that a rational frequencies system generates a finite number of nodal lines, and no chaos occurs without further discussing why the nodal points in this system could not generate chaos when a particle approaches them. Therefore, in the present paper, we investigate numerically the form of the quantum potential in a system of rational frequencies and show that at certain conditions of wavefunction amplitudes, this potential can generate chaotic motion near nodal lines.

The structure of the paper is as follows: In section 2 we derive the equation of Bohmian trajectory for a two-dimensional commensurate harmonic oscillator and the corresponding nodal point. Then we focus on the computation of Lyapunov exponent, which is a chaos indicator in section 3. In section 4, we study the behaviour of the Bohmian trajectory near the nodal point, followed by the effect of a mixture of wavefunction amplitudes for generating chaos near a nodal point in section 5. Finally, in section 6, we discuss our results and conclusions.

2. Rational frequencies system and nodal point

In this study, we focus on the behaviour of a rational frequency system near a moving nodal point. The simplest example of the system is a two-dimensional commensurate harmonic oscillator. The Hamiltonian of this system is given by

$$\hat{H}(x, y) = -\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(m\omega_1^2 x^2 + m\omega_2^2 y^2), \tag{3}$$

where x and y represent the particle's position, m is the mass of a particle, and ω_1 and ω_2 are the oscillator frequencies. For convenience, we set $\hbar = m = 1$ and all quantities are expressed in these natural units, i.e. time is in units of $\frac{1}{\omega}$, position in units of $\left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}$, momentum in units of $(\hbar m\omega)^{\frac{1}{2}}$, and energy in units of $\hbar\omega$.

Let us consider the superposition of this system's three eigenstates as follows:

$$\psi_{0,0} = \frac{\alpha}{\sqrt{\pi}} e^{-\frac{1}{2}(\omega_1 x^2 + \omega_2 y^2)} e^{-\frac{i}{2}(\omega_1 + \omega_2)t}$$
$$\psi_{1,0} = \frac{2x\alpha}{\sqrt{\pi}} e^{-\frac{1}{2}(\omega_1 x^2 + \omega_2 y^2)} e^{-\frac{i}{2}(3\omega_1 + \omega_2)t}$$
$$\psi_{0,1} = \frac{2y\alpha}{\sqrt{\pi}} e^{-\frac{1}{2}(\omega_1 x^2 + \omega_2 y^2)} e^{-\frac{i}{2}(\omega_1 + 3\omega_2)t}$$

with $\alpha = (\omega_1\omega_2)^{\frac{1}{2}}$ and the ground state $\psi_{0,0}$ correspond to an energy equal to $\frac{1}{2}(\omega_1 + \omega_2)$ and the first excited states $\psi_{0,1}$ and $\psi_{1,0}$ associated with an energy equal to $\frac{1}{2}(\omega_1 + 3\omega_2)$ and $\frac{1}{2}(3\omega_1 + 3\omega_2)$ respectively. Such a combination of the eigenstates is crucial because if we consider only two eigenstates, the Bohmian trajectory generated completely depends on a single frequency which means no chaos occurs [5].

By linearly combining the eigenstates, we were able to obtain the resulting time evolution wavefunction $\psi(\mathbf{r}, t)$ given as

$$\left(\frac{\alpha A e^{-\frac{i}{2}(\omega_1+\omega_2)t}}{\sqrt{\pi}} + \frac{2x\alpha B e^{-\frac{i}{2}(3\omega_1+\omega_2)t}}{\sqrt{2\pi}} + \frac{2y\alpha C e^{-\frac{i}{2}(\omega_1+3\omega_2)t}}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}(w_1 x^2 + \omega_2 y^2)} \tag{4}$$

where $\mathbf{r} = (x, y)$ and $A = a + ib$, $B = c + id$, $C = f + ig$ are wavefunction amplitudes. It is necessary for these complex numbers to meet the normalization requirement $|A| + |B| + |C| = 1$. After that, equation (4) was substituted into the Bohmian equation of motion,

$$v(\mathbf{r}, t) = \frac{i}{2}\frac{\psi^*(\mathbf{r}, t)\nabla\ \ \psi(\mathbf{r}, t) - \psi(\mathbf{r}, t)\nabla\ \ \psi^*(\mathbf{r}, t)}{|\psi(\mathbf{r}, t)|^2} \tag{5}$$

allowing us to determine the particle's velocity

$$v_x(\mathbf{r}, t) = -\frac{\sqrt{2}\,\alpha^2\beta_x(t) - 2\alpha^2\gamma(t)y}{G(\mathbf{r}, t)} \tag{6}$$

$$v_y(\mathbf{r}, t) = -\frac{\sqrt{2}\,\alpha^2\beta_y(t) - 2\alpha^2\gamma(t)x}{G(\mathbf{r}, t)} \tag{7}$$

where,

$$\beta_x(t) = (ac + bd)\sin(\omega_1 t) - (ad - bc)\cos(\omega_1 t)$$
$$\beta_y(t) = (af + bg)\sin(\omega_2 t) - (ag - bf)\cos(\omega_2 t)$$
$$\gamma(t) = (cf + dc)\sin(\omega_1 - \omega_2)t + (cg - df)\cos(\omega_1 - \omega_2)t$$
$$\begin{aligned} G(\mathbf{r}, t) = {} & \alpha^2(\alpha^2 + d^2) + 2x^2\alpha^2(c^2 + d^2) + 2y^2\alpha^2(f^2 + g^2) + 2\sqrt{2}\,\alpha^2[(ad - bc)\sin(\omega_1 t) \\ & + (ac + bd)\cos(\omega_1 t)]x + 2\sqrt{2}\,\alpha^2[(ag - bf)\sin(\omega_1 t) + (af + bg)\cos(\omega_1 t)]y \\ & + 4\alpha^2[(dc - cg)\sin(\omega_1 - \omega_2)t + (cf + dg)\cos(\omega_1 - \omega_2)t]xy \end{aligned}$$

and using *Mathematica*'s NDSolve command, we solve these differential forms, which becomes the trajectory equation for particles in the Bohmian framework.

Furthermore, the equations of motion (6) and (7) become singular when the value of $G(\mathbf{r}, t)$ is zero and then we can obtain the equation of a nodal point (x_N, y_N) as follows

$$x_N(t) = \frac{\beta_y(t)}{\gamma(t)}, \qquad y_N(t) = \frac{\beta_x(t)}{\gamma(t)}. \tag{8}$$

Besides that, to study the characteristic of quantum potential near the nodal point, we numerically computed its evolution by substituting the wavefunction (4) into the following quantum potential equation written as

$$Q(\mathbf{r},\, t) = -\frac{1}{2}\left[Re\frac{\nabla^2\psi(\mathbf{r},\, t)}{\psi(\mathbf{r},\, t)} + Im\left(\frac{\nabla\psi(\mathbf{r},\, t)}{\psi(\mathbf{r},\, t)}\right)^2\right] \tag{9}$$

where terms *Re* and *Im* correspond to the real part and the imaginary part of the function $Q(\mathbf{r}, t)$ respectively.

3. Lypunov exponent

Similar to classical trajectories, chaotic and regular motion can also occur in Bohmian trajectories. Unlike conventional quantum mechanics, we can use the classical approach to study chaos in this theory. The rate at which adjacent trajectories in a dynamical system diverge or converge over time is measured by the Lyapunov exponent. Benettin *et al* [22] created a technique especially for bounded systems to calculate this measurement. Let r_0 and s_0 be the initial points of two trajectories in the phase space M that are very close to one another. Their initial separation is represented as

$$|\delta_0| = ||r_0 - s_0||, \tag{10}$$

where $||\cdots||$ is Euclidean norm. The Hamiltonian flow ϕ' maps the trajectories as time passes, resulting in

$$x_t = \phi^t(r_0), \qquad y_t = \phi^t(s_0), \tag{11}$$

and at any given time $t \in \mathbb{R}$, the separation between the two trajectories is

$$|\delta_t| = ||\phi^t(r_0) - \phi^t(s_0)||. \tag{12}$$

We periodically rescale the system after a defined time τ to make sure the distance between trajectories remains within the region where linear approximations hold. To accomplish this, the trajectories are advanced using

$$x_n = \phi^\tau(r_{n-1}), \qquad y_n = \phi^\tau(s_{n-1}), \tag{13}$$

where $n \in \mathbb{N}$ is the step count. The distance between the points is represented as follows after rescaling

$$|\delta_t| = ||\phi^\tau(r_{n-1}) - \phi^\tau(s_{n-1})||. \tag{14}$$

Every step of the trajectory s_n is modified so that its separation from r_n is equal to the original separation $|\delta_0|$. This procedure is repeated n times, producing a sequence of separations $|\delta_n|$. The Lyapunov exponent, which is calculated using the following equation,

$$\lambda = \frac{1}{n\tau}\sum_{i=1}^{n}\ln\left(\frac{\delta_i}{\delta_0}\right) \tag{15}$$

where for δ_i; $i = 1, 2, 3, \ldots, n$. Then measures the average rate of divergence where τ is the amount of time that passes between rescaling operations.

The Lyapunov exponent is a useful tool for characterizing the nature of dynamical systems since it can distinguish between regular and chaotic motion. A system with a positive Lyapunov exponent is said to behave chaotically, as neighbouring trajectories diverge exponentially over time, in contrast to a system with a zero or negative Lyapunov exponent, which is said to exhibit regular or stable motion, where trajectories either converge or maintain a constant separation. To do this computation, we developed an iterative method using *Mathematica* that is highly useful for doing these numerical computations accurately and efficiently.

4. Chaotic orbit near a nodal point

The characteristic of a Bohmian trajectory near a moving nodal point is demonstrated numerically in this section. In figure 1, Bohmian trajectories starting close to nodal lines (green) show mixed regular and chaotic behaviour. Starting with regular behaviour, the trajectories then slowly attract to the centre and move around the nodal lines area with a complex orbital structure. This attraction affects the characteristic of the particle's velocity (figure 3), which appears to change rapidly and is no longer periodic. Here, chaos is introduced when a particle moves closely to the neighbourhood of the nodal point. This is clearly illustrated by the two trajectories in figure 1, which start close to each other and no longer follow the same path when entering the nodal lines area. Based on a computation of the Lyapunov exponent, the transition from regular to chaos can be confirmed by the graph approaching and becoming zero at first, and after some time the graph suddenly jumps and stabilizes at a constant positive value (figure 2). From this result, we identify the behaviour of the particle's velocity is closely related to the onset of chaos since the Lyapunov exponent appears to have a positive value (red graph) at the same time when the velocity of the particle becomes non-periodic (figure 3). Besides, the emergence of chaos strongly depends on the distance between the particle's initial position and the nodal point. If the particle's starting position is far from the nodal point, the motion shows no fundamental difference from the former orbit but in this case, the onset of chaos appears is late because it takes some time for the particle to enter the nodal point's region. This phenomenon can be

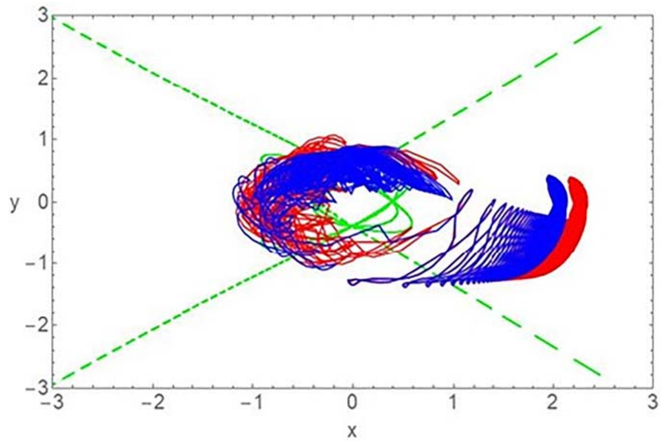

Figure 1. Bohmian trajectory for initial position $x_0 = 2.00$ (blue), 2.20 (red), $y_0 = 0.00$ and $a = 0.37$, $b = -0.02$, $c = 0.44$, $d = 0.49$, $f = -0.49$, $g = 0.22$.

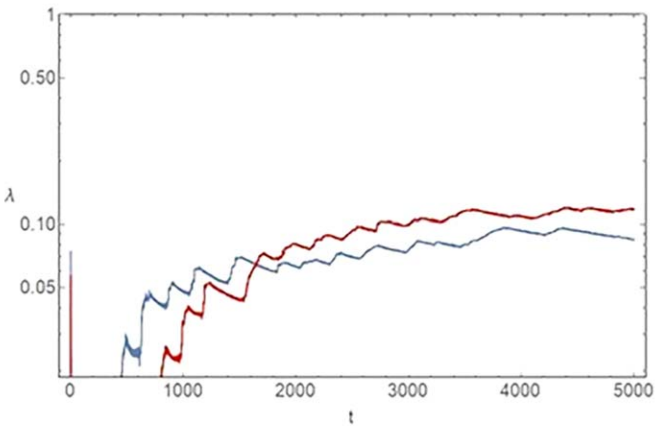

Figure 2. Lyapunov exponent for Bohmian trajectory with initial position $x_0 = 2.00$ (blue), $x_0 = 2.20$ (red) and $y_0 = 0.00$.

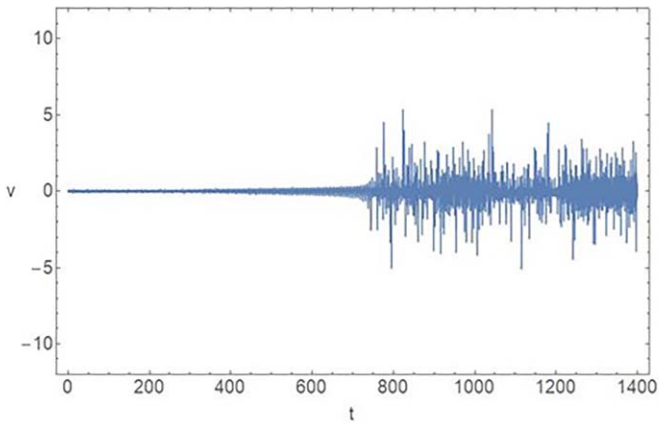

Figure 3. Velocity of Bohmian trajectory starts with regular motion and turns to aperiodic motion near a moving nodal point.

shown in figure 2, that the Lyapunov exponent for a particle that starts at $x_0 = 2.20$ appears to have a positive value later ($t = 800$) compared to a particle starting at $x_0 = 2.00$ that has time ($t = 400$).

In general, we can identify the resemblance of this result with the recent observation of the irrational frequency system [13] that shows the generation of chaos when the particle trajectory is moving very close to the

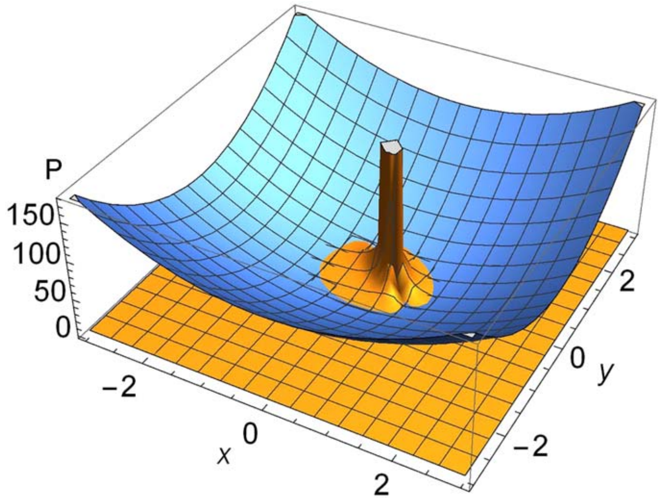

Figure 4. Distribution of quantum potential (yellow) and classical potential (blue) over 2-dimensional space.

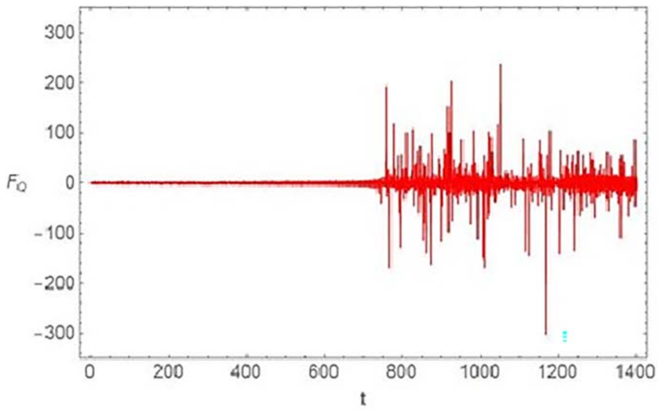

Figure 5. Quantum force starts with smooth evolution and turns to abrupt oscillations and sharp spikes near a moving nodal point.

nodal point. Here, the key role in generating chaotic motion is due to the detailed form of quantum potential very close to the vicinity of a nodal point i.e. quantum potential takes maximum value near this point [17].

Thus, in our study, we are interested in investigating the characteristics of the quantum potential close to a nodal point. To illustrate it, we plot figure 4 to present the quantum and classical potential (P) as a function of 2-dimensional space (x, y). It is clear from this figure that quantum potential changes drastically when approaching the nodal point's area with a maximum value is over 150. This result gives us strong evidence as to why the force exerted on a particle changes rapidly with aperiodic motion in the centre (figure 5) since the value of quantum potential is significantly higher than classical potential compared to other areas. To understand how this relates to chaos, consider that quantum force is defined as

$$F_Q = -\frac{\partial Q}{\partial x} \tag{16}$$

which act as impulsive forces on Bohmian trajectories. These impulses can be analyzed by considering the dynamics of two initially close trajectories, whose separation evolves according to

$$\frac{d^2\delta(t)}{dt^2} = \Delta F_Q, \tag{17}$$

where ΔF_Q, is the difference in quantum forces acting on the trajectories. When an impulsive event takes place, the velocity difference Δv increases because the force F_Q changes at a very rapid rate in a very short amount of time, Δt expressed as

Table 1. Percentage of wavefunction amplitudes for the ground state and first excited states.

Ground state	Excited state	Amplitudes of wavefunction
100%	0%	$a = 0.707, b = -0.707, c = 0.000, d = 0.000, f = 0.000, g = 0.000$
90%	10%	$a = 0.671, b = -0.671, c = 0.158, d = 0.158, f = -0.158, g = 0.158$
80%	20%	$a = 0.632, b = -0.632, c = 0.224, d = 0.224, f = -0.224, g = 0.224$
70%	30%	$a = 0.592, b = -0.592, c = 0.274, d = 0.274, f = -0.274, g = 0.274$
60%	40%	$a = 0.548, b = -0.548, c = 0.316, d = 0.316, f = -0.316, g = 0.316$
50%	50%	$a = 0.500, b = -0.500, c = 0.354, d = 0.354, f = -0.354, g = 0.354$
40%	60%	$a = 0.447, b = -0.447, c = 0.387, d = 0.387, f = -0.387, g = 0.387$
30%	70%	$a = 0.387, b = -0.387, c = 0.418, d = 0.418, f = -0.418, g = 0.418$
20%	80%	$a = 0.316, b = -0.316, c = 0.447, d = 0.447, f = -0.447, g = 0.447$
10%	90%	$a = 0.224, b = -0.224, c = 0.474, d = 0.474, f = -0.474, g = 0.474$
0%	100%	$a = 0.000, b = 0.000, c = 0.500, d = 0.500, f = -0.500, g = 0.500$

$$\Delta v = \int_{0}^{\Delta t} \Delta F_Q dt \approx \Delta F_Q \cdot \Delta t. \tag{18}$$

The amplification of $\delta(t)$ is the net outcome of such a quick increase in Δv. As a result, $\delta(t)$ becomes

$$\delta(t + \Delta t) \approx \delta(t) + \Delta v \cdot \Delta t. \tag{19}$$

From the oscillatory and spiking characteristic of F_Q, it is possible to identify multiple forms of repetitive impulsive phenomena that contribute to the exponential increase of $\delta(t)$, therefore causing $\ln\left(\frac{\delta(t)}{\delta(0)}\right)$ to rise and the Lyapunov exponent λ to increase. A positive value of λ indicates chaotic conditions, which are triggered by the impulsive forces stemming from abrupt change of quantum potential. The phenomena described explain why the Bohmian particle experiences violent and significant changes in movement or behaviour over time. Such dramatic shifts end up destabilizing the particles' movement, which amplifies any gap in the initial conditions set and contributes to chaotic motion.

However, contrary to our observation, most previous papers claim that all trajectories in rational frequency systems are periodic. In this situation, a deeper investigation is essential to find why we found chaotic motion in this type of system.

5. Mixture of wavefunction amplitudes and chaos

One of the factors that we think may affect the behaviour of the Bohmian trajectory is the mixture of wavefunction amplitude. To analyse this, we systematically examine this factor by increasing or decreasing the value of wavefunction amplitudes in terms of percentage followed by plotting the trajectory (table 1). According to the results, in general, when the amplitude of the ground state is much greater than those of the first excited states, the pattern of the Bohmian trajectory becomes similar to the classical trajectory. In other words, we found that the only mixture that produces the trajectory regular everywhere is when we consider the amplitude of the ground state is dominant (70%–100%) compared to the amplitude of the excited states (0%–30%). Here, the nodal points in the centre vanish and there is no abrupt change in quantum potential, implying the Bohmian trajectory for this system is regular. Although there is a situation where the particle crosses the nodal point multiple times as shown in figure 6, it still does not demonstrate any sign of chaotic behaviour. In consequence, the Lyapunov exponent vanishes very fast to zero. The reason for this phenomenon is that classical potential has a relatively high value at the area far from the centre and therefore quantum potential does not have a strong effect on the characteristic of a particle.

For the possibility of chaos, in contrast to the previous case, the amplitude of the first excited states needs to increase significantly greater than that of the ground state for Bohmian trajectory tend to become disordered and chaotic. Specifically, it is necessary to set the mixture of amplitude at least 40% from the ground state and 60% from the excited states. In this case, the nodal line pattern appears to 'shrink' toward the centre (figure 7) as the percentage of wavefunction amplitudes for excited state increases and the chaotic motion occurs when the nodal point reaches the area where the value of quantum potential is significantly greater than classical potential. It is

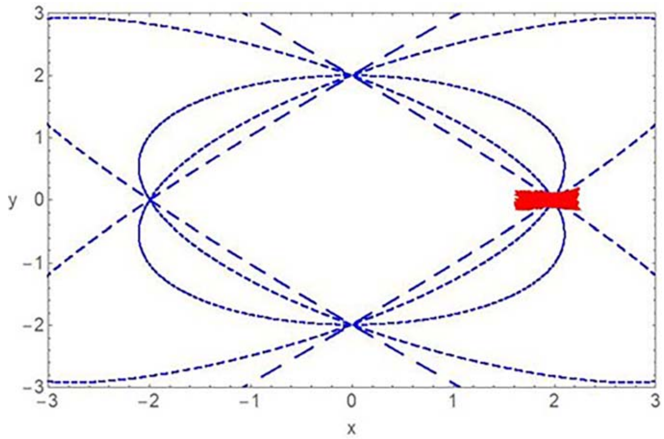

Figure 6. Bohmian trajectory for initial position $x_0 = 2.20, y_0 = 0.00$ and $a = 0.63, b = 0.63, c = 0.22, d = 0.22, f = 0.22, g = 0.22$.

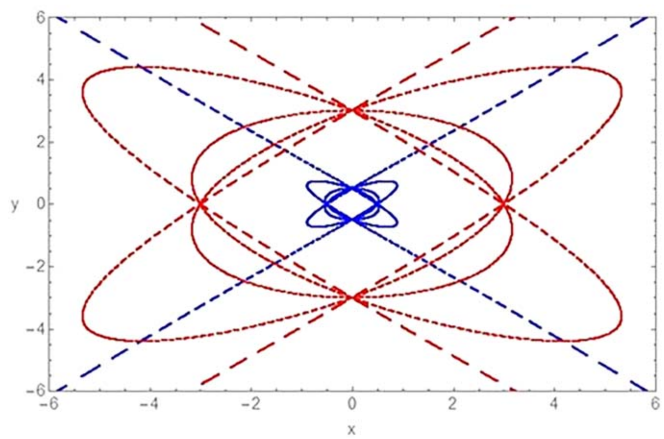

Figure 7. The pattern of red nodal lines appears to 'shrink' to blue nodal lines when the percentage of wavefunction amplitudes for the first exited states increase from 10% to 80%. Chaos occurs when nodal lines near the centre where the quantum potential is significantly higher than classical potential.

clear from these results that the proportion between the ground state and first excited states plays an important part in determining the location of the nodal point which then affects the total potential at a certain area. Besides that, mathematically these results may also be explained by examining the slope of Bohmian trajectory,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t}\frac{\Delta t}{\Delta x} = \frac{v_y(\mathbf{r},\ t)}{v_x(\mathbf{r},\ t)} = \frac{-\sqrt{2}\,\alpha^2\beta_y(t) - 2\alpha^2\gamma(t)x}{-\sqrt{2}\,\alpha^2\beta_x(t) - 2\alpha^2\gamma(t)y}. \tag{20}$$

Note that, besides using the Bohmian equation of motion, we can describe the pattern of the Bohmian trajectory according to this slope equation. This is because, using the slope equation, we can eliminate the term $G(\mathbf{r}, t)$, making it easier to compare with a classical equation of motion. Thus, in the case in which the amplitude of the ground state is too dominant in comparison to those of the first excited states, the terms $\beta_y(t)$ and $\beta_x(t)$ are much greater than the term $\gamma(t)$. Therefore, in this case, we can neglect the later terms and Bohmian's slope equation becomes approximately identical to the slope equation of classical trajectory,

$$\frac{\Delta y}{\Delta x} = \frac{a\omega_1 \sin(\omega_1 t)}{b\omega_2 \cos(\omega_2 t)} \tag{21}$$

where a and b are constants. However, when we gradually increase the amplitude of the first excited states and reduce the amplitude of the ground state, in contrast to the previous case, the term $\gamma(t)$ becomes much greater than the terms $\beta_y(t)$ and $\beta_x(t)$. Hence, the resultant Bohmian's slope equation is different from the classical one and the trajectories show disordered patterns and behave chaotically. Note that, since $\gamma(t)$ is a periodic function, we believe that the source of chaotic motion arises from the analytical solution of the Bohmian equation of motion $(x(t), y(t))$ near the nodal point, which deserves additional investigation in the future.

6. Conclusion

In this paper, we present numerically the condition where the rational frequencies system tends to exhibit chaotic motion. Specifically, we show that chaos in this type of system depends on the specific mixture of wavefunction amplitude, which aligns with the analytical result produced in [15]. However, our main contribution is to demonstrate that the form of quantum potential seems to play an important role in determining the behaviour of Bohmian trajectories not only in an irrational frequency system but also in a rational frequency system. This is the case when the quantum potential changes abruptly due to the presence of a moving nodal point near the centre, creating a significantly higher value relative to the classical potential. In this context, the velocity of a particle changes rapidly near this area and the trajectory diverges from periodic to chaotic motion.

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Data availability statement

The data cannot be made publicly available upon publication because the cost of preparing, depositing and hosting the data would be prohibitive within the terms of this research project. The data that support the findings of this study are available upon reasonable request from the authors.

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