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On the SU(1,1)-based stabilizer formalism

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Abstract.
This work is motivated by the geometry and symmetry of continuous-variable (CV) and open quantum systems. We describe a stabilizer formalism based on the non-compact group SU(1,1). In contrast to the Pauli stabilizer codes, which are finite and discrete, and the GKP code, which uses displacement stabilizers on a flat phase-space lattice, the SU(1,1) approach is naturally connected to hyperbolic geometry. Errors can be organized into elliptic, parabolic, and hyperbolic types according to the subgroup structure of SU(1,1). This provides new classes of stabilizer operations that go beyond the Pauli–Clifford setting and at the same time, can capture the encoding structure of the GKP code. The construction is preliminary, but it suggests a more general framework for building fault-tolerant codes tailored to continuous-variable systems. It is natural to conjecture that SU(1,1)-based stabilizers admit a coset-like decomposition, with elliptic, parabolic, and hyperbolic subgroups playing the role of error classes, in analogy with the Pauli case. This perspective offers a pathway toward defining logical operators and error classification in hyperbolic phase space.

1 Introduction

The stabilizer formalism is one of the most influential frameworks in quantum information theory, providing the mathematical foundation for practical quantum error correction (QEC) and fault-tolerant computation [1, 2, 3, 4, 5]. In this framework, stabilizer groups typically subgroups of the Pauli group generated by tensor products of Pauli operators define code subspaces in which quantum information can be reliably stored. The group-theoretic structure, particularly the role of normal subgroups, organizes how logical states are encoded and how errors can be detected and corrected [6, 7, 8].

In conventional qubit-based architectures, Pauli stabilizer codes and Clifford operations offer a highly successful method for detecting and correcting discrete error models [1, 2]. However, this finite-dimensional setting becomes insufficient when one turns to continuous-variable (CV) systems such as quantum optical modes or oscillator-based platforms. In such systems, relevant noise processes include squeezing, displacement, and continuous shifts in phase space, which lie outside the algebraic scope of the Pauli group [9, 10]. Extensions to CV codes, such as the Gottesman-Kitaev-Preskill (GKP) code, have demonstrated how lattice-like stabilizers in phase space can correct Gaussian displacement errors [9, 10], but a broader algebraic framework is still needed to encompass the full range of non-Pauli error processes encountered in realistic physical systems [11, 12, 13].

To address this gap, we propose a stabilizer formalism grounded in the non-compact Lie group SU(1,1). The generators of $su(1,1)$ naturally describe squeezing operations and hyperbolic phase-space flows, making this group a natural candidate for capturing CV error dynamics [14]. The GKP code, for instance, is an example of how the logical states appear as coherent-state lattices embedded in

Euclidean phase space, which cannot be defined in a finite Hilbert space [15]. In contrast to SU(2), which governs compact Bloch-sphere rotations, SU(1,1) generates non-compact transformations on negatively curved geometries such as the Poincaré disk [16]. This geometric distinction allows us to classify and model a wider class of error operators by leveraging the subgroup decomposition of SU(1,1) into elliptic, parabolic, and hyperbolic elements [17]. These subgroup classes provide a natural taxonomy of error types, extending the stabilizer formalism beyond the Pauli and GKP frameworks.

This paper is structured as follows. Section 2 reviews the standard stabilizer formalism in both finite-dimensional (Pauli, SU(2)) and continuous-variable (GKP, Heisenberg–Weyl) settings. This sets the stage for Section 3, where we introduce the SU(1,1)-based stabilizer framework and highlight its geometric interpretation on the hyperbolic disk. Section 4 discusses the classification of errors according to the elliptic, parabolic, and hyperbolic subgroups of SU(1,1). Finally, Section 5 concludes with prospects, including open questions about the intersection of Pauli, GKP, and SU(1,1) stabilizer formalisms.

2 Preliminaries on stabilizer formalism

Before turning to the SU(1,1)-based constructions, it is useful to recall the two main stabilizer settings that are already well established. The first is the finite-dimensional case (Pauli stabilizers), which underlies most qubit-based error correction. The second is the continuous-variable case (GKP code), where stabilizers are given by displacements on a lattice in phase space. These two examples will serve as reference points when introducing the SU(1,1) formalism in the next section.

2.1 Finite-dimensional (Pauli codes)

The Pauli group provides the natural algebraic setting for finite-dimensional stabilizer codes. Its elements I, X, Y, Z can be understood in three complementary roles. As unitary gates, they act as fundamental logical operations on qubits. As errors, they model bit flips (X), phase flips (Z), and their combinations (Y), forming the canonical error basis. As stabilizers, Abelian subgroups of the Pauli group define code spaces as joint $+1$ eigenspaces. In this way, Pauli gates, Pauli errors, and Pauli stabilizers are not separate objects but different utilization of the same algebraic structure.

From a group-theoretic perspective, the Pauli operators are tied to the compact group SU(2), defined as [18]

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{1}$$

The generators of this algebra are the familiar Pauli matrices

$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

which obey the commutation relations with ϵ_{jkl} as the antisymmetric tensor

$$[S_j, S_k] = i\,\epsilon_{jkl} S_l, \qquad j,k,l \in \{x,y,z\}. \tag{3}$$

Geometrically, these operators generate rotations about orthogonal axes of the Bloch sphere: X about the x-axis, Y about the y-axis, and Z about the z-axis. This correspondence provides a direct geometric interpretation of both intended gates and unintended errors in terms of rotations, with stabilizer conditions specifying invariant subspaces under these symmetries.

The subgroup structure of the Pauli group is central to the stabilizer codes. In particular, the stabilizer subgroup of a code is a normal Abelian subgroup, ensuring that the cosets classify errors systematically. This normal subgroup property underpins the error detection and correction mechanism: distinct cosets correspond to distinct error syndromes.

This finite-dimensional setting highlights the central role of the Pauli group while also pointing toward natural generalizations. Just as the Pauli group admits normal subgroups that stabilize code spaces, the non-compact group SU(1,1) admits subgroups of different types (elliptic, hyperbolic, parabolic), each with distinct geometric meaning. This analogy motivates the extension of stabilizer ideas beyond the Pauli framework, as explored in Section 3.

2.2 Continuous-variable (GKP code)

The GKP code extends the stabilizer framework to continuous-variable systems. Here, instead of qubits, the stabilizers are displacement operators defined on a lattice in phase space, arising from the Heisenberg-Weyl group [18]. This setting leads to an infinite-dimensional Hilbert space equipped with flat, Euclidean geometry. Explicitly, the stabilizers are generated by

$$S_q = e^{i2\sqrt{\pi}\hat{q}}, \qquad S_p = e^{-i2\sqrt{\pi}\hat{p}}, \tag{4}$$

which correspond to lattice translations in position, q and momentum, p. Similarly, $S_q|\psi_L\rangle = |\psi_L\rangle$ and $S_p|\psi_L\rangle = |\psi_L\rangle$ for all logical states $|\psi_L\rangle \in C$ preserving the code space that is the joint $+1$ eigenspace of the chosen subgroup stabilizers.

Formally, the Heisenberg group can be realized as the set of 3×3 upper triangular matrices

$$H(x,p,t) = \begin{pmatrix} 1 & x & t + \frac{1}{2}xp \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}, \qquad (x,p,t) \in \mathbb{R}^3. \tag{5}$$

x, p, t represent the group parameters of the Heisenberg group i.e. x is displacement in position, p is displacement in momentum, and t is the central phase-like coordinate. t captures the non-commutativity of phase-space translations, ensuring that the canonical commutator $[x,p] = iI$ is faithfully represented. Physically, however, the relevant variables are the projected pair $(x,p) \in \mathbb{R}^2$, corresponding to the phase-space plane of position and momentum. In the GKP construction, the stabilizers arise from a *lattice subgroup* of the Heisenberg-Weyl group acting on this plane. Thus, while the full matrix Heisenberg group provides the algebraic backbone, the effective geometry for error correction is that of a two-dimensional Euclidean lattice. The GKP framework illustrates how stabilizers can be generalized to infinite-dimensional Hilbert spaces with Euclidean geometry, but its structure remains closely tied to the flat lattice of the Heisenberg-Weyl group.

With these two frameworks as reference points: a) SU(2) for finite qubits and b) Heisenberg-Weyl for CV lattices. We now introduce the SU(1,1)-based stabilizer formalism. Here, the underlying group structure naturally organizes error processes into elliptic, parabolic, and hyperbolic classes, providing a hyperbolic extension of stabilizer theory beyond lattice-based approaches.

3 SU(1,1)-based stabilizer formalism

We now turn to the SU(1,1) group as a new setting for stabilizers. Unlike SU(2), which is compact, and the Heisenberg–Weyl group, which describes a flat Euclidean lattice, SU(1,1) is non-compact and naturally associated with hyperbolic geometry. This shift allows us to define stabilizers on curved spaces, where group orbits can be discretized into what may be viewed as an SU(1,1) lattice. The formalism is still parallel in spirit to the Pauli and GKP cases, but it brings in new algebraic and geometric structures that suggest richer classes of errors and codes.

The three main stabilizer frameworks can also be understood through their underlying geometry. Pauli codes are compact and live on the Bloch sphere, GKP codes rely on a Euclidean lattice in phase space, and SU(1,1)-based codes are naturally described on the hyperbolic disk.

The non-compact Lie group $\mathrm{SU}(1,1)$ has generators (K_0, K_1, K_2) satisfying [19]

$$[K_0, K_1] = iK_2, \qquad [K_0, K_2] = -iK_1, \qquad [K_1, K_2] = -iK_0. \tag{6}$$

SU(1,1) described a hyperbolic norm translating a non-compact structure

$$K \in SU(1,1) = \Big\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \Big\}. \tag{7}$$

One of the common $\mathfrak{su}(1,1)$ matrices that form the Lie algebra is

$$K_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad K_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{8}$$

These generators provide a natural geometric classification of flows into elliptic, parabolic, and hyperbolic types [20]. For instance, a ladder formulation analogous to the Pauli structure could be written as $K_\pm = K_1 \pm iK_2$ and K_0 is like the number operator in quantum optics [14] (see further discussion in Section 4). This classification parallels the role of Pauli errors (bit, phase, both) and GKP displacements (lattice shifts), but in a curved hyperbolic setting. Recent work has shown that hyperbolic tilings and Fuchsian groups provide a natural setting for quantum codes in curved geometries [13]. Within our framework, this resonates with the role of $\mathrm{SU}(1,1)$, whose representations support stabilizers adapted to hyperbolic error structures.

4 Error Classes from SU(1,1) Subgroups

In the Pauli stabilizer formalism, errors are classified by combinations of the stabilizer group within the Pauli group. Analogously, in the Gottesman–Kitaev–Preskill (GKP) code, errors correspond to displacements modulo a lattice defined by the stabilizers. We emphasize that its application to a stabilizer formalism for classifying error types in quantum systems is a new idea proposed in this work. For $\mathrm{SU}(1,1)$-based codes, the classification of errors naturally arises from the subgroup decomposition of

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SU$(1,1)$ itself. The three principal conjugacy classes of SU$(1,1)$ are elliptic, parabolic, and hyperbolic. They are correspond to qualitatively distinct types of geometric flows, which we interpret as distinct classes of errors.

We define stabilizer-like operators as exponentials of the $\mathfrak{su}(1,1)$ generators (or linear combination)

$$S = \exp(-i\lambda\mathfrak{K}), \qquad \mathfrak{K} \in \{K_0, K_+, K_-\}. \tag{9}$$

Here λ is associated to the parameter that translated the three error classes i.e. θ, τ, ϵ below. For such operators to function as stabilizers of a logical code space $\mathcal{C}$, they must preserve the code subspace ($S|\psi_L\rangle = |\psi_L\rangle$ for all $|\psi_L\rangle \in \mathcal{C}$), commute with one another to form an Abelian subgroup, and enable error detection by distinguishing how errors act relative to the subgroup stabilizers. Thus, just as Pauli stabilizers define error cosets through normal subgroups, SU$(1,1)$ stabilizers classify errors according to their subgroup decomposition. The group elements are obtained by exponentiating the generators with an i factor:

$$U(\vec{\theta}) = \exp\big(i\,\theta_\alpha K_\alpha\big) \in \mathrm{SU(1,1)} \quad \alpha = 0,1,2, \tag{10}$$

which satisfy the unitarity condition, equivalently expressed as

$$U^\dagger M U = M, \quad M = \mathrm{diag}(1,-1),$$

applies only to elements of $SU(1,1)$. The $\mathfrak{su}(1,1)$ algebra, with generators $\{K_0, K_1, K_2\}$ or $\{K_0, K_\pm\}$ [21]

$$[K_0, K_\pm] = \pm K_\pm, \qquad\qquad [K_+, K_-] = -2K_0, \tag{11}$$

supports three distinct subgroup actions:

- **Elliptic subgroup (phase-like errors).** Generated by K_0, these compact rotations

$$g_E(\theta) = e^{i\theta K_0} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \qquad \theta \in \mathbb{R} \tag{12}$$

  correspond to *elliptic errors*, which act as dephasing or phase noise, analogous to Z-type errors in the Pauli framework.

- **Parabolic subgroup (displacement-like errors).** Generated by the nilpotent element $S = K_0 \pm K_2$.

$$\begin{aligned} g_P(\pm\tau) =& e^{i\tau(K_0 \pm K_2)} = I + i\tau(K_0 \pm K_2) \\ =& \begin{pmatrix} 1 \pm \frac{i\tau}{2} & \mp\frac{\tau}{2} \\ \mp\frac{\tau}{2} & 1 \mp \frac{i\tau}{2} \end{pmatrix}, \qquad \tau \in \mathbb{R} \end{aligned} \tag{13}$$

  correspond to *parabolic errors*, which act as displacements in the hyperbolic plane, paralleling displacement errors in GKP codes. However, these matrices are not unitary in the usual Hilbert inner product $g_P^\dagger g_P \neq I$ for $\tau \neq 0$. They are instead satisfy pseudo-unitary i.e. $g_P^\dagger M g_P = M$ with $M = \mathrm{diag}(1,-1)$.

- **Hyperbolic subgroup (squeezing-like errors).** Generated by K_2, these transformations

$$g_H(\epsilon) = e^{i\epsilon K_2} = \begin{pmatrix} \cosh(\epsilon/2) & \sinh(\epsilon/2) \\ \sinh(\epsilon/2) & \cosh(\epsilon/2) \end{pmatrix}, \qquad \epsilon \in \mathbb{R} \tag{14}$$

  correspond to *hyperbolic errors*, which act as squeezing distortions, producing non-compact deformations of phase space.

Geometrically, the one-parameter subgroups of SU(1,1) act on the hyperbolic disk as flows: elliptic elements generate circular orbits around an interior fixed point, parabolic elements generate horocycles tangent to the boundary, and hyperbolic elements generate geodesics joining distinct boundary points [20], where the classification is given for which is projectively isomorphic to SU(1,1). Each flow deforms phase space in a way that parallels familiar physical error processes: elliptic errors $\sim$ phase noise, parabolic errors $\sim$ displacement noise, and hyperbolic errors $\sim$ squeezing noise. Thus, the subgroup structure of SU$(1,1)$ provides a natural error taxonomy and generalizes the stabilizer mechanism beyond the Pauli and GKP cases.

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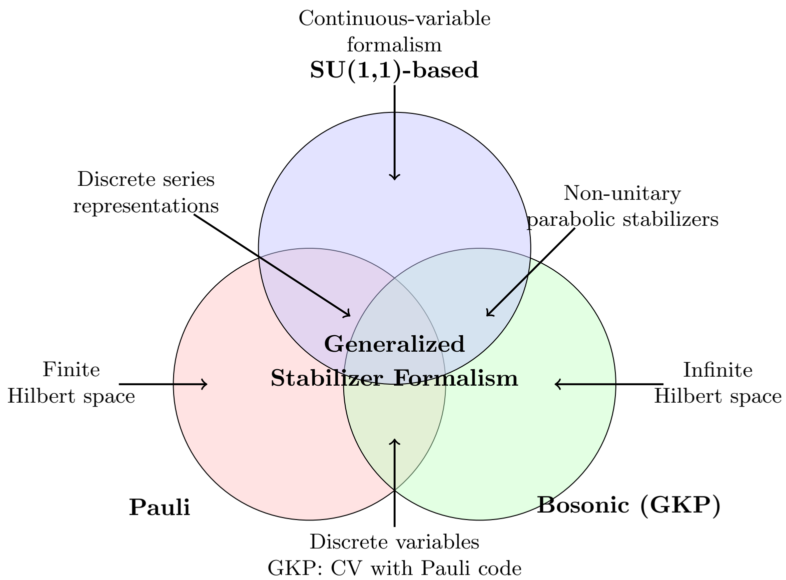

Figure 1: Proposed conceptual landscape of stabilizer formalisms: Pauli (finite-dimensional, discrete), bosonic/GKP (infinite-dimensional, CV), and SU(1,1)-based codes which unify and extend both. The Pauli–SU(1,1) overlap highlights discrete series representations. The parabolic stabilizer are pseudo-unitary but preserving the SU(1,1) indefinite form.

5 Conclusion and Prospects

In this proceeding we outlined a stabilizer formalism based on the group SU(1,1). The motivation was to extend the Pauli stabilizer picture (finite Hilbert space) and the bosonic/GKP construction (continuous variables in flat phase space) towards a framework that also takes into account the hyperbolic geometry of SU(1,1). The classification of errors into elliptic, parabolic and hyperbolic types follows directly from the subgroup structure, and this provides a new way to think about error processes in continuous-variable systems. The Venn diagram (Fig. 1) is conjectured to capture the relationship between the three approaches. The Pauli codes are discrete and finite, the GKP code is continuous-variable and infinite-dimensional, while the SU(1,1) case is linked to hyperbolic orbits. The overlap of Pauli and SU(1,1) suggests a connection with the discrete series representations of SU(1,1) [22], which can be thought of as embedding Pauli-like structures inside a non-compact setting.

We note that the classification of SU(1,1) error flows also admits a natural description in terms of the Iwasawa decomposition of the group [17], which provides a geometric way to relate stabilizer flows to hyperbolic dynamics; a detailed analysis will be presented in future work. The main intersection of all three regions is not yet well understood, but it points to the possibility of hybrid stabilizer codes that are discrete, continuous-variable, and hyperbolic at the same time. These observations are only preliminary, but they open up several directions. One is to make the algebraic side more concrete, for example, by working out explicit codewords using the discrete series. Another is to understand how such codes behave under noise models relevant for open quantum systems. We expect that clarifying this overlap may provide new ideas for fault-tolerant code that mixes discrete, CV, and hyperbolic structures in the quantum error correction model.

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