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ORIGIN-TRANSFER GEOMETRIC CONSTRAINT PURSUIT GAME MODELLED BY INFINITE SYSTEM OF DYADIC DIFFERENTIAL EQUATIONS

(*Permainan Kejar-Mengejar Berkekangan Geometri Pindahan-Asalan Dimodelkan oleh Sistem Persamaan Pembezaan Diadik Tak Terhingga*)

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ABSTRACT

This paper investigates a zero-sum two-person pursuit differential game modelled by an infinite system of dyadic differential equations. The two players in the game are a pursuing player and an evading player; their control functions adhere to geometric constraints, and the control resources available to the pursuing player is more than those of the evading player. The pursuing player is intent on driving the system's state from the initial state ξ^0 to ℓ_2 space origin in a finite period of time while the evading player is counteracting this. For the control problem, we design an admissible control that can steer the system's state to the origin and for the differential game problem, we develop an admissible strategy for the pursuing player that helps realize the objective and the guaranteed pursuit time equation. An illustrative example to show how our results can be applied to determine pursuit completion and compute guaranteed pursuit time in our differential game model is provided.

Keywords: admissible strategy; control problem; geometric constraint; infinite system; pursuit differential game

ABSTRAK

Kertas kerja ini mengkaji permainan pembezaan pengejaran dua orang jumlah sifar yang dimodelkan oleh sistem persamaan pembezaan diadik tak terhingga. Permainan ini melibatkan dua pemain, iaitu pemain yang mengejar dan pemain yang mengelak. Fungsi kawalan mereka mematuhi kekangan geometri, dan pemain yang mengejar mempunyai lebih banyak sumber kawalan berbanding pemain yang mengelak. Pemain yang mengejar berhasrat untuk mengarahkan keadaan sistem daripada keadaan awal ξ^0 ke titik asal dalam ruang ℓ_2 dalam masa terhingga, manakala pemain yang mengelak bertindak untuk menentang usaha ini. Bagi masalah kawalan, kami menghasilkan kawalan yang boleh diterima untuk mengarahkan keadaan sistem ke titik asal. Untuk masalah permainan pembezaan, kami membangunkan strategi yang boleh diterima bagi pemain yang mengejar, yang membantu mencapai objektif dan mendapatkan persamaan masa pengejaran yang terjamin. Contoh ilustrasi turut disediakan untuk menunjukkan bagaimana hasil kami boleh digunakan dalam menentukan kejayaan pengejaran dan mengira masa pengejaran terjamin dalam model permainan pembezaan ini.

Kata kunci: strategi yang boleh diterima; masalah kawalan; kekangan geometri; sistem tak terhingga; permainan pembezaan mengejar

1. Introduction

Game theory deals with studying the mathematical models of strategic interactions between parties, called players in the game. Neumann and Morgenstern (1944) laid the foundational work in modern game theory which developed the concept of zero-sum two-person games. Game theory has applications to real-world problems such as production planning in production networks (Argoneto *et al.* 2008), decision-making in defence and warfare (Ho *et al.* 2022), management of risks associated with natural-gas pipelines in urban areas (Li *et al.* 2023), analysis of cooperation and competition dynamics among Northern Vietnam's container terminals (Nguyen & Kim 2020), minimization of operation costs and improvement of reliability for coordinating electric vehicle aggregators connected to micro-grids in smart cities (Modarresi *et al.* 2024), evaluating the effectiveness of social distancing and mask-wearing strategies during a respiratory disease outbreak (Nabi *et al.* 2024) and many more.

A significant development in the field came with the work of Isaacs (1965), who established differential games as a framework for modelling and analyzing dynamic interactions through continuous-time strategies. Within this realm, pursuit-evasion games represent a significant area, where the pursuing player(s) aim to intercept or capture the evading player(s) who simultaneously seek to avoid capture. This class of games finds application in areas such as defence, surveillance, robotics, engineering, economics and many other areas.

Most pursuit-evasion research have focused on finite-dimensional spaces, typically $\mathbb{R}^n$, $n \geq 2$ (Badakaya *et al.* 2024a; Azamov & Ibaydullayev 2021; Samatov *et al.* 2021; Rilwan *et al.* 2023; Samatov & Akbarov 2024; Umar *et al.* 2024b; Li *et al.* 2024; Dorothy *et al.* 2024). However, many dynamic systems - especially those involving distributed parameters or spatially extended processes - are more accurately represented in infinite-dimensional spaces. Techniques such as spectral decomposition allow certain systems governed by partial differential equations (PDEs) to be reformulated as infinite systems of ordinary differential equations (ODEs), thus necessitating analysis in an infinite-dimensional framework such as a Hilbert space (Tukhtasinov 1995; Satimov & Tukhtasinov 2006; Azamov & Ruziboev 2013; Allahabi & Mahiub 2019).

A critical advancement in the analysis of these games is the introduction of geometric constraints on the control resources of each player. Geometric constraints refer to restrictions on the players' movement, which limit the set of feasible trajectories or control functions they can adopt within the game. These constraints commonly arise from physical limitations, environmental boundaries, or tactical requirements. Research in pursuit-evasion differential games with geometric constraints limiting players' controls in finite-dimensional spaces has received a lot of attention, such as in Badakaya *et al.* (2021), Mamadaliev and Ibaydullaev (2021), Samatov *et al.* (2022), Szöts and Harmati (2023), and Petrov (2023). However, similar studies (on geometric constraints differential game problems) have been conducted in infinite-dimensional spaces as well (Ibragimov *et al.* 2017, 2022a, b; Adamu *et al.* 2022; Badakaya *et al.* 2024b; Umar *et al.* 2024a).

In differential games modelled using an infinite system of ODEs, the pursuit is typically said to be completed when the system's state vector reaches the target state, often the origin, within a finite duration. The pursuing player aims to complete the pursuit while the evading player acts to prevent this from happening. Studies concerning differential game problems where pursuit completion entails guiding the system's state into the origin of a Hilbert space in a finite time include Tukhtasinov (1995), Satimov and Tukhtasinov (2006), and Ibragimov *et al.* (2022a).

Satimov and Tukhtasinov (2006) dealt with differential games modelled by parabolic-type PDEs on a given interval $[0, T]$. The differential games were transformed into those modelled by the infinite system of ODEs:

$$\dot{z}_i = -\lambda_i z_i - u_i + v_i, \qquad z_i(0) = z_i^0, \quad i = 1, 2, \ldots, \tag{1}$$

whereby u_i and v_i, $i = 1, 2, \ldots$ are respectively the pursuing player's and the evading player's control parameters and the parameters λ_i, $i = 1, 2, \ldots$ satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty. \tag{2}$$

The research analyzed four differential games with various control constraints and found sufficient conditions for completing pursuit and evasion. In the game, the pursuing player aims to steer the system's state from a given initial state to the origin in a finite time and the evading player aims to prevent this from happening. The game's guaranteed pursuit time was found to be $T_0 = \frac{\|z_0\|}{\rho-\sigma}$. The studies (Tukhtasinov 1995; Satimov & Tukhtasinov 2006) suggest investigating game problems modelled by Eq. (1)) in a single theoretical framework devoid of PDEs, where the parameters $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots$, instead of them being ordered as in Eq. (2).

Ibragimov *et al.* (2022a) continued the research in this direction by studying the game in Eq. (1), where $\lambda_i > 0$, $i = 1, 2, \ldots$, $z^0 = (z_1^0, z_2^0, \ldots) \in \ell_2$ distinct from the origin of ℓ_2, $u = (u_1, u_2, \ldots) \in \ell_2$ and $v = (v_1, v_2, \ldots) \in \ell_2$ which are respectively the pursuing player's and the evading player's controls, adhere to geometric constraints. They improved the guaranteed pursuit time to $T_1 = \frac{1}{\lambda_1} \ln\left(\frac{\lambda_1\|z_0\|}{\rho-\sigma} + 1\right) < T_0$.

Ibragimov *et al.* (2017) investigated a pursuit differential game modelled by an infinite system of first-order dyadic ODEs:

$$\begin{cases} \dot{x}_i = \mu_i x_i - \lambda_i y_i + u_{i1} - v_{i1}, & x_i(0) = x_i^0 \\ \dot{y}_i = \lambda_i x_i + \mu_i y_i + u_{i2} - v_{i2}, & y_i(0) = y_i^0 \end{cases}, \quad i = 1, 2, \ldots \tag{3}$$

in the ℓ_2 space, where $\mu_i \le 0$, $\lambda_i \in \mathbb{R}$, the initial state is $\xi^0 = (x^0, y^0)^T$, where

$$x^0 = (x_1^0, x_2^0, x_3^0, \ldots) \in \ell_2, \quad \text{and} \quad y^0 = (y_1^0, y_2^0, y_3^0, \ldots) \in \ell_2.$$

The pursuing player has the control parameter $u = (u_{11}, u_{12}, u_{21}, u_{22}, \ldots)$, while that of the evading player is $v = (v_{11}, v_{12}, v_{21}, v_{22}, \ldots)$. Geometric constraints were imposed on these players' control parameters. In the game, the pursuit is said to be completed when the pursuing player transfers the state of the system in Eq. (3) to the origin of the space ℓ_2; nevertheless, the evading player acts to prevent this. In the game, the pursuing player's control resources exceed those of the evading player. In the study, an auxiliary control problem was solved by constructing an admissible control function and obtaining the time to transfer the state of the system to the origin of ℓ_2 space. Then, the pursuit differential game problem was solved by constructing an admissible pursuing player's strategy and establishing an equation for the guaranteed pursuit time in the game.

Ibragimov *et al.* (2022c) studied a differential game of pursuit modelled by the infinite system of dyadic ODEs in Eq. (3) where $\mu_i \ge 0$, $i = 1, 2, \ldots$ instead and $\lambda_i \in \mathbb{R}$, $\xi^0 = (x^0, y^0)^T \in \ell_2$ is the initial state, where $x^0 = (x_1^0, x_2^0, \ldots)$, $y^0 = (y_1^0, y_2^0, \ldots)$. In the study, $u = (u_{11}, u_{12}, u_{21}, u_{22}, \ldots)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \ldots)$ are the pursuing and the evading players' control parameters respectively. The initial state ξ^0 is taken to be distinct from the origin of the ℓ_2 space. The pursuing player in the game (3) aims to steer the system's state into the origin of the ℓ_2- space within a finite duration, while the evading player attempts to prevent this. Integral constraints instead, are prescribed on the players' control functions. The study investigates the optimal pursuit time and explicitly constructs the optimal players' strategies.

The reviews mentioned explore various studies on differential games where player dynamics are modelled with infinite systems of ODEs. From the reviewed literature, observe that even though Ibragimov *et al.* (2017) studied a pursuit game problem modelled by the infinite system of dyadic ODEs in Eq. (3) for $\mu_i \le 0$, $i = 1, 2, \ldots$, and for geometric constraints limiting the players' controls, the paper did not consider for the case where $\mu_i > 0$ and with geometric constrained controls. Again, notice that even though Ibragimov *et al.* (2022c) studied a pursuit game problem modelled by the infinite system of dyadic ODEs in Eq. (3) for $\mu_i \ge 0$, $i = 1, 2, \ldots$, and for integral constraints limiting the players' controls, the research did not consider for the case of geometric constraints. Also, to the best of our knowledge, in the existing literature, there is no study yet on pursuit differential game for the infinite system of dyadic ODEs in Eq. (3) for $\mu_i \ge 0$ and geometric constraints restricting the players' control resources. This is the gap our study intends to fill.

Motivated by filling this mentioned research gap observed in the existing literature, this paper focuses on a zero-sum two-person pursuit differential game problem whose mathematical model is the infinite system of first-order dyadic ODEs in Eq. (3) but with $0 \le \mu_i \le \mu$ instead, for $i = 1, 2, \ldots$, where $\mu \in \mathbb{R}^+$ and $\lambda_i \in \mathbb{R}$. The game is played in the Hilbert space

$$\ell_2 = \{\xi = (\xi_1, \xi_2, \ldots) \; : \; \sum_{i=1}^{\infty} |\xi_i|^2 < \infty, \; \xi_i = (x_i, y_i)^T, \; x_i, y_i \in \mathbb{R}\},$$

with inner product and norm respectively defined as

$$\langle \xi, \eta \rangle = \sum_{i=1}^{\infty} \xi_i \eta_i \quad \text{and} \quad \|\eta\| = \sqrt{\langle \eta, \eta \rangle} = \sqrt{\sum_{i=1}^{\infty} |\eta_i|^2} < \infty, \qquad \eta, \xi \in \ell_2.$$

The initial state is $(x^0, y^0)^T = \xi^0 = (\xi_1^0, \xi_2^0, \ldots)$ with $\xi_i^0 = (x_i^0, y_i^0)^T$. The initial state ξ^0, assumed to be in ℓ_2, is taken to be distinct from the origin of the space ℓ_2. In our game, the pursuing player and the evading player use the respective control parameters

$$\begin{aligned} & u = (u_1, u_2, \ldots) = (u_{11}, u_{12}, u_{21}, u_{22}, \ldots) \in \ell_2, \\ \text{and} \quad & v = (v_1, v_2, \ldots) = (v_{11}, v_{12}, v_{21}, v_{22}, \ldots) \in \ell_2 \end{aligned}$$

to control the system's state in the game modelled by Eq. (3). These control parameters u and v are functions of time t, and can also be written as $u(t)$ and $v(t)$. The players (a pursuing player and an evading player), operate under geometric constraints, with the pursuing player having greater control resources than the evading player.

To address the control problem, we develop a strategy that guides the system into the origin in ℓ_2 space. We then examine a pursuit game where the pursuing player aims to return the system's state to the origin of ℓ_2 space within a finite time, while the evading player seeks to prevent this. We derive sufficient conditions for successful pursuit within a guaranteed time frame.

This is how the rest of the article is organized: in Section 2, we set the stage by introducing the mathematical model governing the control system, defining key terms, notations and stating the research problems. Section 3 presents the results, divided into Subsection 3.1, which addresses the control problem solution, and Subsection 3.2, focusing on the pursuit differential game solution. Section 4 provides a solid example illustrating our results, The paper concludes in Section 5.

2. Statement of the Problem

Let $\varsigma, \varrho_0, \sigma_0 > 0$, and

$$\begin{aligned} \xi(t) &= (\xi_1(t), \xi_2(t), \ldots) = (x_1(t), y_1(t), x_2(t), y_2(t), \ldots) \\ \xi_i(t) &= (x_i(t), y_i(t))^T, \quad |\xi_i(t)| = \sqrt{x_i^2(t) + y_i^2(t)}, \quad i = 1, 2, \ldots, \\ \|\xi(t)\| &= \sqrt{\sum_{i=1}^{\infty} x_i^2(t) + y_i^2(t)}, \quad \|\xi^0\| = \sqrt{\sum_{i=1}^{\infty} (x_i^0)^2 + (y_i^0)^2} \end{aligned}$$

Denote $\omega(t)$, $t \in [0, \vartheta]$ by $\omega(\cdot)$.

These definitions below are crucial in the discourse:

Definition 2.1. The function

$$\omega[0, \vartheta] \to \ell_2, \quad \omega(\cdot) = (\omega_1(\cdot), \omega_2(\cdot), \ldots)$$

with coordinates $\omega_i = (\omega_{i1}(t), \omega_{i2}(t))$, $i = 1, 2, \ldots$, which are measurable, is termed an admissible control if the geometric constraint criterion

$$\|\omega(t)\| \equiv \sum_{i=1}^{\infty} |\omega_i(t)|^2 \equiv \sum_{i=1}^{\infty}\sum_{j=1}^{2} \omega_{ij}^2(t) \leq \varsigma^2, \quad t \in [0, \vartheta], \quad \varsigma > 0 \qquad i = 1, 2, \ldots$$

is satisfied.

We denote the set of all such admissible controls by $S(\varsigma)$.

Definition 2.2. We refer to the functions $u(\cdot) \in S(\varrho_0)$ and $v(\cdot) \in S(\sigma_0)$ as the pursuing and the evading players' admissible controls, respectively. The quantities ϱ_0^2 and σ_0^2 are the instantaneous control resource limit available to the pursuing player and the evading player respectively.

Definition 2.3. An admissible strategy of the pursuing player is a function $U : [0, \vartheta] \times \ell_2 \to \ell_2$ of the form

$$U(t, v) = (U_1(t, v), U_2(t, v), \ldots) \quad U_i(t, v) = (U_{i1}(t, v), U_{i2}(t, v))^T$$

with components

$$U_i(t, v(t)) = v_i(t) + \omega_i(t), \quad v_i(t) = (v_{i1}(t), v_{i2}(t)), \quad \omega_i(t) = (\omega_{i1}(t), \omega_{i2}(t))$$

satisfying, for every $v(\cdot) \in S(\sigma_0)$, the constraint criterion

$$\sqrt{\sum_{i=1}^{\infty} |U_i(t, v(t))|^2} \leq \varrho_0, \quad |U_i(t, v(t))|^2 = \sum_{j=1}^{2} U_{ij}^2(t, v(t)),$$

where $\omega(\cdot) = (\omega_1(\cdot), \omega_2(\cdot), \ldots) \in S(\varrho_0 - \sigma_0)$.

Definition 2.4. The time $\vartheta > 0$ is termed a guaranteed time of pursuit in the game modelled by Eq. (3) if we can find a pursuing player's admissible strategy such that for any evading player's admissible control, we have $\xi(\theta) = 0$ at some time θ, where $0 \leq \theta \leq \vartheta$.

Let $C(0, \vartheta; \ell_2)$ be the space of continuous functions $\xi(t) \in \ell_2$, $0 \leq t \leq \vartheta$. It is proved in Ibragimov and Kuchkarova (2021) that if $0 \leq \mu_i \leq \mu$, $\mu \in \mathbb{R}^+$, $\lambda_i \in \mathbb{R}$ and $\omega(\cdot) \in S(\varsigma)$, then the infinite system of dyadic ODEs:

$$\begin{cases} \dot{x}_i = \mu_i x_i - \lambda_i y_i + \omega_{i1}(t), & x_i(0) = x_i^0 \\ \dot{y}_i = \lambda_i x_i + \mu_i y_i + \omega_{i2}(t), & y_i(0) = y_i^0, \end{cases} \qquad i = 1, 2, \ldots \tag{4}$$

has a unique solution $\xi(t) = (\xi_1(t), \xi_2(t), \ldots)$, $0 \leq t \leq \vartheta$ in $C(0, \vartheta; \ell_2)$. Indeed, for $i = 1, 2, \ldots$,

$$\xi_i(t) = \Psi_i(t)\xi_i^0 + \int_0^t \Psi_i(t-s)\omega_i(s)ds \tag{5}$$

where

$$\Psi_i(t) = \begin{pmatrix} e^{\mu_i t}\cos\lambda_i t & -e^{\mu_i t}\sin\lambda_i t \\ e^{\mu_i t}\sin\lambda_i t & e^{\mu_i t}\cos\lambda_i t \end{pmatrix}, \qquad i = 1, 2, \ldots.$$

The following property helps us to analyse the trajectory $\xi(t)$, $t \in [0, \vartheta]$ in the space ℓ_2.

Property 2.5. It is easy to ascertain that $\Psi_i(t)$ satisfies:

(a) $\Psi_i^{-1}(t) = \Psi_i(-t)$,
(b) $\Psi_i(t-s) = \Psi_i(t)\Psi_i(-s) = \Psi_i(-s)\Psi_i(t)$,
(c) $|\Psi_i(t)\xi_i| = |\Psi_i^T(t)\xi_i| = e^{\mu_i t}|\xi_i|$,
(d) $\Psi_i(t)\Psi_i^T(t) = e^{2\mu_i t}I_2$,

where we use Ψ^T to represent the matrix transpose of Ψ and I_2 is the identity 2 by 2 matrix.

The research problems and objectives guiding the study are formally stated thus:

Problem 2.6. Find a solution to the associated control problem. That is to say, design an admissible control that can steer the state of the controlled system in Eq. (4) from any nonzero initial state $\xi^0 \in \ell_2$ into the origin within a finite time.

Problem 2.7. Find sufficient conditions that solve the differential game problem. In other words, develop an admissible strategy for the pursuing player that guarantees completion of pursuit from any nonzero initial state $\xi^0 \in \ell_2$ regardless of the control the evading player applies, the condition on the control resources ϱ_0^2 and σ_0^2 under which this happens, and a formula for the guaranteed pursuit time.

3. Results

3.1. *The control problem*

In differential games, it is standard practice to address the associated control problem before resolving the pursuit differential game. Accordingly, we formulate a control $\omega(\cdot)$ that adheres to geometric constraints. We then use this control to steer the control system state into the origin within some finite duration.

For $i = 1, 2, \ldots$, let

$$\phi_i(t) = \int_0^t e^{-2\mu_i s} ds = \begin{cases} \frac{1}{2\mu_i}(1 - e^{-2\mu_i t}), & \mu_i > 0 \\ t, & \mu_i = 0, \end{cases} \qquad t > 0 \tag{6}$$

The following proposition deals with the convergence of the series crucial to our main result.

Proposition 3.1. The series $\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)}$ converges for fixed $t > 0$.

Proof. By the definition of $\phi_i(t)$ in Eq. (6), we analyze the behaviour of $\phi_i(t)$ and check for convergence. To establish the convergence of $\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)}$, we split the sum into two parts:

$$\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)} = \sum_{i\,:\,\mu_i>0} \frac{|\xi_i^0|^2}{\phi_i^2(t)} + \sum_{i\,:\,\mu_i=0} \frac{|\xi_i^0|^2}{\phi_i^2(t)}. \tag{7}$$

- For the terms where $\mu_i > 0$: For large t, $\phi_i(t) \sim \frac{1}{2\mu_i}$, so $\frac{1}{\phi_i^2(t)} \sim 4\mu_i^2$. Since $\xi^0 \in \ell_2$ implies $\sum_{i=1}^{\infty} |\xi_i^0|^2 < \infty$, the series $\sum_{i\,:\,\mu_i>0} \frac{|\xi_i^0|^2}{\phi_i^2(t)}$ converges because the terms behave as constants multiplied by $|\xi_i^0|^2$.
  For small t, $\phi_i(t) \sim t$ and thus $\frac{1}{\phi_i^2(t)} \sim \frac{1}{t^2}$. Here, $\sum_{i\,:\,\mu_i>0} \frac{|\xi_i^0|^2}{\phi_i^2(t)} \sim \frac{1}{t^2}\sum_{i\,:\,\mu_i>0} |\xi_i^0|^2$, which converges as long as $t > 0$.
- For the terms where $\mu_i = 0$: Here, $\phi_i(t) = t$, so $\frac{|\xi_i^0|^2}{\phi_i^2(t)} = \frac{|\xi_i^0|^2}{t^2}$. Then, $\sum_{i\,:\,\mu_i=0} \frac{|\xi_i^0|^2}{\phi_i^2(t)} = \frac{1}{t^2}\sum_{i\,:\,\mu_i=0} |\xi_i^0|^2$, which converges because $\sum_{i=1}^{\infty} |\xi_i^0|^2 < \infty$ due to $\xi^0 \in \ell_2$.

Since both partial sums in Eq. (7) converge for any fixed $t > 0$, we conclude that $\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)}$ converges for any fixed $t > 0$. $\square$

The proposition stated below is needed in formulating the main theorems in the paper.

Proposition 3.2. Suppose that the equation

$$\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)} = \varsigma^2, \qquad \varsigma,\, t > 0 \tag{8}$$

has a positive root, then the root must be unique.

Proof. Suppose for contradiction that Eq. (8) has roots at t_1 and t_2, $t_1 \neq t_2$. Then, this means that

$$\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t_1)} = \varsigma^2 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t_2)} = \varsigma^2.$$

Assume without loss of generality that $t_1 < t_2$. Define

$$f(t) = \sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)}, \quad t > 0.$$

We begin by showing that the function $f(t)$ is strictly monotone.

From the expression for $\phi_i(t)$ in Eq. (6), observe that for each i and

- for $\mu_i > 0$, $\phi_i(t)$ is strictly increasing in t because $1 - e^{-2\mu_i t}$ is increasing in t;
- for $\mu_i = 0$, $\phi_i(t) = t$ is also strictly increasing in t.

Hence, $\phi_i(t)$ is strictly increasing in t for each i. This implies that $\frac{1}{\phi_i^2(t)}$ is strictly decreasing in t for each i. The series $f(t) = \sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)}$ is a sum of strictly decreasing functions in t. Hence, $f(t)$ itself is also strictly decreasing in t. This implies that if $t_1 < t_2$, then $f(t_1) > f(t_2)$. Given that both $f(t_1) = \varsigma^2$ and $f(t_2) = \varsigma^2$, this leads to a contradiction because a strictly decreasing function cannot have the same value at two different points. Thus, the positive root of Eq. (8) is unique. $\square$

Theorem 3.3. *Let $\vartheta_1 > 0$ be the unique root of Eq.* (8)*, then the control*

$$\omega_i(t) = \begin{cases} -\frac{1}{\phi_i(\vartheta_1)} \Psi_i^T(-t)\xi_i^0, & 0 \le t \le \vartheta_1, \\ 0, & t > \vartheta_1 \end{cases}, \quad i = 1, 2, \ldots \tag{9}$$

is admissible and transfers the state of the controlled system in Eq. (4) *to the origin of ℓ_2 space at the time ϑ_1.*

Proof. To show that the control in Eq. (9) is admissible, we prove that it satisfies geometric constraint criterion. Now,

$$\begin{aligned} \sum_{i=1}^{\infty} |\omega_i(t)|^2 &= \sum_{i=1}^{\infty} \left| -\frac{1}{\phi_i(\vartheta_1)} \Psi_i^T(-t)\xi_i^0 \right|^2 \\ &= \sum_{i=1}^{\infty} \frac{1}{\phi_i^2(\vartheta_1)} \left| \Psi_i^T(-t)\xi_i^0 \right|^2 \end{aligned} \tag{10}$$

Applying Property 2.5 (c) in Eq. (10), we obtain that

$$\begin{aligned}\sum_{i=1}^{\infty}|\omega_i(t)|^2 &= \sum_{i=1}^{\infty}\frac{1}{\phi_i^2(\vartheta_1)}\left|\Psi_i^T(-t)\xi_i^0\right|^2\\ &= \sum_{i=1}^{\infty}\frac{1}{\phi_i^2(\vartheta_1)}e^{-2\mu_i t}\left|\xi_i^0\right|^2\end{aligned}$$

Since $t \geq 0$,

$$\begin{aligned}\sum_{i=1}^{\infty}|\omega_i(t)|^2 &= \sum_{i=1}^{\infty}\frac{1}{\phi_i^2(\vartheta_1)}e^{-2\mu_i t}\left|\xi_i^0\right|^2\\ &\leq \sum_{i=1}^{\infty}\frac{1}{\phi_i^2(\vartheta_1)}\left|\xi_i^0\right|^2\\ &= \varsigma^2.\end{aligned}$$

That is, $\omega(\cdot) \in S(\varsigma)$. This shows that the designed control in Eq. (9) conforms to geometric constraint and thus, is admissible.

Next, we show that the system's state (Eq. (4)) can be guided from ξ_i^0 to the ℓ_2 origin at the instant $t = \vartheta_1$ by utilizing the control (Eq. (9)), that is $\xi(\vartheta_1) = 0$. Now, observe that if we use Property 2.5 (b) in Eq. (5), we find that

$$\begin{aligned}\xi_i(\vartheta_1) &= \Psi_i(\vartheta_1)\xi_i^0 + \int_0^{\vartheta_1}\Psi_i(\vartheta_1)\Psi_i(-s)\omega_i(s)ds\\ &= \Psi_i(\vartheta_1)\left[\xi_i^0 + \int_0^{\vartheta_1}\Psi_i(-s)\omega_i(s)ds\right]\\ &= \Psi_i(\vartheta_1)\eta_i(\vartheta_1).\end{aligned}$$

where

$$\eta_i(\vartheta_1) = \xi_i^0 + \int_0^{\vartheta_1}\Psi_i(-s)\omega_i(s)ds. \tag{11}$$

Thus, to show that $\xi(\vartheta_1) = 0$, we only need to establish that $\eta(\vartheta_1) = 0$, where $\eta(\cdot) = (\eta_1(\cdot), \eta_2(\cdot), \ldots)$.

Deploying the control (Eq. (9)) in Eq. (11) gives

$$\begin{aligned}\eta_i(\vartheta_1) &= \xi_i^0 + \int_0^{\vartheta_1}\Psi_i(-s)\left[-\frac{1}{\phi_i(\vartheta_1)}\Psi_i^T(-s)\xi_i^0\right]ds\\ &= \xi_i^0 - \frac{1}{\phi_i(\vartheta_1)}\int_0^{\vartheta_1}\Psi_i(-s)\Psi_i^T(-s)\xi_i^0 ds. \end{aligned}\tag{12}$$

Applying Property 2.5 (d) inside the integrand in Eq. (12) and simplifying, we have that

$$\begin{aligned}\eta_i(\vartheta_1) &= \xi_i^0 - \frac{1}{\phi_i(\vartheta_1)}\left(\int_0^{\vartheta_1} e^{-2\mu_i s} ds\right)\xi_i^0 \\ &= \xi_i^0 - \frac{\phi_i(\vartheta_1)}{\phi_i(\vartheta_1)}\xi_i^0 \\ &= \xi_i^0 - \xi_i^0 \\ &= 0.\end{aligned}$$

Since $\eta_i(\vartheta_1) = 0$ for each i, we have that $\eta(\vartheta_1) = 0$, from which we conclude that $\xi(\vartheta_1) = 0$. Thus, with the designed control (Eq. (9)), the state of the system in Eq. (4) is transferred from the initial state ξ_i^0 into the ℓ_2 origin at the instant $t = \vartheta_1$. □

Having dealt with this preliminary step, we can now turn our attention to solving the pursuit game problem in Eq. (3).

3.2. *The pursuit differential game*

In this subsection, we examine the pursuit differential game whose mathematical model is given by Eq. (3), which describes the motion dynamics of a point in ℓ_2 space controlled by the pursuing and evading players with control parameters $u(\cdot)$ and $v(\cdot)$ respectively. We aim to fulfill the pursuing player's objective by developing an admissible strategy to bring the point to the origin at a certain time, regardless of the evading player's admissible control. The pursuit game is examined under geometric constraints.

The game model in Eq. (3) has the solution $\xi_i(t)$, $i = 1, 2, \ldots$, given by

$$\xi_i(t) = \Psi_i(t)\xi_i^0 + \int_0^t \Psi_i(t)\Psi_i(-s)\left[U_i(s, v_i(s)) - v_i(s)\right] ds. \tag{13}$$

with the pursuing player's strategy contained therein.

Let the equation

$$\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(t)} = (\varrho_0 - \sigma_0)^2, \qquad \varrho_0, \sigma_0 > 0 \tag{14}$$

have a root at $t = \vartheta_2 > 0$. Then, this root is unique, from Proposition 3.2.

Theorem 3.4. *Let $\varrho_0^2 > \sigma_0^2$. The unique root ϑ_2 of Eq.* (14) *is a guaranteed pursuit time in the game in Eq.* (3).

Proof. To prove the theorem, we first formulate a strategy for the pursuing player in the game; then, we check for the admissibility of the strategy; and lastly, we show that the pursuit is completed whenever the pursuing player utilizes that strategy.

We begin by offering to the pursuing player, the following strategy:

$$U_i(t, v_i(t)) = \begin{cases} v_i(t) - \frac{1}{\phi_i(\vartheta_2)}\Psi_i^T(-t)\xi_i^0, & 0 \le t \le \vartheta_2; \\ 0, & t > \vartheta_2, \end{cases} \qquad i = 1, 2, \ldots \tag{15}$$

where $v(\cdot) \in S(\sigma_0)$ is an admissible control of the evading player. Next, we show that the pursuing player's strategy (Eq. (15)) is admissible; that is, geometric constraints are satisfied by the pursuing player's strategy. From Eq. (15),

$$\left(\sum_{i=1}^{\infty} |U_i(t,\, v_i(t))|^2\right)^{1/2} = \left(\sum_{i=1}^{\infty} \left|v_i(t) - \frac{1}{\phi_i(\vartheta_2)}\Psi_i^T(-t)\xi_i^0\right|^2\right)^{1/2} \tag{16}$$

The RHS of Eq. (16) when estimated with Minkowskii's inequality gives

$$\begin{aligned}\left(\sum_{i=1}^{\infty} |U_i(t,\, v_i(t))|^2\right)^{1/2} &\le \left(\sum_{i=1}^{\infty} |v_i(t)|^2\right)^{1/2} + \left(\sum_{i=1}^{\infty} \left|\frac{1}{\phi_i(\vartheta_2)}\Psi_i^T(-t)\xi_i^0\right|^2\right)^{1/2}\\ &= \left(\sum_{i=1}^{\infty} |v_i(t)|^2\right)^{1/2} + \left(\sum_{i=1}^{\infty} \frac{1}{\phi_i^2(\vartheta_2)}e^{-2\mu_i t}|\xi_i^0|^2\right)^{1/2}\end{aligned}$$

Since $t \ge 0$, we have that

$$\begin{aligned}\left(\sum_{i=1}^{\infty} |U_i(t,\, v_i(t))|^2\right)^{1/2} &\le \left(\sum_{i=1}^{\infty} |v_i(t)|^2\right)^{1/2} + \left(\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{\phi_i^2(\vartheta_2)}\right)^{1/2}\\ &= \sigma_0 + \varrho_0 - \sigma_0 = \varrho_0.\end{aligned}$$

This verifies the admissibility of the pursuing player's strategy (Eq. (15)).

Next, we show that the pursuit can be completed whenever the pursuing player applies the strategy in Eq. (15). Now, applying the strategy (Eq. (15)) in Eq. (13), we get that

$$\begin{aligned}\xi_i(\vartheta_2) &= \Psi_i(\vartheta_2)\xi_i^0 + \int_0^{\vartheta_2} \Psi_i(\vartheta_2)\Psi_i(-s)\left[U_i(s,\, v_i(s)) - v_i(s)\right] ds\\ &= \Psi_i(\vartheta_2)\xi_i^0 + \int_0^{\vartheta_2} \Psi_i(\vartheta_2)\Psi_i(-s)\left[-\frac{1}{\phi_i(\vartheta_2)}\Psi_i^T(-s)\xi_i^0\right] ds\\ &= \Psi_i(\vartheta_2)\left\{\xi_i^0 - \frac{1}{\phi_i(\vartheta_2)}\int_0^{\vartheta_2} \Psi_i(-s)\Psi_i^T(-s)\xi_i^0 ds\right\}\\ &= \Psi_i(\vartheta_2)\left\{\xi_i^0 - \frac{1}{\phi_i(\vartheta_2)}\left(\int_0^{\vartheta_2} e^{-2\mu_i s} ds\right)\xi_i^0\right\}\\ &= \Psi_i(\vartheta_2)\left[\xi_i^0 - \frac{\phi_i(\vartheta_2)}{\phi_i(\vartheta_2)}\xi_i^0\right]\\ &= \Psi_i(\vartheta_2)\left[\xi_i^0 - \xi_i^0\right]\\ &= 0.\end{aligned}$$

Thus, pursuit is completed at $t = \vartheta_2$ in the game. We then conclude that ϑ_2 is a guaranteed pursuit time in the game. □

4. Example

We provide the following example to demonstrate how our findings can be analyzed.

Example 4.1. Consider a two-player pursuit differential game modelled by Eq. (3) with the parameters $\mu_i = \frac{1}{2}$, $\lambda_i = 1000$, $\varrho_0 = 3$, $\sigma_0 = 1$, and the initial state $\xi^0 = (x^0, y^0)^T$, where

$$x^0 = \left(\frac{1}{i}\right)_{i=1}^{\infty}, \quad y^0 = \left(\frac{1}{i^2}\right)_{i=1}^{\infty},$$

with the control parameters of the pursuing and evading players conforming to geometric constraints. Specifically, consider a differential game where the pursuing player and the evading player control the states of the dynamic system:

$$\begin{matrix} \dot{x}_i = \frac{1}{2}x_i - 1000y_i + u_{i1} - v_{i1}, & x_i(0) = \frac{1}{i}, \\ \dot{y}_i = 1000x_i + \frac{1}{2}y_i + u_{i2} - v_{i2}, & y_i(0) = \frac{1}{i^2}, \end{matrix} \quad i = 1, 2, \dots$$

where $u = u(t)$ and $v = v(t)$ are the respective control inputs, satisfying the geometric constraints $\|u(t)\| \leq 3$ and $\|v(t)\| \leq 1$ for all t while the game is on. The initial configuration of the system is given by:

$$\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{1}{9} \end{pmatrix}, \dots\right),$$

and the objective of the pursuing player is to drive the system's states from this initial configuration to the zero configuration:

$$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots\right)$$

within a finite time, while the evading player attempts to prevent this. The pursuit is considered complete if and only if the pursuing player realizes his objective. Note that completing the pursuit in the game is challenging since even if one coordinate of the states remains nonzero, the pursuit is considered incomplete.

Since

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \zeta(2) = \frac{\pi^2}{6} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{i^4} = \zeta(4) = \frac{\pi^4}{90} < \infty,$$

it follows that $\xi^0 \in \ell_2$. Substituting $\mu_i = \frac{1}{2}$ into Eq. (6), we obtain:

$$\phi_i(t) = 1 - e^{-t}, \quad i = 1, 2, \dots.$$

Since $\varrho_0 > \sigma_0$, or equivalently $\varrho_0^2 > \sigma_0^2$, the pursuing player has an instantaneous control resource advantage over the evading player. Thus, the hypothesis of Theorem 3.4 is satisfied. According to Theorem 3.4, the guaranteed pursuit time ϑ_2 satisfies Eq. (14), which in this case becomes:

$$\sum_{i=1}^{\infty} \frac{|\xi_i^0|^2}{(1 - e^{-t})^2} = (\varrho_0 - \sigma_0)^2 = 4.$$

Substituting $\sum_{i=1}^{\infty}(x_i^0)^2 = \frac{\pi^2}{6}$ and $\sum_{i=1}^{\infty}(y_i^0)^2 = \frac{\pi^4}{90}$ into the equation yields:

$$\frac{1}{(1 - e^{-t})^2}\left(\frac{\pi^2}{6} + \frac{\pi^4}{90}\right) = 4.$$

Solving for $t > 0$, we find:

$$\vartheta_2 = -\ln\left(1 - \sqrt{\frac{\pi^2}{24} + \frac{\pi^4}{360}}\right).$$

Thus, for any arbitrary control of the evading player $v(t)$, if the pursuing player adopts the strategy in Eq. (15), then the pursuit in the game is guaranteed to be completed at time ϑ_2.

5. Conclusions

In this paper, we presented and examined a zero-sum two-person pursuit differential game modelled by an infinite system of first-order dyadic ODEs in the ℓ_2 space. By establishing admissible control functions under geometric constraints that can transfer states to the origin, we developed strategies enabling the pursuing player to drive the system's state into the origin in a finite time, despite the counteractions of the evading player. For the pursuit differential game problem, we derived sufficient conditions for completing the pursuit from any nonzero initial state in ℓ_2, including the guaranteed pursuit time equation. An example was provided to illustrate our findings, demonstrating how to determine pursuit completion and calculate the guaranteed pursuit time in this complex, infinite-dimensional system.

It should be noted that this research considered two participants in the pursuit game: a pursuing player and an evading player, For future research, we recommend studying the game problem for many pursuing players and many evading players. Pursuit game problems where mixed constraints limit the control resources of the players in the game in Eq. (3) can also be investigated.

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References

Adamu J., Halliru A.S. & Abdulhamid B.M. 2022. On some pursuit differential game problem in a Hilbert space. *Journal of the Nigerian Society of Physical Sciences* **4**(1): 83–87.

Allahabi F. & Mahiub M.A. 2019. A problem of pursuit game with various constraints on controls of players. *International Journal of Partial Differential Equations and Applications* **6**(1): 13–17.

Argoneto P., Perrone G., Renna P., Lo Nigro G., Bruccoleri M. & Noto La Diega S. 2008. *Production Planning in Production Networks*. London, UK: Springer-Verlag London Limited.

Azamov A.A. & Ibaydullayev T.T. 2021. A pursuit-evasion differential game with slow pursuers on the edge graph of a simplex. I. *Automation and Remote Control* **82**: 1996–2005.

Azamov A.A. & Ruziboev M.B. 2013. The time-optimal problem for evolutionary partial differential equations. *Journal of Applied Mathematics and Mechanics* **77**(2): 220–224.

Badakaya A.J., Abdullahi H. & Salimi M. 2024a. A pursuit game in a closed convex set on a Euclidean space. *Differential Equations and Dynamical Systems* **32**: 1215–1224.

Badakaya A.J., Abdulrasheed A.A. & Aguda A. 2021. On two pursuit differential game problems with state and geometric constraints in a Hilbert space. *Uzbek Math. J.* **65**(3): 5–16.

Badakaya A.J., Tsoho S.M. & Salimi M. 2024b. On some ℓ-catch pursuit differential games with different players' dynamic equations. *Differential Equations and Dynamical Systems* pp. 1–13.

Dorothy M., Maity D., Shishika D. & Von Moll A. 2024. One Appolonius circle is enough for many pursuit-evasion games. *Automatica* **163**: 111587.

Ho E., Rajagopalan A., Skvortsov A., Arulampalam S. & Piraveenan M. 2022. Game theory in defence applications: A review. *Sensors* **22**(3): 1032.

Ibragimov G., Akhmedov A., Izzati P.N. & Abdul Manaf N. 2017. Pursuit differential game described by infinite first order 2-systems of differential equations. *Malaysian Journal of Mathematical Sciences* **11**(2): 181–190.

Ibragimov G., Ferrara M., Alias I.A., Salimi M. & Ismail N. 2022a. Pursuit and evasion games for an infinite system of differential equations. *Bull. Malays. Math. Sci. Soc.* **45**: 69–81.

Ibragimov G., Kazimirova R. & Pansera B.A. 2022b. Evasion differential game of multiple pursuers and one evader for an infinite system of binary differential equations. *Mathematics* **10**(23): 4448.

Ibragimov G. & Kuchkarova S. 2021. Existence and uniqueness of solutions of an infinite system of 2-block of differential equations. *Bull. Inst. Math.* **4**(23): 70–75.

Ibragimov G., Kuchkarova S., Hasim R.M. & Pansera B.A. 2022c. Differential games for an infinite system of two-block differential equations. *Mathematics* **10**(14): 2541.

Isaacs R. 1965. *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*. New York, NY: John Wiley and Sons.

Li J., Li C. & Zhang Y. 2024. Guidance strategy of motion camouflage for spacecraft pursuit-evasion game. *Chinese Journal of Aeronautics* **37**(3): 312–319.

Li X., Ma J., Han Z., Zhang Y. & Yang M. 2023. Application of game theory in risk management of urban natural gas pipelines. *J. Loss Prev. Process Ind.* **83**: 105037.

Mamadaliev N.A. & Ibaydullaev T.T. 2021. On the modified third method in the pursuit problem for differential-difference equations of neutral type. *Russian Mathematics* **65**(11): 18–28.

Modarresi J., Ahmadian A., Diabat A. & Elkamel A. 2024. A game theory based scheduling approach for charging coordination of multiple electric vehicles aggregators in smart cities. *Energy* **313**: 133674.

Nabi K.N., Ovi M.A. & Kabir K.M.A. 2024. Analyzing evolutionary game theory in epidemic management: A study on social distancing and mask-wearing strategies. *PLoS ONE* **19**(6): e0301915.

Neumann J.v. & Morgenstern O. 1944. *Theory of Games and Economic Behaviour*. NJ: Princeton University Press.

Nguyen M.D. & Kim S.J. 2020. Application of game theory to analyze the competition and cooperation scenarios among container terminals in Northern Vietnam. *The Asian Journal of Shipping and Logistics* **36**(1): 13–19.

Petrov N.N. 2023. On the problem of pursuing two coordinated evaders in linear recurrent differential games. *Journal of Optimization Theory and Applications* **197**(3): 1011–1023.

Rilwan J., Fotia P. & Ferrara M. 2023. On game value of a differential game problem with Grönwall-type constraints on players control functions. *Decisions in Economics and Finance* **46**(1): 319–333.

Samatov B.T. & Akbarov A.K. 2024. Differential game with a "life line" under the Grönwall constraint on controls. *Vestnik of Saint Peterburg University. Applied Mathematics. Computer Science. Control Processes* **20**(2): 265–280.

Samatov B.T., Horilov M.A. & Akbarov A.A. 2022. Differential game: "Life line" for non-stationary geometric constraints on controls. *Lobachevskii Journal of Mathematics* **43**(1): 237–248.

Samatov B.T., Umaraliyeva N.T. & Uralova S.I. 2021. Differential games with the Langenhop type constrains on controls. *Lobachevskii Journal of Mathematics* **42**(12): 2942–2951.

Satimov N.Y. & Tukhtasinov M. 2006. Game problems on a fixed interval for in controlled first-order evolution equations. *Math. Notes* **80**(3-4): 578–589.

Szöts J. & Harmati I. 2023. Optimal strategies of a pursuit-evasion game with three pursuers and one superior evader. *Robotics and Autonomous Systems* **161**: 104360.

Tukhtasinov M. 1995. Some problems in the theory of differential pursuit games in systems with distributed parameters. *J. Appl. Math. Mech.* **59**(6): 935–940.

Umar B.A., Waziri U., Yusha'u A. & Gwani A.A. 2024a. Differential game of two pursuers chasing one evader with different forms of constraints in ℓ_2-space. *Gadau Journal of Pure and Applied Sciences* **3**(1): 72–78.

Umar B.M., Rilwan J., Aphane M. & Muangchoo K. 2024b. Pursuit and evasion linear differential game problems with generalized integral constraints. *Symmetry* **16**(5): 513.

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