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Exploring quasi-geometric frameworks for quantum error-correcting codes: a systematic review

Valentine Nyirahafashimana[1,2] · Nurisya Mohd Shah[1,3] · Umair Abdul Halim[1,4] · Mohamed Othman[1,5]

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Abstract

This study investigates quasi-geometric strategies for improving quantum error correction in quantum computing, utilizing geometric principles to improve error detection and correction while maintaining computational efficiency. A comprehensive review of 20 studies, selected from 2988 publications spanning 2019 to 2024, reveals significant progress in quasi-cyclic codes, quasi-orthogonal codes, and quasi-structured geometric codes, highlighting their growing importance in quantum error correction and information theory. The findings indicate that quasi-orthogonal codes that employ coefficient vector differential quasi-orthogonal space-time frequency coding demonstrated a 1.20 dB gain at a bit error rate of 10^{-4}, while reducing computational complexity. Quasi-structured geometric codes offered energy-efficient solutions, facilitating multi-state orthogonal signaling and reliable linear code construction. Furthermore, quasi-cyclic low-density parity-check codes with optimized information selection surpassed traditional forward error correction codes, achieving superior quantum error rates of 10^{-5} at 10.00 dB and 10^{-6} at 15.00 dB. Performance analysis showed that the effectiveness of error correction depends more on the frequency of six-length cycles than on girth, suggesting a new direction for optimization. The study emphasizes the transformative potential of quasi-geometric strategies in improving quantum communication by focusing on bit and quantum bit error rates within both stabilizer and classical frameworks. Future work focuses on integrating hybrid quantum-classical codes to raise error resilience and efficiency, addressing challenges like decoding instability, and limited orthogonality to enable reliable and computational quantum communication systems.

Keywords Quantum computing · Quantum error correction · Quasi-cyclic codes · Quasi-orthogonal codes · Weak condition

Valentine Nyirahafashimana, Nurisya Mohd Shah and Mohamed Othman contributed equally to this work.

Extended author information available on the last page of the article

1 Introduction

In quantum computing, error correction is a critical challenge due to quantum states fragile nature and susceptibility to decoherence and other errors [76]. Traditional approaches to quantum error correction (QEC) [22–25] involve encoding quantum information into larger Hilbert spaces, using specific error-detecting and error-correcting codes (ECCs) [7, 26, 27]. The geometric approach, particularly through the perspective of holonomy and geometric phases, provides an alternative strategy for enhancing QEC [77]. It is also reported that quasi-geometric strategies (QGS) utilize relaxed geometric conditions to improve the reliability and implementation of QEC schemes. QEC is crucial in developing strong quantum systems [28], with research funding for quantum technologies increasing in recognition of its importance. This is reflected in QEC codes (QECCs), which exist within a larger Hilbert space [23, 28, 30]. The ongoing pursuit of fully fault-tolerant quantum systems relies on effective error correction but remains a challenging goal. Mathematical techniques such as hash functions, symmetric zero-knowledge proofs, and ECCs have been proposed for quantum-safe cryptosystems [31]. Within this context, QGS emerge as key players in advanced QEC, particularly in quantum codes like quasi-cyclic codes (QCCs), quasi-structured geometric codes (QSGCs) and quasi-orthogonal codes (QOCs) [69], where quasi-geometric principles make better error correction by exploiting geometric structures.

QCCs, characterized by cyclic properties in their codewords, are linear block codes where each block is a cyclic shift of the preceding one [14, 32]. These codes provide structured and efficient solutions for QEC, facilitating hardware implementations and streamlined decoding algorithms. In the work [60], authors demonstrate that QCCs can effectively construct asymmetric QECCs, achieving parameters beyond the quantum Gilbert–Varshamov bound. They explore the algebraic structure of 2- and 3-generators QCCs, identifying nested codes to raise error correction. A systematic method for determining code parameters is provided, optimizing their performance and suggests future research on improving decoding efficiency, particularly under weak geometric conditions. In parallel, QOCs, primarily used in wireless communication systems, especially multiple-input multiple-output (MIMO) systems, aim to balance the benefits of orthogonality with practical considerations of complexity and performance [1, 2, 104]. These codes relax the requirement for exact orthogonality between logical states, providing reliable error correction by encoding logical qubits into approximately orthogonal subspaces, ensuring resilience against noise and imperfections in quantum systems. Lastly, QSGCs utilize geometric structures, including algebraic curves [4, 105], to refine error correction. Blending principles from algebraic geometry with structured approaches like cyclic or quasi-cyclic designs codes optimize error correction through geometric symmetries and transformations. The integration promotes practical and scalable QEC techniques, essential for developing advancing fault-tolerant quantum computing and information processing [34, 35]. Although these techniques face challenges such as the necessity for large key sizes and high computational overhead, QGS offer a promising path for advanced QEC. They use geometry and algebra to outline

error correction principles that emphasize creating more efficient and practical QEC methods, reducing the computational complexity and resource demands placed on current techniques. They also use geometric and topological principles to develop error protection [57]. After that, authors provide geometric codes like the surface code, which encodes qubits on a 2D grid [75], and topological codes such as toric and color codes, using global topological properties for error correction [68, 84]. Additionally, their constructions approximate these methods through quasi-2D and approximate topological codes, refining error detection, flexibility, and fault tolerance, but with high-resource demands [57, 67]. By combining structured grids and topological features, these techniques balance resilience with practical implementation constraints.

QGS codes, such as QCCs, QOCs, and QSGCs, provide innovative ways to link geometric and transformation properties with system behavior, elaborating on error resilience and redefining the role of normal subgroups in error correction [83]. The distinction between compact and non-compact groups is key in finite vs. infinite-dimensional Hilbert spaces [73]. In finite-dimensional spaces like qubit systems, behavior is discrete, while non-compact groups are used for continuous-variable systems, such as coherent and squeezed states [72]. Otherwise, compact groups like SU(2) offer a stabilizer framework for continuous-variable QEC [70, 71]. QCCs excel in discrete error correction [60], while QSGCs are suited for continuous-variable systems [80]. Stabilizer codes, using Pauli operators $(\sigma_x, \sigma_y, \sigma_z)$ from SU(2), preserve coherence through syndrome measurements [82]. Integrating QCCs into quantum protocols can complement stabilizer codes, improving both error correction and system performance [23]. Normal subgroups is a crucial in the quasi-geometric symmetries of QEC [57]. For example, SU(2) with its normal subgroup U(1) stabilizes rotations around the Z-axis on the Bloch sphere [74], and the group $G = \mathbb{Z}_n \times \mathbb{Z}_n$ with normal subgroup $N = \mathbb{Z}_n$ stabilizes shifts on a circular or toroidal surface. In quasi-cyclic error correction, a group C_n with normal subgroup C_k stabilizes cyclic shifts, reflecting the quasi-geometric nature of these codes. The relationship between QSGCs and group properties could offer new insights for increasing error stabilization and correction.

Authors in [100] designed quasi-orthogonal space-time block codes with full diversity and fast maximum-likelihood decoding, demonstrating how such constellations can be adapted to fault?tolerant quantum links. Lv [101] presented explicit constructions of asymmetric quantum error?correcting codes (AQECCs) derived from quasi?cyclic codes, yielding practical binary and ternary AQECCs. Likewise, [102] extended classical BCH, RS, and RM cyclic codes into the quantum setting via tailored generator polynomials, and [103] introduced asymmetric entanglement?assisted QECCs, combining entanglement with asymmetry to boost correction efficiency in quantum communication and computation. As noted in [7], developing high-quality quantum codes from classical codes is a key objective in quantum information science and quantum computing. Classical families, such as quasi-orthogonal, quasi-cyclic, and low-density parity-check (LDPC) codes, offer proven techniques for error detection and correction, which can be systematically adapted to the quantum setting. This transition involves tailoring classical methods to meet the unique constraints of quantum information, including coherence, entanglement, and no-cloning. The fun-

Table 1 Comparative advantages of QGS versus traditional QEC methods

Advantage	Traditional QEC	QGS
Qubit overhead	High (concatenated, surface)	Low (adaptive tessellation)
Connectivity constraints	Strict 2D grids	Irregular graphs matching hardware topology
Decoder complexity	Superlinear	Near-linear or polylogarithmic
Noise-bias adaptation	Limited	Intrinsic via geometric alignment
Modular scalability	Rigid	Highly modular and hierarchical

damental principles of error detection, syndrome measurement, and decoding remain central—now applied to quantum bits (qubits) rather than classical bits. In particular, classical insights into decoding thresholds and syndrome extraction continue to inform quantum protocols, such as stabilizer codes and syndrome-based decoding, thereby improving the error resilience of quantum systems. Quantum error correction [97, 98] is essential for fault-tolerant quantum computing [100], but traditional codes such as concatenated codes [91], lattice stabilizers [97], and topological schemes [96] often require many qubits, high decoding costs as error rates increase [94], strict 2D layouts, and large code distances [68, 89, 90, 92, 93].Quasi-geometric strategies address these limitations by embedding logical qubits in irregular layouts that adapt cell sizes to local error rates and hardware connectivity. This approach reduces qubit overhead, allows near-linear decoding, and improves thresholds under biased noise without complex code deformations. These strategies offer several advantages. First, they reduce the number of physical qubits needed to reach a given code distance by tailoring cell sizes to local error conditions. Their irregular tessellations align naturally with hardware architectures such as ion traps or superconducting qubits, minimizing swap operations and simplifying routing. Decoding scales near-linearly or poly-logarithmically with system size, unlike the super-linear decoding growth of large surface codes. By matching geometry to known noise biases (e.g., stronger dephasing), quasi-geometric codes achieve higher error thresholds than isotropic codes. Finally, their modular design enables flexible combinations of optimized blocks, supporting incremental scaling from prototypes to large quantum processors. Table 1 shows that QGS cut qubit counts and ease hardware layout by using flexible, irregular tessellations. Thus, QGS removes major QEC bottlenecks and enable hardware-aware, efficient code design. This review will chart current QGS methods, compare their trade-offs, and point to promising research directions in fault-tolerant quantum computing.

In this study, we evaluate the contributions of QGS to QEC by analyzing their strengths, limitations, and methodologies, while identifying directions for future research. Classical error correction techniques form the foundational basis for quantum methods, helping to address challenges unique to quantum coherence and superposition. Our aim is to explore the role of weak geometric conditions critical to QGS, and

examine how QOCs, QCCs, and QSGCs can be integrated with geometric strategies particularly in qubit mapping. To support this, we review key findings, challenges, and proposed solutions from the literature, evaluate performance metrics indicators such as quantum error rate(QER) and bit error rate (BER), error correction and efficiency performance, and apply systematic data analysis methods to assess the effectiveness of QGS-based QEC. The paper is structured as follows: Section 2 outlines the systematic review methodology, including research questions, database searches (Scopus, ScienceDirect, PubMed Quantum), and inclusion/exclusion criteria for the 2019–2024 period. Section 3 examines weak geometric conditions in QGS for QEC and compares the sensitivity of the three QGS codes to these conditions. Section 4 analyzes 20 selected studies, highlighting design challenges, solutions, and performance comparisons between QGS and traditional QEC codes. Section 5 concludes with a summary of findings and their implications for future QEC development.

2 Methods and material

The methods and materials for this systematic review were derived by examining and evaluating previously published works, as discussed in the state of the art. The selected articles were analyzed based on the techniques employed and their achieved performance. This section explains the publication standards followed in this study, focusing on key aspects such as formulating research questions, using a systematic search strategy across databases like Scopus, ScienceDirect, and PubMed Quantum, and applying methods for quality appraisal, data extraction, and analysis. Additionally, the study protocol is documented in the Open Science Framework at [https://osf.io/af5jw], last access 29th June 2025.

2.1 Research question formulation

The comprehensive review is guided by research questions developed during the preliminary phase. These questions align with the research objective of presenting a conceptual framework for QGS to advanced QEC. The formulated research questions are: (1) What are the key QGS contributing to the advancement of QEC techniques? and (2) What are the effectiveness and limitations of QGS in advancing QEC. The purpose of this review is to develop QGS, such as quasi-cyclic, quasi-structured geometric, and quasi-orthogonal codes, to advance QEC and enhance the reliability of quantum communication systems. Additionally, this work aims to assess the effectiveness and limitations of these strategies in improving QEC techniques and their practical applications, while proposing methods to overcome these limitations.

2.2 Data source and search strategy

The study primarily sourced articles from three main multidisciplinary search engines Scopus, ScienceDirect, and the PubMed quantum database covering publications from 2019 to 2024. A comprehensive search encompassing title, abstract, and keyword fields

Table 2 Systematic review search engine, string, and acquisition date

Database	Search string and keywords	No of articles	Date of acquisition
Scopus, PubMed quantum, and science direct	main search terms using document title, abstract, and keywords: *"geometric codes"* **OR** *"quasi-cyclic codes"* **OR** *"quasi-geometric strategies"* **OR** *"quasi-orthogonal code"* **AND** *"quantum computing"* **AND** *"quantum error correction"* **AND** *"weak condition"*	130,309 and 2549	12-13/04/2024

was conducted across all databases. The search strings, tailored to retrieve relevant articles, were executed in April 2024. This systematic search yielded a total of 2988 potentially related articles, which were subsequently downloaded for further analysis see Table 2.

2.3 Data extraction and analysis

The studies analyzed in this research were sourced from Scopus, ScienceDirect, and the PubMed quantum database. Articles published between 2019 and 2024 were identified using targeted search keywords, as illustrated in Fig. 1 and Table 2. This comprehensive search yielded 2,988 studies, which were screened through specific inclusion and exclusion criteria. Titles were initially assessed to focus on QGS codes, including QOCs, QCCs, and QSGCs, followed by full-text screening. To ensure reliability, non-peer-reviewed articles, duplicates, pre-2019 publications, studies with unclear methodologies, non-English texts, and inaccessible articles were excluded. During the initial screening, 103 duplicate articles were removed, and 2,859 additional studies were discarded after title and abstract reviews for irrelevance. Further examination excluded one study due to unclear methodology, another for lack of access, and four others for not evaluating the research theme. Ultimately, 20 articles met the inclusion criteria, as summarized in Fig. 1. The selected studies underwent quality appraisal by a field expert following guidelines from [21], which ranked articles as high, moderate, or low quality. Only high and moderate-quality articles were included after evaluating their themes, objectives, and results. The expert confirmed the inclusion of all 20 articles in the review. Subsequent analysis involved extracting relevant data from abstracts, discussions, conclusions, and main content, which was systematically tabulated in Microsoft Excel. These 20 articles were grouped based on publication year: 2 articles in 2019, 3 in 2020, 4 in 2021, 6 in 2022, 4 in 2023, and 1 in 2024 (Fig. 3). The articles were categorized into thematic groups: 3 focused on QOCs, 3 on QSGCs, 11 on QCCs, and 3 addressed mix of QSGCs and QCCs with subcode variations (Fig. 2). Notably, 7 of these 20 articles focused on broader theoretical frameworks, algorithmic approaches, or general applications in quantum computing and information science, rather than specific advancements in QEC or QGS. Key contributions, methodolo-

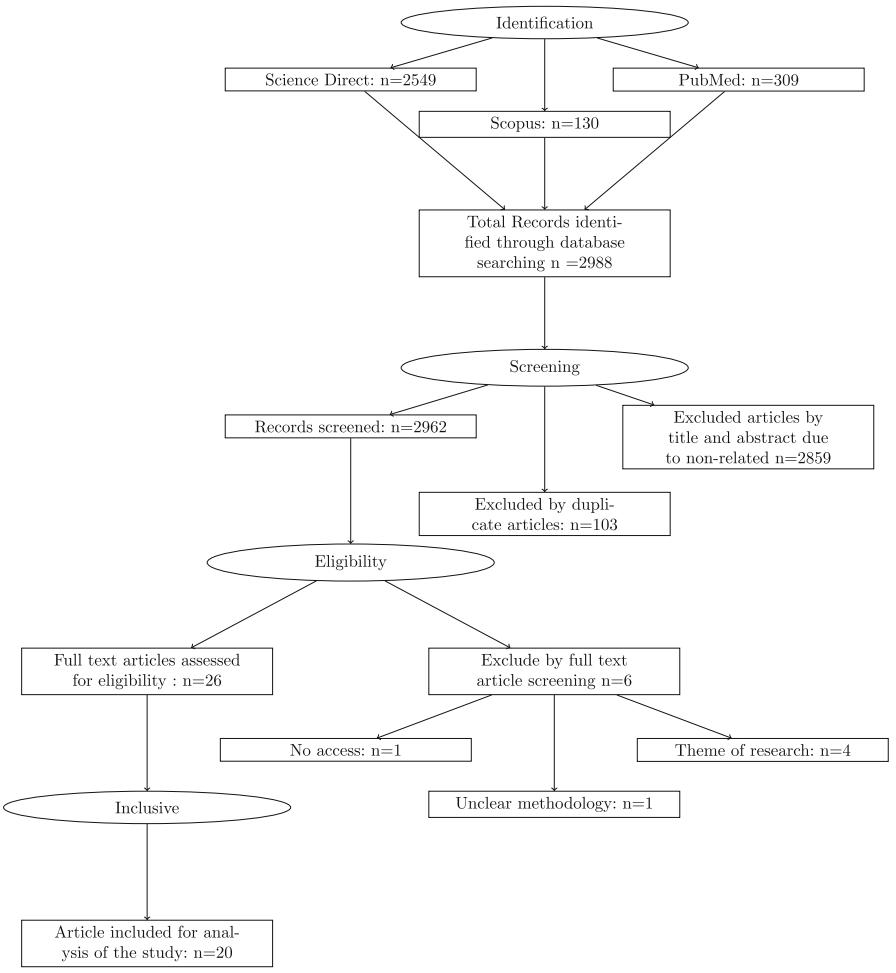

Fig. 1 Literature selection involved searching sciencedirect, PubMed quantum, and scopus with keywords, along with a manual search for studies in five-year frame time from 2019 to 2024. After removing duplicates and conducting a review, 20 studies were selected based on full-text analysis

gies, stabilizer formalism, subcodes, limitations and the strengths of the codes were extracted alongside additional innovations and findings in the field.

3 Background and analysis of weak geometric conditions in quasi-geometric strategies for QEC

This section focuses on the essential weak geometric conditions that form the foundation of QGS in QEC. We examines how QOCs, QCCs, and QSGCs can be combined with geometric strategies, especially for mapping qubits. These methods help correct

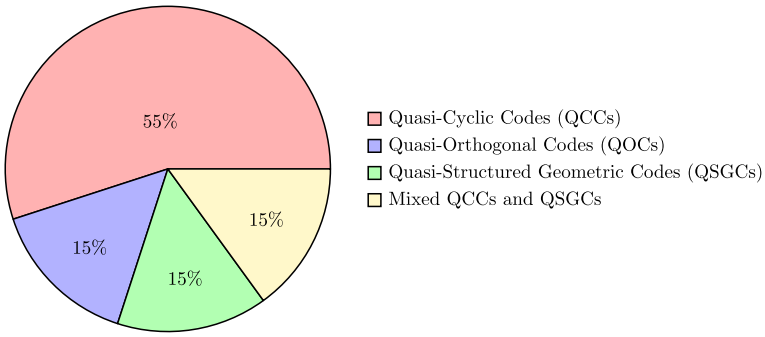

Fig. 2 Percentage distribution of categorized articles

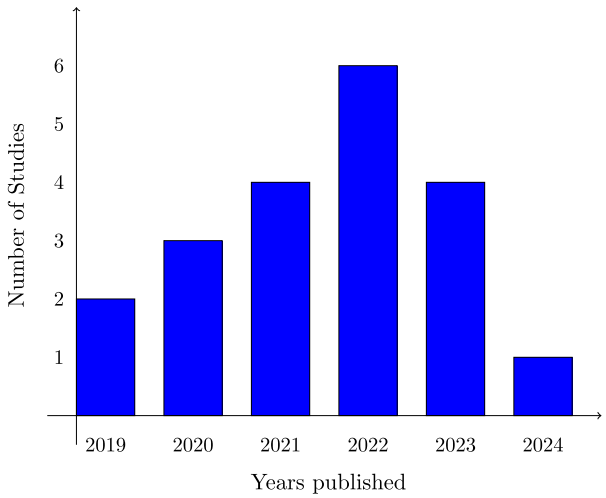

Fig. 3 Distribution of quasi-geometric strategy codes over the period 2019-2024

errors using geometric techniques, improving the reliability of quantum communication systems.

The geometric approach, based on differential geometry and topology, offers a mathematical framework for understanding the evolution and manipulation of quantum states [36, 40]. Authors highlight the importance of geometric phases and holonomies in designing fault-tolerant quantum gates and algorithms [37, 38]. In quantum systems, the state is represented as a vector in a complex Hilbert space, typically $\mathcal{H} = \mathbb{C}^{2^n}$ for an n-qubit system [78, 79]. Quantum gates, unitary operators acting on the Hilbert space, correspond to rotations and transformations that preserve quantum information [26, 39]. During adiabatic evolution, a quantum system acquires a geometric phase, known as the Berry phase [41], which is associated with the curvature of the parameter space. For cyclic evolution, the Berry phase γ is given by:

$$\gamma = \oint_C \langle \psi(t) | \nabla | \psi(t) \rangle \cdot d\mathbf{R}, \tag{1}$$

where $\mathcal{C}$ is a closed path in the parameter space $\mathbf{R}$, and the integral reflects the purely geometric nature of the phase, dependent only on the path taken, not the speed of traversal [26].

In QEC, the weak condition within the geometric approach offers a more flexible error detection and correction process, allowing for greater error tolerance while still enabling the recovery of quantum information [42, 43]. This flexibility build up ECCs but may require additional resources or more complex decoding algorithms [42, 45]. The author [46] describes this as a flexible criterion that facilitates effective error correction without adhering strictly to traditional QEC standards. Techniques like Approximate QEC, relaxed fidelity requirements, and subsystem codes protect critical quantum information, even under noise or imperfect syndrome measurements [44, 47, 48]. Moreover, QGS utilize geometric principles to improve QEC, making it more practical for real-world applications [50]. This approach also extends to the geometric properties of the quantum state space, particularly its curvature [51, 52]. By easing constraints on curvature, the approach balances the need for stability in quantum gate operations with practical implementation needs, often controlling the curvature of the quantum state space. This is especially relevant in holonomic quantum computation (HQC) [53]. The curvature tensor F, which ensures stability in geometric phases, is defined as:

$$F = dA + A \wedge A, \tag{2}$$

where d is the exterior derivative and $\wedge$ is the wedge product. By permitting controlled, nonzero curvature, this approach simplifies quantum gate construction while maintaining computational accuracy [49]. For instance, a relaxed constraint on curvature, expressed as:

$$||F|| \leq \epsilon, \tag{3}$$

ensures that any deviations in the geometric phase remain within acceptable limits. This flexible geometric method supports key quantum computation strategies, such as using geometric phases for noise-resistant operations, quasi-geometric error correction, adiabatic quantum processes to minimize errors during system evolution, and holonomic quantum operations for resilient quantum gate implementation. Additionally, stabilizer codes and topological protection hold geometric principles to safeguard quantum information from local disturbances [65, 66].

In this field, the geometry-based framework facilitates the evolution of quantum states under a Hamiltonian $H(t)$, producing a unitary transformation $U(t)$ with a total phase ϕ, which includes both the dynamical phase ϕ_d and the geometric (Berry) phase γ [54], defined as

$$\phi = \phi_d + \gamma. \tag{4}$$

This approach enhances the accessibility and reliability of geometric methods, allowing practical applications in fault-tolerant quantum gates. QEC has a crucial role

in reducing decoherence and error effects, with traditional codes like Shor and surface codes protecting information by encoding it in redundant spaces [55]. However, integrating geometric phases into ECCs through QGS strengthens error correction, creating fault-tolerant logical qubits that resist physical qubit errors [56]. The use of bounded curvature in geometric phases further enhances stability and error protection. Additionally, quasi-orthogonal, quasi-cyclic, and quasi-structured geometric codes, when combined with geometric principles, significantly elaborate on QEC performance. For example, QOCs feature nearly orthogonal codewords, which simplify differentiation and raise error resilience [57, 58]. This is mathematically represented as

$$\langle \mathbf{c}_i, \mathbf{c}_j \rangle \approx 0 \quad \text{for} \quad i \neq j, \tag{5}$$

minimizing overlap between codewords and improving error correction efficiency. By combining these codes with geometric phases, they inherit the fault tolerance characteristic of the geometric approach, leading to more stable quantum information systems [59]. In parallel, QCCs support their cyclic structure to facilitate efficient encoding and decoding processes [63]. A QCCs generator matrix G describes how each row is a cyclic shift of the previous one, and the codeword $\mathbf{c}$ is represented as $\{c_0, c_1, \ldots, c_{n-1}\}$, with cyclic shifts defined by

$$\mathbf{c} = \{c_{(i+k) \mod n} \mid k \in \{0, 1, \ldots, n-1\}\}. \tag{6}$$

The integration of the quasi-geometric framework with these codes reinforce their ability to implement more efficient error correction protocols, especially in practical framework, where error tolerance is key [60]. In a similar, QSGCs combine the principles of geometric quantum computation with structured coding strategies to maintain stability and reliability against errors [64]. These codes use geometric phases and holonomies to protect quantum systems from perturbations. The geometric phase accumulated along a path in parameter space is given by

$$\phi = i \int_{\gamma} \langle \psi(t) | \frac{d}{dt} | \psi(t) \rangle \, dt, \tag{7}$$

where $\gamma(t)$ represents the path [61]. By ensuring that the curvature K of the space is constrained by $K \leq \epsilon$, where ϵ is a small positive constant integer, these codes enable practical implementations while maintaining error resistance. This strategy intensify QEC effectiveness, improving error resilience in quantum systems. By allowing controlled rather than zero curvature, it simplifies implementation, unlike holonomic quantum computation, which requires exact zero curvature [62]. This flexibility eases the development of quantum gates and error-correcting circuits, exploiting intrinsic geometric properties of quantum state space to boost powerfully built, and making quantum systems more practical for real-world applications [36]. Table 3 compares three QGS codes based on structure, key features, and sensitivity to weak geometric conditions. QCCs, with their cyclic structure, offer efficient decoding but are mildly sensitive to geometric irregularities. Whereas, QOCs provide high error resistance

Table 3 Comparison of quasi-geometric strategy codes based on weak geometric conditions

Code type	Structure	Key features	Weak geometric condition
QCCs	Cyclic structure with partial repetition	Efficient for decoding, simple design	Mildly sensitive to geometric irregularities
QOCs	Orthogonal bases with quasi properties	High error resistance, strong design	Moderately sensitive to weak geometric constraints
QSGCs	Structured with flexible geometric design	Versatile and adaptable	Least sensitive due to flexible geometry

and robustness through orthogonal bases but show moderate sensitivity to weak geometric constraints. Then, QSGCs are the most versatile, with adaptable designs and minimal sensitivity, making them ideal for applications involving geometric imperfections. Figure 4 illustrates the sensitivity levels of the three QGS codes to weak geometric conditions, rated on a scale from 1 to 3, with lower values indicating less sensitivity. QCCs, with a sensitivity level of 2, can handle minor geometric irregularities but require careful design. QOCs, rated 3, offer robust error resistance but are more sensitive to precise geometric conditions. In contrast, QSGCs, rated 1, are the least sensitive, showcasing flexibility and resilience to weak geometric constraints. In practical terms, QSGCs are suitable for systems with uncontrolled geometric constraints, such as quantum communication networks with hardware imperfections. However, the hardiness of QOCs comes at the expense of increased sensitivity, making them less adaptable in variable conditions. A hybrid approach supporting the strengths of these codes can boost reliability and performance in QEC applications.

4 Critical and in-depth analysis of key milestones and influential studies

This section reviews key findings on QGS, focusing on challenges, overcome, and recent breakthroughs. It highlights the strengths and weaknesses of QCCs, QOCs, and QSGCs, their connection to stabilizer groups, and their importance in QEC. It also examines weak geometric conditions, QER, and BER to evaluate code efficiency and performance in quantum communication. Table 4 provides a summary of 20 studies on QGS codes, classified into QOCs, QCCs, and QSGCs.

4.1 Effect of quasi-orthogonal codes

Table 4 outlines three studies [1–3] examining the effects of QOCs on quasi-geometric approaches in advanced QEC. One study [1] showed that the differential distributed quasi-orthogonal space-time-frequency coding (DQOSTFC) scheme add to quasi-geometric methods in QEC, outperforming traditional designs in cooperative

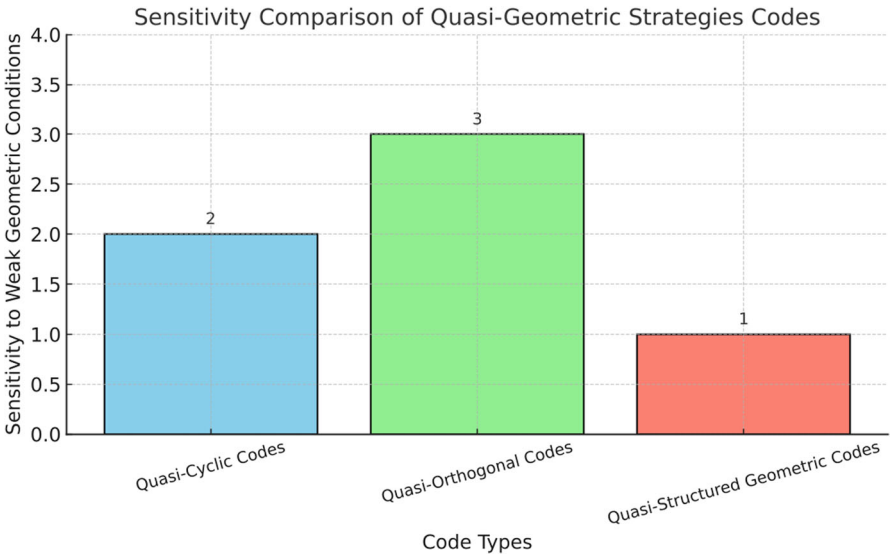

Fig. 4 Sensitivity comparison of quasi-geometric strategy codes. Quasi-orthogonal codes are more sensitive to weak geometric conditions but provide stronger error resistance compared to other methods

networks. It achieves a 1.20 dB gain at a BER of 10^{-4}, excelling over other coding schemes like QODSTC-orthogonal frequency-division multiplexing (OFDM) and distributed space–frequency coding (DSFC), especially in high Signal-to-Noise Ratio (SNR) and challenging multipath fading conditions. By loosening strict orthogonality requirements, this approach maintains strong performance even when channel conditions are less stable, assuming quasi-static channel coefficients over certain time slots and frequency subcarriers. A key measure, the diversity product ζ_c, assesses code quality, where positive values indicate maximum diversity:

$$\zeta_c = \frac{1}{2} \min \left\{ \left\| \left(\tilde{V}_{k+1} - V_{k+1} \right) \right\|_1 \right\}, \tag{8}$$

with $\tilde{V}_{k+1}$ and V_{k+1} as matrices related to the scheme. To achieve quasi-orthogonality (QO), coefficient vector rotation (CVR) is applied:

$$\Phi = D \cdot \text{diag} \left[1, e^{j\theta_1}, \ldots, e^{j\theta_{(\Gamma/2)-1}} \right], \tag{9}$$

where Φ serves as a CVR to ensure QO, with D as a Hadamard matrix and Γ setting the length of the coefficient vector. The diagonal matrix $\text{diag}\left[1, e^{j\theta_1}, \ldots, e^{j\theta_{(\Gamma/2)-1}}\right]$ includes phase shifts up to $e^{j\theta_{(\Gamma/2)-1}}$, maintaining balanced rotation for QO, where θ_i are phase angles. Future research aims to explore multi-user networks, focusing on energy efficiency, scalability, and security, with applications in massive MIMO systems. Another study [2] introduced quaternion orthogonal designs (QODs) for space-time block codes (STBCs) in dual-polarized antenna systems, employing quater-

Table 4 Quasi-geometric strategies and codes used in previous studies, with all acronyms defined in the table footnotes

Types of Codes	Cite	Methods	Subcodes	SF	Challenges	Overcome
QOCs	[1]	Co-efficient vector differential DQSTFC scheme	DQSTFC	✗	Complex construction, decoding, sensitivity to channel conditions, adaptability, high overhead	Optimize design, improve power, increase adaptability, enhance interference management
	[2]	QODs for STBCs in Dual-Polarized MIMO Systems	STBCs	✗	Complex implementation, balancing performance, computational demands	Simplify implementation, innovate architectures, continuously evaluate
	[3]	MTACs and Constacyclic Codes	Self-Dual MTACs, SOMTACs	✓	Complex analysis, limited versatility, lack of practical feasibility	Simplify analysis, assess feasibility, compare with existing technologies
QSGCs	[4]	Construction and decoding algorithms of AGCs	AGCs, MDSCs	✗	Encoding/decoding complexity, resource intensity, real-time performance	Optimized algorithms, hardware acceleration, modular design, real-time optimizations
	[5]	Probabilistic methods and geometric group results	MTASOCs, ACDCs	✓	Complex construction, limited practical implementation	Develop simpler algorithms, enhance error correction, and use advanced computational tools
	[6]	Exploring linear codes with 1D Euclidean hulls from function fields	EAQECCs, GGCs, and AGCs	✓	Theoretical complexity, limited practical applicability, implementation challenges	Simplify models, increase applicability, and develop user-friendly tools
QCCs	[7]	Decomposition of the defining set of Constacyclic Codes for EAQEC	EAQMDSCs and Constacyclic Codes	✓	Limited code diversity, complexity in decomposition, scalability issues	Develop hybrid codes, optimize decomposition algorithms, improve scalability through layered code generation
	[8]	GPU-based BP decoder for QC-MET-LDPC codes in CV-QKD	QC-MET-LDPC and MET-LDPC Codes	✗	Memory shortages, suboptimal GPU thread utilization, scalability limitations	Optimize memory management, improve thread scheduling, enhance QC-MET-LDPC performance

Table 4 continued

Types of Codes	Cite	Methods	Subcodes	SF	Challenges	Overcome
	[9]	Polynomial inversion in QC-MDPC codes for BIKE KEM	QC-MDPC Codes	✗	Key generation complexity, computational overhead, security considerations	Use pre-computation techniques, hardware accelerators, and adaptive parameter tuning
	[10]	Embedding cycles in LDPC codes for better performance	QC-BLDPCCs	✗	Managing short cycles, input quality dependency, enhancing girth	Develop cycle-avoidance algorithms, improve preprocessing, optimize code design for better girth
	[11]	High-parallel, low-latency encoding for 5 G QC-LDPC codes	5 G QC-LDPCCs	✗	High complexity and resource requirements for QC-LDPC coding in 5 G	Simplify algorithms, use custom hardware, optimize resource management with dynamic allocation
	[12]	Constructs MDPC codes using projective bundles and desarguesian projective planes with bit flipping	MDPC Codes	✗	Complexity and limited generalizability in communication systems	Optimizing algorithms, modular designs, adaptable architectures, hardware and software tailoring
	[13]	Joint iterative algorithm for 5 G QC-LDPC-encoded full-duplex transmissions with MSE and BER metrics	5 G QC-LDPCCs	✗	Limited practical application and generalizability in real-world communication	Extensive literature review, novel design techniques, real-world testing, industry-academic partnerships
	[14]	Algorithm to determine minimal CPM size for GC-QC-LDPCCs design, focusing on 8-girth cycle analysis	GC-QC-LDPCCs	✗	Complexity in construction, resource intensity, decoding complexity, sensitivity to channel conditions	Streamline algorithms, optimize resources, adaptive algorithms, channel estimation, flexible architectures
	[15]	Development of additive codes using QCCs for QEC, focusing on larger distances for error correction	QACs, ACDCs	✓	Complexity in construction using QCCs and combinatorial methods	Benchmarking, further research, collaboration, improving comparison with linear codes

Table 4 continued

Types of Codes	Cite	Methods	Subcodes	SF	Challenges	Overcome
	[16]	QEC coding with block jacket and circulant permutation matrices for efficient error correction	QQCCs	✓	Limited exploration of long-length QQCCs for quantum information processing	Advanced simulations, interdisciplinary collaboration, secure funding for research on long-length QQCCs
	[17]	Algorithm for optimal bit pattern selection with QC-LDPCCs for better wireless communication	DQC-LDPCCs	✗	Complexity in LBLCs and sensitivity to channel conditions	Efficient algorithms, adaptive designs, benchmarking, error correction optimization, simulations
QCCs and QSGCs	[18]	Construction of SC-LDPC and QC-LDPC codes for CVQKD systems using ATSC LDPC codes	SC-LDPCCs, QC-LDPCCs and AGCs	✗	High implementation complexity of SC-LDPCCs, limited flexibility for QC-LDPCCs	Optimizing algorithms, using specialized hardware such as FPGA and ASIC, modular design, and adaptive algorithms
	[19]	EXIT chart analysis optimizing QC-LDPCCs degree distribution, improving system efficiency	QC-LDPCCs and FECCs	✗	QC-LDPCC complexity for optimal performance, FECCs overhead affecting system complexity	Advanced optimization algorithms, efficient code design, simulations, adaptive techniques for real-time metrics
	[20]	Energy consumption model for QC-LDPCCs in optical systems, with a decoding circuit and energy evaluation	QC-LDPCCs and FECCs	✓	Decoding complexity, error propagation, hardware implementation, performance-complexity trade-offs	Advanced algorithms, error mitigation, hardware optimization, balanced code structures, and adaptive methods

ACDCs, additive complementary dual codes; AGCs, algebraic geometry codes; ASICs, application-specific integrated circuits; BER, bit error rate; CPMs, circulant permutation matrices; CVQKD, continuous-variable quantum key distribution; DQC-LDPCCs, distributed quasi-cyclic low-density parity-check codes; EAQECCs, entanglement-assisted quantum error-correcting codes; FECCs, forward error correction codes; FPGAs, field-programmable gate arrays; G-QC-LDPCCs, generalized QC-LDPC codes; GGCs, geometric goppa codes; MDSCs, maximum distance separable codes; MTACs, multi-twisted additive codes; MTASOCs, multi-twisted additive self-orthogonal codes; QC-BLDPCCs, QC-based LDPCCs; QC-LDPCCs, quasi-cyclic LDPCCs; QQCCs, quantum quasi-cyclic codes; SC-LDPCCs, Spatially Coupled LDPCCs; SF, Stabilizer Formalism

nion algebra to apply space, time, and polarization diversities. This design reduces computational complexity and achieves superior error performance by implementing a linear, decoupled decoder structure. The study confirms the built up diversity gains and strongest of these designs, demonstrating significant improvements in BER for quaternionic versus complex channel models such as (2×1) and (2×2), respectively. For instance, the QOCs exhibit a 13.00 dB gain at a BER of 10^{-5} with dual-polarized antennas. This approach holds promise for future research on higher-dimensional QODs and practical applications in MIMO systems. In study [3], authors expanded on multi-twisted additive Codes (MTACs) over finite fields, extending constacyclic codes to enable versatile algebraic structures. The paper develops criteria for self-dual and self-orthogonal MTACs based on trace bilinear forms, contributing to systematic classification and analysis of these codes.

It introduces δ-self-dual and δ-self-orthogonal MTACs, quantified as:

$$M = e_1 \prod_{v=1}^{\mu_v} e_2 \prod_{w=e_1+1}^{\eta_w} \left(\sum_{k_1=0}^{\eta_w} \binom{\eta_w}{k_1} q^{t\beta} \sum_{k_2=0}^{\beta} \binom{\beta}{k_2} q^t \right), \tag{10}$$

where e_1 and e_2 are indices for code constituents, with μ_v denoting the number of self-reciprocal components and η_w the dimension of components from reciprocal pairs. The parameter β is the dimension of an isotropic subspace, q is the field size, and t is the extension degree. The outer products iterate over the decomposition described in Theorem 3.2 in [3]. The double summation counts totally isotropic subspaces: $\binom{\eta_w}{k_1} q^{t\beta}$ and $\binom{\beta}{k_2}$ give the number of dimension-k_1, k_2 subspaces, and the inner sum accounts for orthogonal extensions respectively. This structure captures the algebraic and combinatorial elements of the code enumerating self-orthogonal MTACs over finite fields.

Equation (10) calculates the total number M of distinct δ-self-orthogonal MTACs of length n over F_{q^t}, where F_{q^t} represents an extension field based on a prime power q raised to t, $\beta = (\eta_w - k_1)$, and with parameters such as e_1, μ_v, e_2, and $t = d_w$ reflecting code length and orthogonality. This approach advances coding theory by creating efficient MTACs and exploring their applications. The study highlights that QER performance depends on code structure, decoding algorithms, and channel conditions, with self-dual codes providing supplemented error correction. These studies emphasized the adaptability and effectiveness of QGS in QEC, particularly through QO, quaternionic designs, and multi-twisted structures in quantum communication systems.

4.2 Effect of quasi-structured geometric codes

Three recent studies [4–6] summarized in Table 4 explore the effectiveness of QSGCs in advanced QEC. The energy efficiency of algebraic geometry codes (AGCs) for noise-resistant systems is emphasized, where discrete messages are transmitted using M-ary orthogonal signals, with $M > 0$ [4]. These codes, which exhibit comparable computational complexity to Reed-Solomon codes, highlight benefits in telecommuni-

cations but also emphasize performance limitations in QSGCs when certain parameters $[[n, k, d]]$ are suboptimal. Performance constraints are governed by key relationships, including the genus-degree relationship, where the genus g and degree d of a curve follow the inequality:

$$g \leq \frac{(d-1)(d-2)}{2}. \tag{11}$$

This relationship restricts the number of points on a curve, impacting code performance. Code length n, dimension k, and minimum distance d are also governed by the structure of the curve, where n is the number of points on the curve, $k = (n - g)$, and $d \geq (n - k + 1)$, reflecting the singleton bound for maximum distance separable (MDS) codes. Additionally, energy efficiency is estimated based on the SNR and coding gain, with AGCs demonstrating a gain of 6.00 dB compared to uncoded transmission and a 0.80 dB improvement over MDS codes. At comparable SNRs, AGCs achieve BER between 10^{-5} and 10^{-6}, outperforming MDS codes, which exhibit BERs from 10^{-4} to 10^{-5} under similar conditions. Future research should focus on integrating AGCs practically to boost noise immunity and code-based security in telecommunication networks. Besides, a previous study [5] investigated multi-twisted additive self-orthogonal codes (MTASOCs) and additive complementary dual codes (ACDCs) over finite fields, focusing on their asymptotic goodness and potential in practical applications. This study generalized 1-generator multi-twisted additive codes (MTACs), which extend constacyclic codes, and uses probabilistic methods for analyzing their properties. To satisfy desired asymptotic properties, specific rate and distance conditions must be met. The code rate R is defined as:

$$R = \frac{\dim_{F_{q^t}}(C)}{m}, \tag{12}$$

where $\dim_{F_{q^t}}(C)$ is the code's dimension over the field F_{q^t}, and m represents the code length. The distance condition relates the minimum distance d to the relative Hamming distance Δ through:

$$\Delta = \frac{wt(C)}{m},$$

here $wt(C)$ denotes the Hamming weight of the code. Otherwise, to achieve asymptotic goodness, the condition $h_{q^t}(\delta) < (1 - R)$ must be holden, with $h_{q^t}(\delta)$ as the q-ary entropy function at the relative distance δ. MTACs generally exhibit stronger error correction capabilities than ACDCs, with QER for MTACs approximated by:

$$\mathrm{QER}_{\mathrm{MTACs}} \approx Q\left(\sqrt{\frac{\mathrm{d}}{N_0}}\right), \tag{13}$$

where d is the minimum distance, and N_0 is the noise power spectral density and $Q = \langle q \rangle$, the cyclic subgroup of generated by q. For instance, if $d = 5$ and $N_0 = 1$,

the QER for MTACs is approximately 0.02. Other hand, MTACs efficiency, defined as:

$$R_{\text{MTACs}} = \frac{\dim_{F_{q^t}}(C)}{n}, \tag{14}$$

can reach 0.90 for well-structured codes. Otherwise, ACDCs also offering good error correction, have slightly higher QERs, approximated by:

$$\text{QER}_{\text{ACDCs}} \approx Q\left(\sqrt{\frac{d_{\text{ACD}}}{N_0}}\right), \tag{15}$$

where, for $d_{\text{ACDCs}} = 4$ and $N_0 = 1$, the QER for ACDCs is approximately 0.16 while efficiency for ACDCs, defined as:

$$R_{\text{ACDCs}} = \frac{\dim_{F_{q^t}}(C_{\text{ACD}})}{n}, \tag{16}$$

is typically around 0.85 due to increased redundancy. MTACs, with lower QER and higher efficiency, prove more suitable for applications demanding high data rates and minimal errors. Future work should explore high-dimensional extensions, error correction across diverse environments, and code optimization in quantum communication. Authors in [6] investigated linear codes with one-dimensional Euclidean hulls, broadening the applications of algebraic geometric codes (AGCs) to entanglement-assisted quantum error-correcting codes (EAQECCs). EAQECCs, which utilize shared entanglement between sender and receiver, are described by the singleton bound:

$$d \leq (n - k + 1 + c), \tag{17}$$

where n is code length, k is dimension, d is minimum distance, and c is the number of pre-shared entangled pairs. In AGCs, ensuring the divisor D satisfies specific conditions is essential, with dimension k expressed as:

$$k = (g + 1) - \frac{\deg(D)}{m}, \tag{18}$$

here g represents the genus of the curve and m the degree of the divisor. EAQECCs achieve QERs of 10^{-5} to 10^{-7} and efficiencies from 0.50 to 0.80. Comparatively, AGCs and GGCs offer QERs from 10^{-4} to 10^{-6} with efficiencies between 0.60 and 0.90, influenced by code structure. Moreover, EAQECCs show the lowest QER, making them ideal for high-reliability applications, while AGCs and Goppa codes offer a balance between error correction and efficiency. These studies highlighted key advancements in QSGCs, such as AGCs for energy efficiency, constacyclic and additive codes for asymptotic goodness, and EAQECCs for improving QEC in quantum transmission. Future research should focus on higher-dimensional hulls and progressive coding strategies to optimize QEC in quantum applications.

4.3 Effect of quasi-cyclic codes in advanced QEC

Figure 2 and Table 4 emphasize that research on QCCs dominates over QOCs and QSGCs, representing 55% of QCCs related studies from 2019 to 2024. Specifically, 11 out of 20 studies focus on Quantum QGS using QCCs protocols. Notably, two studies from 2019 to 2020 [7, 8] improved QCCs performance. Following this, three studies [9–11] in 2021, three more [12–14] in 2022, and three recent studies in 2023 [15–17] explored QCCs advancements in quantum computing, highlighting their role in strengthening QEC strategies.

The foundational studies [7, 8] examined QCCs contribution to QEC improvements. In 2019, the authors of one of the study [7] introduced eight classes of entanglement-assisted quantum maximum distance separable (EAQMDS) codes via classical constacyclic codes, expanding the entanglement-assisted quantum error-correcting (EAQEC) framework. By decomposing the defining set of constacyclic codes, the study demonstrated that many QECCs, including maximum distance separable (MDS) codes, can be derived from the stabilizer formalism. For codes with parameters $[n, k, d]_q$, the singleton bound in (17) applies, where equality denotes an MDS code. In EAQEC codes with parameters codewords, the entanglement-assisted singleton bound becomes $(n + c - k) \geq 2(d - 1)$; if equality holds for $d \leq (\frac{n+2}{2})$, the code is classified as EAQMDS. These bounds ensure robust error correction by optimizing entanglement use. EAQMDS codes generally achieve lower QERs, between 10^{-3} and 10^{-5}, and higher efficiencies, ranging from 0.50 to 0.75. EAQEC codes, in comparison, have QERs around 10^{-3} to 10^{-4} and efficiencies from 0.40 to 0.60, making EAQMDS codes particularly effective for applications needing high error correction performance. In 2020, the findings obtained in another previous study [8] assessed a GPU-based belief propagation decoder for quasi-cyclic multi-edge type low-density parity-check (QC-MET-LDPC) codes, achieving better rectification of errors; speed and throughput across 128 codewords at code rates of 0.10, 0.05, and 0.02. This study addressed the weak condition in QC-MET-LDPC codes, which hinges on the degree distribution of variable and check nodes, defined by multivariate polynomials. Then, the degree distribution function for variable nodes is represented by

$$\nu(r, x) = \sum_{b} \nu_{b,d} r^b x^d, \tag{19}$$

where b and d represent channel and edge types, while r and x are their corresponding parameters. Together, r and x enable the function $\nu(r, x)$ to capture variable node contributions based on these connections. For check nodes, it is defined as

$$\mu(x) = \sum_{d} \mu_d x^d, \tag{20}$$

where μ_d is the probability for check nodes of type d. Optimizing these distributions is essential for maximizing QEC performance, particularly in continuous-variable quantum key distribution (CV-QKD) systems. QC-MET-LDPC codes exhibited notable decoding speeds 64.11 Mbits/s for 0.10, 48.65 Mbits/s for 0.05, and 39.51 Mbits/s for

0.02 outperforming MET-LDPC codes, which achieved about 9.17 Mbits/s. This structured approach also yields better BER, making QC-MET-LDPC codes advantageous for secure, high-speed CV-QKD. The authors further examined how QC-MET-LDPC codes align with classical coding theory and QKD but require design adjustments to fit stabilizer codes, merging classical and quantum coding strategies for advanced applications.

Advancements in QCCs continue to drive improvements in key generation, decoding performance, and data transmission. Three studies evaluated QCCs [9–11], demonstrating significant progress in these areas. Some researchers [9] introduced an optimized polynomial inversion algorithm for quasi-cyclic moderate-density parity-check (QC-MDPC) codes, accelerating key generation for the bit flipping key encapsulation (BIKE)-2 scheme. The algorithm achieved speedups of 11.40× over NTL and 83.50× over OpenSSL, contributing to BIKE-2's selection for round 3 of the NIST post-quantum cryptography standardization project. Despite these improvements, the authors highlight the weak condition in geometric approaches for QC-MDPC codes, particularly inefficiencies arising from the structure of polynomials, including their support and Hamming weight. These inefficiencies impact key generation in schemes like BIKE, necessitating efficient algorithms to optimize polynomial inversion. The authors highlighted inefficiencies in polynomial inversion for QC-MDPC codes, crucial for key generation in schemes like BIKE. These inefficiencies stem from the polynomial structure, particularly its support and Hamming weight. Let $R = \mathbb{F}_2[x]/(r)$ be a binary polynomial ring, where r is an irreducible polynomial of degree n. A polynomial a in the ring R is represented as

$$a = \sum_{j \in \mathrm{supp}(a)} x^j, \tag{21}$$

where $\mathrm{supp}(a) \subseteq \{0, 1, \ldots, n-1\}$ is the support of a, denoting the positions of nonzero coefficients (i.e., bits set to 1). The characteristic two Frobenius map, known as k-squaring , is expressed as

$$a^{2^k} = \sum_{j \in \mathrm{supp}(a)} x^{j \cdot 2^k} \mod r, \tag{22}$$

illustrating how bit permutations impact polynomial inversion. This expression shows that squaring a polynomial in this ring corresponds to a modular exponentiation of each index, effectively permuting the positions of nonzero bits. The initial observation in [9] formalized this behavior as a permutation of exponents modulo n. This property is central to the fast inversion algorithm, which exploits these deterministic bit permutations to efficiently compute inverses in R without performing costly full polynomial multiplications.

Efficient algorithms are needed to address these computational challenges in QC-MDPC codes. QC-MDPC codes exhibit strong error correction performance with BER ranging from 10^{-5} to 10^{-1} under varying noise levels, and high efficiency, especially in post-quantum cryptography applications. Future research will focus on further optimizing speed, testing across cryptographic schemes, and integrating these codes

into cryptographic libraries. Similarly, another study [10] introduced the Girth-Cycle-Embedding (GCE) algorithm to construct low-density parity-check (LDPC) codes with desired girth, which improves performance in the error-floor region and results in energy savings of 4% to 28%. The GCE algorithm outperforms other methods like Progressive Edge Growth (PEG) and QC-based algorithms, achieving lower BER and reducing the number of decoding iterations, which contributes to energy efficiency. However, the weak condition of the GCE algorithm lies in its inability to guarantee the desired girth, leading to short cycles that can degrade performance. Despite this, QC-based algorithms, including GCE, demonstrate strong error correction in high-SNR regions, with reduced iterations and parallel hardware implementation, making them ideal for wireless sensor networks. Future work will focus on optimizing the algorithm's scalability and efficiency for broader applications. Specifically, a previous study [11] proposed a 5 G QC-LDPC encoder with parallel cyclic redundancy check calculations, targeting high throughput and low latency for uplifted enhanced mobile broadband (eMBB) and ultra-reliable low-latency communication (URLLC) scenarios. The study emphasized the limitations in design and performance due to geometric constraints in QC-LDPC codes, particularly with respect to the lifting size Z, related to the maximum code block length $K = K \times Z$ and base graph matrices (H_{BG1} and H_{BG2}), affect error correction and encoding complexity, with the parity-check matrix $H = P \cdot H_{BG}$, where P is the permutation matrix, and the rate R of a code is expressed as the ratio of the number of information-carrying bits, k, to the total number of encoded bits, n, while the error probability is $P_e \approx Q\left(\sqrt{\frac{E_b}{N_0}}\right)$, here Q is the Q-function, E_b is the energy per bit, and N_0 is the noise power spectral density. This approach enabled throughput ranging from 62 to 257.90 Gbps with low latency, achieving BERs between 10^{-5} and 10^{-9} at moderate- to high-SNR levels, making it suitable for 5 G NR applications. QC-LDPC codes also demonstrate high efficiency, with code rates between 0.50 and 0.83. Their superior achievement in terms of BER and throughput, when compared to traditional LDPC and Turbo codes, positions QC-LDPC codes as highly effective for modern high-speed communication systems. Future developments will focus on hardware optimization and adaptation to emerging communication standards. Together, these studies highlight the continuous progress in QC-based error correction, emphasizing their importance in both quantum and classical cryptographic systems, and their growing applicability in high-speed communication technologies like 5 G.

Meanwhile, the three studies [12–14] collectively examine QCCs and their effectiveness within uses in quantum computation and transmission, particularly focusing on error correction and encoding efficiency. A construction for moderate-density parity-check (MDPC) codes via projective bundles in Desarguesian projective planes was introduced [12]. This QC structure, with index 2, supports efficient encoding and optimal error correction with a single decoding round. In QC-MDPC codes, the weak condition of projective planes revolves around minimizing the maximum column intersection s_H in the parity-check matrix H, which directly affects error correction capability. The correctable error count depends on reducing s_H in the H, calculated as

$$s_H = \max\{|C_i \cap C_j| : i \neq j\}. \tag{23}$$

Within the errors corrected $\leq (\frac{v}{2 \cdot s_H})$, here v is code length, while code dimension depends on H's rank, linking it to constituent matrices A and B, where the rank inequalities, linking the rank of H to its constituent matrices A and B, are expressed as: $\text{rk}(H) \geq (\text{rk}(A) + \text{rk}(B) - 1)$. QC-MDPC codes offer high decoding efficiency and are ideal for cryptography, correcting up to 100% of errors detected despite higher BERs (10^{-1} to 10^{-3}). LDPC codes, with lower BERs (10^{-5} to 10^{-9}) and superior error correction, are preferred for communication systems; future research aims to enhance MDPC designs and decoding algorithms. In study [13], authors developed a semi-blind algorithm for 5 G and IoT applications that combines self-interference cancelation with channel estimation in QC-LDPC-encoded transmissions. The code parameters $[[n, k, d]]$ describe the structure of the code, where k represents the information sequence length and n denotes the total codeword length, determining the code rate. Additionally, the average degree of variable nodes, $\bar{u}_v$, is calculated as $\bar{u}_v = \frac{1}{N} \sum_{i=1}^{N} d_{v,i}$, with $d_{v,i}$ indicating the degree of the i-th variable node in the code, offering insights into the code's connectivity properties. Similarly, the average check node degree, $\bar{u}_c$, is expressed as $\bar{u}_c = \frac{1}{M} \sum_{j=1}^{M} d_{c,j}$, where $d_{c,j}$ denotes the degree of the j-th check node, impact QC-LDPC performance, achieving low BER such as 10^{-5} at 10 dB SNR. High SNRs in 5 G achieve reliable low BERs, though careful parameter tuning is essential to avoid accomplishment degradation. In contrast, a different study [14] examined QCCs in computing with quantum technology with LDPC codes, introducing an algorithm to determine the minimum circulant permutation matrix (CPM) size required to achieve girth 8 in full-length row multiplier (FLRM) matrices. By avoiding specific cycle lengths in the exponent matrix E, LDPC efficacy is enhanced as decoding errors are minimized. Conditions were set to avoid both 4-cycles and 6-cycles, making better code reliability. Meanwhile, to prevent 4-cycles, the condition $(a_j - a_i)(b_y - b_x) \neq 0 \pmod{P}$, is applied, where a_j, a_i and b_y, b_x are elements from indexed sets, P is a prime modulus, and J and L are matrix indices. To further improve decoding efficiency, certain submatrices in E must have a nonzero determinant in order to prevent 6-cycles. LDPC codes, with their larger minimum distance, are especially advantageous in high-SNR communication systems due to their superior error correction capabilities. These developments in MDPC, QC-LDPC, and GC-LDPC codes address the changing needs of 5 G, IoT, and quantum technology applications. Better comprehension of these codes capabilities can be achieved by exploring advanced constructs, analyzing additional design parameters, and evaluating their performance in real-world scenarios through future research.

Three studies [15–17] focused on applying QCCs to improve QEC within quantum processing, introducing innovative constructions and error-correcting techniques tailored to quantum information applications. In one study [15] proposed the development of quantum additive codes (QACs), represented as $(n, \frac{k}{2}, d_{min})_{q^2}$, where k denotes even dimension, the codes are defined over the finite field $GF(q^2)$, and d_{min} represents the minimum distance of the QACs. These codes, with built up codewords parameters outperform traditional quaternary linear codes in error correction due to the extended field structure. Using QCCs alongside combinatorial techniques, this research demon-

strated the superior error-correcting abilities of QACs and Additive Complementary Dual Codes (ACDCs) when compared to equivalent linear codes. Additionally, the study suggested QACs are particularly suitable for applications requiring high reliability, achieving QER between 10^{-4} and 10^{-6}, and offering efficiencies from 0.70 to 0.85. The authors provide a theoretical lower bound on symplectic distance for 1-generator QCCs with even index, presenting a foundation for future research to explore the performance and versatility of additive codes in quantum information and memory systems. Building on this, the next study [16] introduced a method for constructing long-length Quantum QCCS (QQCCs) using block jackets and circulant permutation matrices, simplifying the coding process for high-quality quantum codes of length $O(n^2)$. The QQCCs structure proves effective in applications such as quantum machine learning and large-scale data processing, achieving a balance of low QER (typically between 10^{-4} and 10^{-6}) and high efficiency (between 0.60 and 0.90) essential for data-intensive quantum communication systems. The study emphasized the structured redundancy and symmetry inherent in block jackets and circulant matrices, which reduce decoding complexity and promote error resilience. Future directions include optimizing semi-parallel architectures and developing tailored QECCs for large-scale processing, underlining the practical benefits of QQCCs for advanced quantum coding techniques. Complementing these findings, after that study [17] presented a Distributed Quasi-Cyclic LDPC Coded Spatial Modulation (DQC-LDPCC-SM) scheme, involving source, relay, and destination nodes. This approach leveraged QCCs structures to optimize BER performance across different relay and antenna configurations, with a joint iterative decoding algorithm that utilizes a three-layer Tanner graph. Simulations showed that the DQC-LDPCC-SM scheme outperforms polar-coded cooperative spatial modulation by 3.10 dB under various SNR conditions, even with non-ideal source-relay channels. The study emphasized optimized relay encoding algorithms for improved error correction and suggests future advancements in relay node selection and hybrid coding, crucial for quantum information processing, machine learning, and scalable data processing. These studies collectively emphasize the efficacy of QCCs-based methods in achieving strong QEC performance. By advancing the construction of QACs, QQCCs, and DQC-LDPCC-SM schemes, they lay a powerful foundation for further exploration of high-quality quantum codes in both theoretical and applied quantum computing contexts.

4.4 Effect of mixed of QCCs and QSGCs

Three studies [18–20] explored the development and performance of QCCs and QSGCs, focusing on their applications in quantum computing and error correction. Use of low-density parity-check (LDPC) codes, specifically spatially coupled (SC)-LDPC and quasi-cyclic (QC)-LDPC codes, in continuous-variable quantum key distribution (CVQKD) systems is examined in the study [18]. These LDPC codes, developed from ahead Television Systems Committee standards, demonstrate high reconciliation efficiency, exceeding 0.96 with a frame error rate of 0.20. This high efficiency develops CVQKD systems by supporting higher secret key rates, extending transmission distances, and supporting security. Prospective research could expand on this by exploring

other LDPC configurations, moved forward error correction techniques, and practical security measures. A critical factor in QC-LDPC performance is the girth, defined as the length of the shortest cycle in the parity-check matrix, which impacts decoding efficiency. Otherwise, for effective decoding to be maintained, QC-LDPC codes must meet the girth threshold:

$$g_{\mathrm{QC}} \geq g_{\mathrm{LDPC}}, \tag{24}$$

where g_{QC} and g_{LDPC} represent the girth of the QCCs and original LDPC codes, respectively. Additionally, the base matrix's girth is constrained by $g_{\mathrm{base}} \geq 4$ to ensure reliable decoding. In comparing SC-LDPC and QC-LDPC codes, SC-LDPC codes are more resilient at low SNR due to their higher girth and efficiency in error correction, while QC-LDPC codes, though easier to implement, may face performance limitations due to girth constraints. A prior study [19] introduced a subcarrier mapping strategy tailored for forward error correction (FEC)-encoded symbols in bandwidth-limited intensity-modulation and direct-detection optical OFDM systems. This strategy, by allocating systematic and parity-check symbols across subcarriers with varying reliability, improves symbol protection and overall system efficiency. The approach, applicable to QC-LDPC and OFDM systems, extends to other soft-decision FEC and multi-carrier systems, demonstrating versatility. A significant factor in QC-LDPC codes under weak conditions is the degree distribution for variable and check nodes, expressed as:

$$\sum_{i=1}^{\infty} \lambda_i = 1 \quad \text{and} \quad \sum_{j=1}^{\infty} \rho_j = 1, \tag{25}$$

where λ_i and ρ_j are the probabilities of variable and check nodes with degrees i and j, respectively. Performance is often gauged by the decoding threshold, the minimum SNR required for error-free decoding, and EXIT chart analysis, which examines mutual information between transmitted and received symbols. Optimizing degree distributions in bandwidth-limited systems can develop high efficiency by balancing symbol diversity and simplifying design. At the same time, another study [20] compared energy efficiency in short-reach optical systems using both LDPC-coded and uncoded schemes, examining 70 Gbaud on-off-keying (OOK), 35 Gbaud pulse-amplitude modulation (PAM4), 50 Gbaud OOK, and 25 Gbaud PAM4 transmissions. It found that uncoded methods excel in slightly band-limited cases, while LDPC codes need to balance transmitter and decoder energy in more severe conditions. Among these, PAM4 modulation is more energy-efficient than OOK, especially for short-reach applications. For longer-reach applications, LDPC codes demonstrate lower power usage due to raise error correction capabilities. The study highlighted the limitations of geometric approaches in QCCs and QSGCs, focusing on the influence of minimum distance d_{min} on error correction capacity

$$e = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor \tag{26}$$

and the decoding complexity, which increases with iterations: $C = I \cdot O$, where I is number of iterations, and O stands computational cost per iteration. Therefore, convergence rates also impact performance, with more iterations needed to achieve a target QER when the rate is low:

$$N = \frac{1}{\text{Rate}} \cdot \log\left(\frac{1}{\text{Target QER}}\right). \tag{27}$$

Comparative analysis shows that QC-LDPC codes outperform forward ECCs (FECCs) in QSGCs at higher SNRs. For example, at 10 dB, QC-LDPC codes achieve a QER of 10^{-5}, while FECCs show QERs between 10^{-4} and 10^{-3}. QC-LDPC codes maintain higher efficiency, generally between 0.75 and 0.90, compared to FECCs, which range from 0.60 to 0.80. This advantage in QER and efficiency renders QC-LDPC codes favorable in scenarios demanding robust error correction and efficient data transmission. Collectively, this review highlights the role of QCCs and QSGCs in improving the reliability of quantum communication. QCCs, with their cyclic structure, enable efficient encoding and decoding but may struggle under weak conditions. In contrast, QSGCs, based on a complex algebraic curve stabilizer framework, offer reliable error correction and noise immunity, particularly in high error environments. Future research should focus on refining energy-efficient LDPC strategies, exploring adaptive mechanisms, and employing advanced coding techniques to strengthen error correction, ensuring continued progress in quantum information systems and optical communications.

In advanced QEC, applying weak geometric conditions with curvature $K \leq \epsilon$ is vital to enhance QGS, offering more flexibility in the design and evaluation of QCCs, QOCs and QSGCs. This approach improves performance across various systems and error models. However, this method also emphasizes the need to balance improvements in error correction with potential trade-offs in code optimality and complexity, highlighting the importance of optimizing QGS for various quantum computing applications. The review focuses on the theoretical and numerical evaluation of the error correction performance for QGS and traditional codes through simulation. Although experimental data could improve the findings, numerical methods provide a reliable comparison of error correction capacity and efficiency under controlled conditions. Experimental validation is less feasible because it requires significant resources and large-scale infrastructure. To ensure reliable results, well-established numerical approaches are applied, including Monte Carlo simulations and iterative testing, which are widely used in the literature to assess error correction codes. These methods, validated in previous studies [85–88], offer a solid foundation for comparing QGS and conventional codes. The key performance metrics considered include: *Error Correction Performance* (P_e), which measures the percentage of errors that the code can successfully correct, calculated as:

$$P_e = \frac{\text{Correctly Decoded Blocks}}{\text{Total Blocks}} \times 100. \quad \text{(Error Correction Performance)} \tag{28}$$

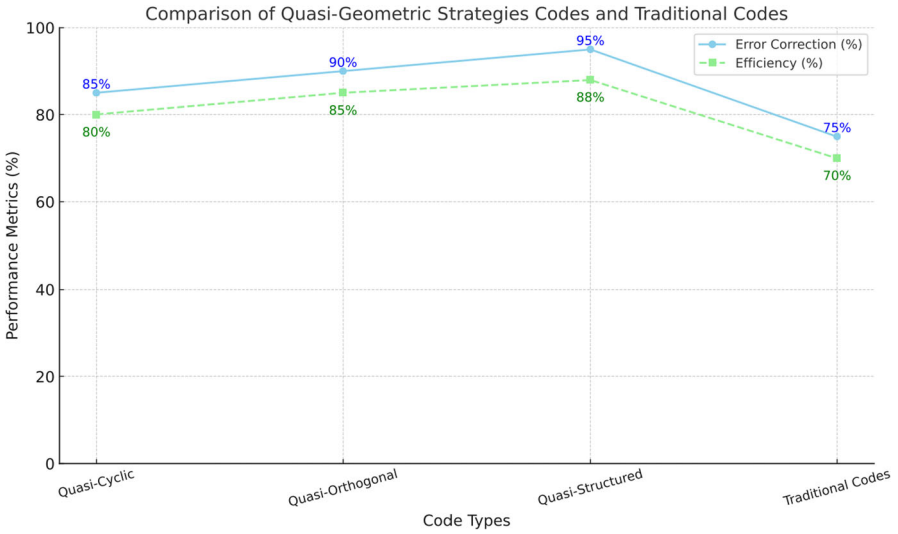

Fig. 5 Comparison of error correction capabilities and efficiency between QGS codes and traditional codes

Efficiency metric, which compares the error correction performance to the theoretical maximum performance of the code:

$$\text{Efficiency} = \frac{P_e}{P_{\max}} \times 100, \quad \text{(Efficiency Metric)} \tag{29}$$

where $P_{\max}$ is considered to be 100% in ideal conditions. These metrics align with industry standards and provide a comprehensive comparison of QGS and traditional codes. Although experimental data could strengthen the study, numerical results offer a solid basis for future experimental validation. Figure 5 compares the performance of QGS codes and traditional codes in terms of error correction and efficiency. QOCs demonstrate the highest error correction capability, while QCCs provide strong performance with a simpler design. QSGCs, though slightly less effective than QOCs, outperform traditional codes significantly due to their adaptable geometric design. QSGCs are highly efficient, making them ideal for resource-constrained systems, while QCCs strike a balance between performance and simplicity. QOCs, despite their validity, require higher computational resources. Traditional codes lag in both metrics, highlighting their limitations in modern quantum systems. Performance metrics include error correction capability, measured as the percentage of corrected errors, and efficiency, reflecting effective resource usage. QSGCs are well-suited for constrained systems, QOCs excel in high-fidelity applications, and QCCs provide a balanced solution, emphasizing the superiority of QGS codes over traditional methods for advancing QEC.

5 Conclusion and further work

The study analyzed quasi-geometric strategies and their contributions to quantum error correction, focusing on key challenges in encoding, decoding, and efficiency. It examined three activation protocols: quasi-orthogonal, quasi-cyclic, and quasi-structured geometric codes and their relevance to weak condition geometric approaches. These strategies improved quantum error correction and communication systems, as shown in studies in the specified time frame 2019-2024. Quasi-cyclic codes improved the efficiency of encoding and decoding with their cyclic properties, while quasi-structured geometric codes improved error correction and energy efficiency. Quasi-orthogonal codes reduced computational overhead and improved bit error rates under weak conditions. However, challenges such as limited orthogonality and decoding instability still hinder optimization. Despite these advancements, the weak geometric conditions pose difficulties for practical integration into stabilizer frameworks. Future work should validate QGS on real quantum hardware to confirm the benefits seen in simulations. This is essential for applying these methods in quantum cryptography and secure communication. Further research can investigate large-scale integration, qubit mapping, hybrid quantum-classical approaches, AI-driven error correction, and advanced mathematical tools to improve reliability and strengthen quantum computing networks.

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Declarations

Conflict and interest The authors have declared no Conflict of interest.

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Authors and Affiliations

Valentine Nyirahafashimana[1,2] · Nurisya Mohd Shah[1,3] · Umair Abdul Halim[1,4] · Mohamed Othman[1,5]

✉ Nurisya Mohd Shah
risya@upm.edu.my

Valentine Nyirahafashimana
nyirahafashimanav@ulk.ac.rw

Mohamed Othman
mothman@upm.edu.my

1 Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia (UPM), UPM Serdang, 43400 Serdang, Selangor Darul Ehsan, Malaysia

2 UKL Polytechnic Institute, Kigali Independent University (ULK), Kigali Campus, 102 KG 14 Ave Gisozi, Kigali, Rwanda

[3] Department of Physics, Faculty of Science, Universiti Putra Malaysia (UPM), UPM Serdang, 43400 Serdang, Selangor Darul Ehsan, Malaysia

[4] Centre of Foundation Studies in Science, Universiti Putra Malaysia (UPM), UPM Serdang, 43400 Serdang, Selangor Darul Ehsan, Malaysia

[5] Department of Communication Technology and Network, Universiti Putra Malaysia (UPM), UPM Serdang, 43400 Serdang, Selangor Darul Ehsan, Malaysia