

Original articles

A novel two-derivative multistep collocation method with fitting-techniques with application to Duffing problem

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ABSTRACT

The general k -step fifth-order two-derivative linear multistep collocation method (TDLMM5) using collocation technique with Gegenbauer polynomial as basis function is derived for direct integrating second-order ordinary differential equation in the form $u''(t) = f(t, u(t))$ with periodic solution. Fifth-order two-derivative linear multistep method with various collocation points and off-set points is developed using collocation and interpolation approach. Order, stability, consistency and convergence of TDLMM5 are analyzed and discussed. Then, trigonometrically-fitting technique is adapted into TDLMM5 by setting $u(t)$ as the linear combination of the functions $\{\sin(\lambda t), \cos(\lambda t)\}$, $\lambda \in \mathbb{R}$ and turn the coefficients of TDLMM5 into frequency-dependent. Numerical experiment is conducted to verify the proposed method is superior compared to other existing methods in the literature with similar order. Additionally, the trigonometrically-fitted TDLMM5, denoted as TFTDLMM5, is applied to the well-known damped and driven oscillator problem, known as the Duffing problem. The outcome demonstrates that the proposed method is still successful in modeling this real-world application problem.

1. Introduction

High-order differential equations (ODEs) find extensive utility in forecasting and predicting the evolution of scientific phenomena and application issues, particularly within engineering and physics domains. Examples include their application to electric circuits, damped oscillation and vibration, the study of the Pleiades constellation, classical mechanics, and quantum mechanics [1–5]. The strategic importance of these equations lies in their ability to model systems with high accuracy and predict future states under varying conditions. Advanced numerical methods and computational techniques are often employed to solve these differential equations, especially when analytical solutions are not feasible. The development and refinement of such methods are crucial for enhancing the precision and reliability of predictions in these critical areas. [6–8]. Numerous studies have been undertaken to devise effective techniques for the integration of second-order ordinary differential equations, particularly those exhibiting specific patterns or properties.

In this article, our emphasis lies in the development of a trigonometrically-fitted two-derivative linear multistep method. This method is founded on frequency evaluation techniques and is designed for the solution of a specific class of second-order ordinary differential equations exhibiting periodic solutions as follows:

$$u''(t) = f(t, u(t)),$$

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$$u(t_0) = u_0, \quad u'(t_0) = u'_0, \quad t \geq t_0. \quad (1.1)$$

where $u''(t)$ represents the second derivative of $u(t)$ with respect to time (or some independent variable t).

In recent, high-order direct multistep methods with various basis functions and various kind of efficient techniques have been developed and analyzed to effectively solve high-order ODEs and application problems [9–12]. Awoyemi et al. [13] particularly proposed four-point hybrid linear multistep method using Taylor's series approach. The construction of proposed method involves the interpolation of the power series approximate solution and the collocation of the differential system. The method demonstrates both consistency and zero stability. Numerical results reveal its superior accuracy in comparison to existing methods. Baccouch [14] proposed superconvergent discontinuous Galerkin (DG) method for integrating nonlinear second-order initial-value problems. It establishes optimal a priori error estimates in the L^2 -norm and proves that the method achieves a convergence order of p and superconvergence at downwind points with piecewise polynomials of degree p . Ibrahim et al. [15] introduced two-point variable-order block multistep methods (BMM) with order 3, 4 and 5 for direct solving general class of second-order ODEs. They derived BMM through backward difference interpolation polynomial with two solutions are simultaneously generated at each step. Numerical experiments validated the efficiency of the proposed method by generating consistently low global errors. Rasedee et al. [16] further extended the previous research work by proposing variable-order multistep method with Newton-Gregory backward difference polynomial as basis function in predictor–corrector mode. The predictor–corrector algorithm is further enhanced by incorporating a variable order step-size algorithm to minimize computational costs. Stability and convergence of the proposed method are also established. Singla et al. [17] constructed a hybrid block method as an implicit numerical integrator with adaptive step-size for directly solving general second-order ODEs. Optimization of intermediate points is applied to develop this new efficient method and the numerical results indicate that the new scheme is a strong alternative to existing solvers with similar characteristics. Ayinde et al. [18] developed multistep method by employing interpolation within a finite range with the combination of exponential and trigonometric functions as basis function. Zero stability, absolute stability region and consistency of the proposed method are investigated. The proposed method has been tested numerically and proven to be more reliable when compared to existing linear multistep methods that require the reduction of higher-order equations to a system of first-order equations.

Numerous studies have focused on the advancement of collocation methods for solving differential equations [19–21]. Ahsan et al. [22] introduced a Haar wavelet-based collocation technique specifically designed to address inverse problems with unknown space-dependent heat sources. Their method, which combines finite-difference and Haar wavelet approximations, produces a well-conditioned system of algebraic equations, resulting in a stable and convergent solution that closely aligns with the exact solution, as shown through various numerical examples. A year later, Ahsan et al. [23] enhanced the precision and order of convergence of the established Haar wavelet collocation method. The proposed method is applied to various nonlinear ordinary differential equations under diverse conditions, demonstrating improved stability, convergence and accuracy compared to existing collocation methods. Then, high-order collocation technique based on Haar wavelets are further derived by Ahsan et al. [24] and applied to solve fourth-order nonlinear differential equations with nonlocal integral boundary conditions. The quasi-linearization technique is used to linearize nonlinear fourth-order differential equations, which are then efficiently solved using Haar wavelets.

To further enhance the existing multistep method for increased accuracy and order, there are some schemes incorporate a two-derivative term, which corresponds to the derivative of the f -evaluation. Hojjati et al. [25] presented new class of second derivative multistep method with improved stability region. The second derivative extended backward differentiation formula is employed in a predictor–corrector mode for the method. Yakubu et al. [26] developed two-step multistep method with second-derivative term in the integration of first-order stiff systems. The inclusion of second derivative terms in the methods grants greater flexibility in formulating a set of techniques that exhibit high stability, convergence and larger regions of absolute stability. Lee et al. [27] proposed block hybrid collocation multistep method with four collocation points for direct solving third-order ODEs. They enhanced the existing block method by incorporating the first derivative of a third-order function into the general formulas of the existing method, using the power series method as the basis function. Ramos and Rufai [28] utilized collocation and interpolation techniques in a modified Falkner-type method, incorporating third-order derivatives, to solve systems of second-order initial-value problems. Their approach has been shown to exhibit stability and convergence properties. Majidi et al. [29] derived second derivative linear multistep method using super-future point technique through backward differentiation formulae for solving first-order ODEs. They modified the existing scheme by having same Jacobian matrix for all stages which hugely reduce the computational cost in numerical approximation. Also, the stability and accuracy properties of proposed method are enhanced while simplifying the original structure and computational complexity.

There are some researchers focus on deriving implicit multistep method with the aim to achieve unconditionally stable and better convergence compared to explicit multistep method [30–33]. Tumba et al. [34] presented novel implicit quarter step first derivative block hybrid method in the integration of first-order stiff ODEs. The method acquires consistent, convergent, A-stable and zero-stability properties. Absolute stability region of proposed method is larger and numerical efficiency is greater than other existing explicit multistep methods. Ekoru et al. [35] developed implicit second derivative hybrid linear multistep method with nested predictors based on interpolation and collocation approach using polynomial basis function. Their method is proved to have A-stability property using the boundary locus approach for step-length less than 6. Yakubu et al. [36] introduced multivalued multistep implicit methods with extra-derivatives evaluations in intermediate off-step points located between the well-known step-points. The stability regions of proposed method in the complex plane reveal better stability and convergence, exhibiting expansive regions of absolute stability when compared to existing methods without two-derivative term. The proposed method also demonstrates exceptional performance across a wide range of both linear and nonlinear stiff systems in the form of first-order ODEs, attributed to their high order of accuracy and stiffly accurate characteristic properties.

In the pursuit of integrating differential equations with specific solutions effectively, such as exponential and periodic solutions, many researchers have devised linear or block multistep methods incorporating fitting techniques. You et al. [37] developed a novel set of phase-fitted and P-stable symmetric extended linear multistep (SELM) methods for solving initial value problems involving second-order oscillatory differential equations. Derived from the harmonic fitting condition and order requirements, novel explicit and implicit SELM methods with different orders have been formulated. Mansor et al. [38] presented two-point implicit block multistep methods with trigonometrically-fitting technique for integrating second-order ODEs in the form of $y'' = f(x, y)$ with periodic solutions. Their explicit counterparts are also trigonometrically-fitted, allowing the methods to function as predictor–corrector pairs during implementation for numerical approximations. Numerical results indicate that the new pairs of trigonometrically-fitted methods are superior in terms of accuracy and execution time compared to existing methods. Rufai et al. [39] proposed multistep method with two off-grid point that is formulated through the collocation of a derivative function at both equidistant grid points and off-grid points. Simultaneous approximations to the solution and its derivative are obtained at every point in the interval integration. The inclusion of higher derivatives enhances the method's order, thus improving both its accuracy and stability properties. Simos [40] presented new methodology for the Adams–Bashforth methods with phase lag, amplification factor, phase-fitted and amplification-fitted technique. Stability analysis is conducted towards the proposed methods with different fitting techniques and fitted frequency. All proposed methods are applied to solve first-order application problems with periodic solutions.

In the development of method for integrating second-order ODEs, existing methods include various multistep approaches with fitting techniques and different collocation points. However, there is a notable lack of studies focusing on the development of extra-derivative multistep methods with fitting techniques. Additionally, limited research has been conducted on analyzing the numerical properties of such methods, particularly in terms of stability analysis. It is well-established that incorporating a two-derivative term is an effective way to enhance the accuracy of numerical methods. Furthermore, the difference between the interpolating points and collocating points significantly affects the error generated in subsequent steps and selecting an appropriate difference is crucial for deriving an efficient proposed method. Hence, to address these drawbacks, we propose a two-derivative multistep method with trigonometrical fitting techniques and appropriate distance between the interpolating points and collocating points for solving second-order ODEs with oscillatory solutions. Additionally, a comprehensive stability analysis will be conducted to evaluate the method's robustness and ensure its effectiveness for a wide range of periodic problems.

In the current research field, numerous articles explore the use of various intermediate points. For instance, Omar [41] proposed a ninth-order third-derivative block multistep method utilizing an intermediate point of $3/2$ and employing a power series polynomial as the basis function. Similarly, Soomro et al. [42] developed a fifth-order 3-point block backward differentiation formula incorporating an intermediate point of $5/2$ with a Lagrange polynomial as the basis function. Below are some similarities and differences between these two multistep methods and the proposed method:

Similarities of existing multistep methods:

- Use of Intermediate Points: Similar to existing multistep methods that utilize intermediate points (e.g., $1/2$ or $3/2$), the proposed method incorporates intermediate points to enhance the accuracy of approximations.
- Multistep Framework: Like other multistep methods, the proposed method calculates the solution at multiple steps in a single iteration, reducing computational effort compared to one-step methods.
- Targeted problem: Similar to the existing methods mentioned above, the proposed method specifically targets second-order ODEs.

Differences from other existing multistep methods:

- Fitting Techniques: The proposed method will incorporate trigonometrical fitting techniques, which are also utilized in some existing methods to enhance accuracy for solving ODEs with oscillatory or periodic solutions.
- Choice of Basis Function: Unlike existing methods that typically use power series (e.g., Taylor series) or Lagrange polynomials, the proposed method employs Gegenbauer polynomials, which offer unique orthogonality properties.
- Two-Derivative Term: The proposed method incorporates a two-derivative function in its formulation, whereas the majority of the existing methods rely solely on the f -function, without including the g -function (the derivative of f).

One advantage of using Gegenbauer polynomials is their flexibility in the weighting parameter. Gegenbauer polynomials are orthogonal over the interval $[-1, 1]$ with the weight function $(1 - x^2)^{\alpha - \frac{1}{2}}$. By choosing a suitable α , such as $\alpha = \frac{1}{4}$ for oscillatory solutions, the polynomials can be tailored to match the characteristics of the problem, thereby improving accuracy.

In this study, explicit fifth-order two-derivative multistep method is proposed, denoted as TDLMM5 method for the direct integration of second-order ODEs with periodic solutions. The derivation of TDLMM5 method is constructed with Gegenbauer polynomial up to order five as basis function. The method relies on collocating the differential equation at x_k and x_{k+1} and interpolating the approximate solution through Gegenbauer polynomial at the grid points. The stability region, consistency and convergence properties of the proposed method are thoroughly examined and discussed. Then, trigonometrically-fitting technique is implemented into the proposed method to generate frequency-dependent coefficients that exist in TDLMM5 method for making the proposed method have greater accuracy for solving second-order ODEs with periodic solutions. Numerical tests of both the proposed methods and selected existing multistep methods for the integration of second-order ordinary differential equations with periodic solutions, encompassing renowned application problems such as two-body problem and Pleiades problem, are presented and discussed in this article. The final section concludes with a discussion and summary of the findings.

2. Development of trigonometrical-fitted two-derivative multistep collocation method

2.1. Derivation of two-derivative multistep collocation method

We begin with the general formulation of two-derivative linear multistep method for solving problem (1.1), comprises the derivative of the solution $u'(t) = f'_t(t, u(t)) + f'_u(t, u(t))u' = g(t, u(t), u'(t))$ as below:

$$\sum_{j=0}^k \alpha_j u(t+jh) + \sum_{j=0}^{m_1} \alpha_{v_j} u(t+v_j h) = h^2 \left(\sum_{j=0}^k \beta_j f_{n+j} + \sum_{j=0}^{m_2} \beta_{v_j} f_{n+v_j} \right) + h^3 \left(\sum_{j=0}^k \gamma_j g_{n+j} + \sum_{j=0}^{m_3} \gamma_{v_j} g_{n+v_j} \right), \quad (2.2)$$

where $a_j, \alpha_{v_j}, \beta_j, \beta_{v_j}, \gamma_j, \gamma_{v_j} \in \mathbb{R}$, $m_1, m_2, m_3, k \in \mathbb{Z}^+$ and v_j is non-integer.

To construct TDLMM, we approximate the solution by employing the interpolating function, denoted as $U(t)$, with Gegenbauer polynomials serving as basis functions. According to Stein and Weiss [43], Gegenbauer polynomials can be expressed in relation to their generating function as below:

$$\frac{1}{(1-2zx+x^2)^\xi} = \sum_{n=0}^{\infty} T_n^{(\xi)}(z)x^n, \quad (2.3)$$

where $0 \leq |z| \leq 1, |x| \leq 1, \xi > 0$. The polynomials, $T_n^{(\xi)}(z)$ fulfill the recursive Eq. (2.3), can be expressed as

$$\begin{aligned} T_0^{(\xi)} &= 1, \\ T_1^{(\xi)} &= 2\xi z, \\ (n+1)T_{n+1}^{(\xi)}(z) &= 2(n+\xi)zT_n^{(\xi)}(z) - (n+2\xi-1)T_{n-1}^{(\xi)}. \end{aligned} \quad (2.4)$$

We designate ξ as $\frac{1}{4}$, and subsequently generate the specific Gegenbauer polynomials.

$$\begin{aligned} T_0 &= 1, \\ T_1 &= \frac{1}{2}z, \\ T_2 &= \frac{5}{8}z^2 - \frac{1}{4}, \\ T_3 &= \frac{15}{16}z^3 - \frac{5}{8}z, \\ T_4 &= \frac{195}{128}z^4 - \frac{45}{32}z^2 + \frac{5}{32}, \\ T_5 &= \frac{663}{256}z^5 - \frac{195}{64}z^3 + \frac{45}{64}z. \end{aligned} \quad (2.5)$$

To derive the TDLMM formula, we approximate the solution using an interpolating function in the following equation:

$$U(t) = \sum_{j=0}^{\sigma_1+\sigma_2+\eta-1} a_j T_j(t), \quad (2.6)$$

where $t \in [t_0, b]$, a_j are unknown real coefficients to be determined, T_j is Gegenbauer polynomial, σ_1 and σ_2 the number of interpolations for second derivative and third derivative respectively, η distinct collocation points with $\eta > 0$.

Then, the second derivative of third derivative of interpolating function will be as follows:

$$\begin{aligned} U''(t) &= f(t) = \sum_{j=0}^{\sigma_1+\sigma_2+\eta-1} a_j T_j''(t), \\ U'(t) &= g(t) = \sum_{j=0}^{\sigma_1+\sigma_2+\eta-1} a_j T_j'(t). \end{aligned} \quad (2.7)$$

The continuous approximation is formulated by enforcing the following conditions:

$$\begin{aligned} U(t_{n+\eta}) &= u_{n+\eta}, \quad \eta = 0, 1, \frac{3}{2}, 2, \\ U''(t_{n+\sigma_1}) &= f_{n+\sigma_1}, \quad \sigma_1 = 0, 1, \\ U'(t_{n+\sigma_2}) &= g_{n+\sigma_2}, \quad \sigma_2 = 0, 1, \frac{3}{2}. \end{aligned} \quad (2.8)$$

Here, we solve $U(t_n), U(t_{n+1}), U''(t_n), U''(t_{n+1}), U'(t_n)$ and $U'(t_{n+\frac{3}{2}})$ simultaneously to obtain coefficients $a_j, j = 0, 1, \dots, 5$, then substituting the values a_j into (2.6) and yield the continuous method as follows:

$$U(t) = \sum_{j=0}^1 \alpha_j u_{n+j} + h^2 \sum_{j=0}^1 \beta_j f_{n+j} + h^3 \left(\gamma_0 g_n + \gamma_{\frac{3}{2}} g_{n+\frac{3}{2}} \right). \quad (2.9)$$

Later, we express α_j , β_j and γ_j as continuous coefficients, written as continuous function of x , where $x = \frac{t-t_n}{h}$, to generalize the solution across the interval from t_n to t_{n+1} . This process involves the following steps:

1. Continuous approximation: The interpolating function, $U(t)$ was initially expressed in terms of Gegenbauer polynomials. To simplify the expression and make it applicable across any subinterval $[t_n, t_{n+1}]$, the variable x is introduced. By letting $x = \frac{t-t_n}{h}$, where $h = t_{n+1} - t_n$, the functions can be written as continuous polynomials in x .
2. Substitution and simplification: Once x is substituted, the expressions for the coefficients α_j , β_j and γ_j are derived by expanding the original functions into polynomial forms of x .
3. Polynomial fitting: Coefficients α_j , β_j and γ_j are obtained by matching the coefficients of the interpolating polynomials with the collocation points and their derivatives.

Hence, we obtain

$$\begin{aligned}\alpha_0 &= 1 - x, & \alpha_1 &= x, \\ \beta_0 &= -\frac{39}{100}x + \frac{1}{2}x^2 - \frac{3}{20}x^4 + \frac{1}{25}x^5, \\ \beta_1 &= -\frac{11}{100}x + \frac{3}{20}x^4 - \frac{1}{25}x^5, \\ \gamma_0 &= -\frac{59}{900}x + \frac{1}{6}x^3 - \frac{23}{180}x^4 + \frac{2}{75}x^5, \\ \gamma_{\frac{3}{2}} &= \frac{2}{225}x - \frac{1}{45}x^4 + \frac{1}{75}x^5.\end{aligned}\quad (2.10)$$

Let $x = 2$, we get

$$u_{n+2} = -u_n + 2u_{n+1} + h^2 \left(\frac{1}{10}f_n + \frac{9}{10}f_{n+1} \right) + h^3 \left(\frac{1}{90}g_n + \frac{4}{45}g_{n+\frac{3}{2}} \right). \quad (2.11)$$

To obtain the equation for u'_{n+2} , (2.6) is differentiated with respect to t , substituted by $x = \frac{t-t_n}{h}$ and setting $x = 2$, resulting in:

$$u'_{n+2} = \frac{1}{h} \left[-u_n + u_{n+1} + h^2 \left(\frac{1}{100}f_n + \frac{149}{100}f_{n+1} \right) + h^3 \left(-\frac{19}{900}g_n + \frac{82}{225}g_{n+\frac{3}{2}} \right) \right]. \quad (2.12)$$

In a similar manner, we concurrently solve for $U(t_n)$, $U(t_{n+1})$, $U''(t_n)$, $U''(t_{n+1})$, $U'(t_n)$, and $U'(t_{n+1})$ to derive new coefficients a_j , where $j = 0, 1, \dots, 5$. Subsequently, by substituting these values of a_j into (2.6), we derived continuous method successfully as outlined below:

$$U(t) = \sum_{j=0}^1 \alpha_j u_{n+j} + h^2 \sum_{j=0}^1 \beta_j f_{n+j} + h^3 \sum_{j=0}^1 \gamma_j g_{n+j}. \quad (2.13)$$

Subsequently, when we define α_j , β_j , and γ_j as continuous functions of x with $t_n = t - xh$, we acquire

$$\begin{aligned}\alpha_0 &= 1 - x, & \alpha_1 &= x, \\ \beta_0 &= -\frac{7}{20}x + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{10}x^5, \\ \beta_1 &= -\frac{3}{20}x + \frac{1}{4}x^4 - \frac{1}{10}x^5, \\ \gamma_0 &= -\frac{1}{20}x + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{1}{20}x^5, \\ \gamma_{\frac{3}{2}} &= \frac{1}{30}x - \frac{1}{12}x^4 + \frac{1}{20}x^5.\end{aligned}\quad (2.14)$$

When $x = \frac{3}{2}$, we obtain

$$u_{n+\frac{3}{2}} = -\frac{1}{2}u_n + \frac{3}{2}u_{n+1} + h^2 \left(\frac{3}{32}f_n + \frac{9}{32}f_{n+1} \right) + h^3 \left(\frac{3}{128}g_n + \frac{1}{128}g_{n+1} \right). \quad (2.15)$$

In order to derive the equation for $u'_{n+\frac{3}{2}}$, we differentiate (2.6) with respect to t , substituted by $x = \frac{t-t_n}{h}$, $h = t_{n+1} - t_n$ and setting $x = \frac{3}{2}$, leading to:

$$u'_{n+\frac{3}{2}} = \frac{1}{h} \left[-u_n + u_{n+1} + h^2 \left(\frac{49}{160}f_n + \frac{111}{160}f_{n+1} \right) + h^3 \left(\frac{29}{320}g_n + \frac{167}{960}g_{n+1} \right) \right]. \quad (2.16)$$

Therefore, the complete formula for the two-step fifth-order TDLMM, denoted as TDLMM5 method is provided below:

$$\begin{aligned}u_{n+\frac{3}{2}} &= -\frac{1}{2}u_n + \frac{3}{2}u_{n+1} + h^2 \left(\frac{3}{32}f_n + \frac{9}{32}f_{n+1} \right) + h^3 \left(\frac{3}{128}g_n + \frac{1}{128}g_{n+1} \right), \\ u_{n+2} &= -u_n + 2u_{n+1} + h^2 \left(\frac{1}{10}f_n + \frac{9}{10}f_{n+1} \right) + h^3 \left(\frac{1}{90}g_n + \frac{4}{45}g_{n+\frac{3}{2}} \right), \\ hu'_{n+\frac{3}{2}} &= -u_n + u_{n+1} + h^2 \left(\frac{49}{160}f_n + \frac{111}{160}f_{n+1} \right) + h^3 \left(\frac{29}{320}g_n + \frac{167}{960}g_{n+1} \right),\end{aligned}$$

$$hu'_{n+2} = -u_n + u_{n+1} + h^2 \left(\frac{1}{100} f_n + \frac{149}{100} f_{n+1} \right) + h^3 \left(-\frac{19}{900} g_n + \frac{82}{225} g_{n+\frac{3}{2}} \right). \quad (2.17)$$

2.2. Trigonometrically-fitted two-derivative linear multistep method

In developing the TFTDLM5 method with a trigonometric fitting technique, we replace some coefficients in TDLMM5 method with frequency-dependent coefficients, K_i and L_i , $i = 1, 2, 3, 4$ as follows:

$$\begin{aligned} u_{n+\frac{3}{2}} &= -\frac{1}{2}u_n + \frac{3}{2}u_{n+1} + h^2 \left(K_1 f_n + \frac{9}{32} f_{n+1} \right) + h^3 \left(K_2 g_n + \frac{1}{128} g_{n+1} \right), \\ u_{n+2} &= -u_n + 2u_{n+1} + h^2 \left(\frac{1}{10} f_n + K_3 f_{n+1} \right) + h^3 \left(\frac{1}{90} g_n + K_4 g_{n+\frac{3}{2}} \right), \\ hu'_{n+\frac{3}{2}} &= -u_n + u_{n+1} + h^2 \left(L_1 f_n + \frac{111}{160} f_{n+1} \right) + h^3 \left(L_2 g_n + \frac{167}{960} g_{n+1} \right), \\ hu'_{n+2} &= -u_n + u_{n+1} + h^2 \left(\frac{1}{100} f_n + L_3 f_{n+1} \right) + h^3 \left(-\frac{19}{900} g_n + L_4 g_{n+\frac{3}{2}} \right). \end{aligned} \quad (2.18)$$

Initially, we set $u_n = e^{i\lambda t}$, $u_{n+p} = e^{i\lambda t} e^{piv}$, $p = 1, \frac{3}{2}, 2$, and λ represents the frequency. Subsequently, we determine the first derivative, u'_{n+p} , second derivative, f_{n+p} and third derivative, g_{n+p} using the formula u_n and u_{n+p} as mentioned above. Then, exponential functions $e^{i\lambda t}$ and $e^{-i\lambda t}$ are integrated at each stage, where $v = \lambda h$ and $\lambda \in \mathbb{R}$, we obtain the equations corresponding to u , hu' below:

$$\begin{aligned} e^{\pm \frac{1}{2}iv} &= -\frac{1}{2}e^{\mp iv} + \frac{3}{2} - v^2 \left(K_1 e^{\mp iv} + \frac{9}{32} \right) \mp iv^3 \left(K_2 e^{\mp iv} + \frac{1}{128} \right), \\ e^{\pm iv} &= \mp e^{-iv} + 2 - v^2 \left(\frac{1}{10} e^{\mp iv} + K_3 \right) \mp iv^3 \left(\frac{1}{90} e^{\mp iv} + K_4 e^{\pm \frac{1}{2}iv} \right), \end{aligned} \quad (2.19)$$

$$\begin{aligned} \pm iv e^{\pm \frac{1}{2}iv} &= -e^{\mp iv} + 1 - v^2 \left(L_1 e^{\mp iv} + \frac{111}{160} \right) \mp iv^3 \left(L_2 e^{\mp iv} + \frac{167}{960} \right), \\ \pm iv e^{\pm iv} &= -e^{\mp iv} + 1 - v^2 \left(\frac{1}{100} e^{\mp iv} + L_3 \right) \mp iv^3 \left(-\frac{19}{900} e^{\mp iv} + L_4 e^{\pm \frac{1}{2}iv} \right). \end{aligned} \quad (2.20)$$

The relation $\cos(v) = \frac{e^{iv} + e^{-iv}}{2}$ and $\sin(v) = \frac{e^{iv} - e^{-iv}}{2i}$ are substituted in the Eqs. (2.19) corresponding to u , we get hyperbolic functions of v below:

$$\begin{aligned} \cos\left(\frac{v}{2}\right) &= -\frac{1}{2}\cos(v) + \frac{3}{2} - v^2 \left(K_1 \cos(v) + \frac{9}{32} \right) - v^3 \left(K_2 \sin(v) \right), \\ \sin\left(\frac{v}{2}\right) &= \frac{1}{2}\sin(v) + v^2 K_1 \sin(v) - v^3 \left(K_2 \cos(v) + \frac{1}{128} \right), \\ \cos(v) &= 1 - v^2 \left(\frac{1}{20} \cos(v) + \frac{1}{2} K_3 \right) + v^3 \left[-\frac{1}{180} \sin(v) + \frac{1}{2} K_4 \sin\left(\frac{1}{2}v\right) \right], \\ \sin(v) &= \sin(v) + v^2 \left(\frac{1}{10} \sin(v) \right) - v^3 \left[\frac{1}{90} \cos(v) + K_4 \cos\left(\frac{1}{2}v\right) \right]. \end{aligned} \quad (2.21)$$

Further solving (2.21) and apply Taylor series expansion, we obtain the frequency-dependent parameters of K_i , $i = 1, 2, 3, 4$

$$\begin{aligned} K_1 &= \frac{3}{32} + \frac{11}{15360}v^4 - \frac{491}{3440640}v^6 + \frac{8611}{1238630400}v^8 - \frac{12259}{72666316800}v^{10} + \mathcal{O}(v^{12}), \\ K_2 &= \frac{3}{128} + \frac{13}{30720}v^4 - \frac{311}{8847360}v^6 + \frac{115}{99090432}v^8 - \frac{9511}{435997900800}v^{10} + \mathcal{O}(v^{12}), \\ K_3 &= \frac{9}{10} - \frac{1}{720}v^4 + \frac{1}{11200}v^6 + \frac{17}{3628800}v^8 + \frac{139}{299376000}v^{10} + \mathcal{O}(v^{12}), \\ K_4 &= \frac{4}{45} + \frac{1}{7200}v^4 + \frac{1}{67200}v^6 + \frac{173}{116121600}v^8 + \frac{827}{5474304000}v^{10} + \mathcal{O}(v^{12}). \end{aligned} \quad (2.22)$$

In a similar manner, we incorporate the relationship between $\cos(v)$ and $\sin(v)$ into the Eqs. (2.20), which correspond to hu' . As a result, we obtain trigonometric functions of v as follows:

$$\begin{aligned} v \sin\left(\frac{v}{2}\right) &= \cos(v) - 1 + v^2 \left(L_1 \cos(v) + \frac{111}{160} \right) + v^3 \left(L_2 \sin(v) \right), \\ v \cos\left(\frac{v}{2}\right) &= \sin(v) + v^2 L_1 \sin(v) - v^3 \left(L_2 \cos(v) + \frac{167}{960} \right), \\ v \cos(v) &= \sin(v) + v^2 \left(\frac{1}{100} \sin(v) \right) + v^3 \left[\frac{19}{900} \cos(v) - L_4 \cos\left(\frac{1}{2}v\right) \right], \\ v \sin(v) &= \cos(v) - 1 + v^2 \left(\frac{1}{100} \cos(v) + L_3 \right) - v^3 \left[\frac{19}{900} \sin(v) + L_4 \sin\left(\frac{1}{2}v\right) \right]. \end{aligned} \quad (2.23)$$

Subsequently, the coefficients above are used to generate parameters L_i , $i = 1, 2, 3, 4$ through Taylor series expansion

$$\begin{aligned} L_1 &= \frac{49}{160} + \frac{23}{5760}v^4 - \frac{3071}{3225600}v^6 + \frac{50111}{928972800}v^8 - \frac{203597}{136249344000}v^{10} + \mathcal{O}(v^{12}), \\ L_2 &= \frac{29}{320} + \frac{4181}{1612800}v^4 - \frac{23561}{92897280}v^6 + \frac{394019}{40874803200}v^8 - \frac{8731067}{42509795328000}v^{10} + \mathcal{O}(v^{12}), \end{aligned}$$

$$\begin{aligned}
L_3 &= \frac{149}{100} - \frac{13}{7200}v^4 + \frac{223}{336000}v^6 + \frac{37}{1451520}v^8 + \frac{10609}{2993760000}v^{10} + \mathcal{O}(v^{12}), \\
L_4 &= \frac{82}{225} + \frac{607}{504000}v^4 + \frac{1907}{18144000}v^6 + \frac{19649}{1824768000}v^8 + \frac{1810859}{1660538880000}v^{10} + \mathcal{O}(v^{12}).
\end{aligned} \quad (2.24)$$

As v approaches 0, the coefficients K_i and L_i , $i = 1, 2, 3, 4$ of the proposed methods with fitting technique will revert to the coefficients of the classical form.

3. Numerical analysis of TFTDLMM5 method

Referring to Ibrahim and Nasarudin [44] and generalization from the theorem of Henrici [45], we introduce the zero stability associated with TFTDLMM method in Definition 1.

Definition 1 (Zero stable). A block multistep method with order p is zero stable provided the roots, R_i for $i = 1, \dots, k$ of the characteristic polynomial, $\rho(R)$ such that:

$$\rho(R) = \det \left[\sum_{m=1}^k K^{(m)} R^{(k-m)} \right], \quad K^{(0)} = I, \quad (3.25)$$

satisfies the following conditions:

1. $|R_m| \leq 1$ for $m = 1, 2, \dots, k$,
2. If R_m is a repeated root, then the multiplicity of the root of modulus 1 must be at most p ,

where I is identity matrix and $K^{(m)}$ is $k \times k$ matrix that appears in the construction of the block multistep method's characteristic polynomial.

Zero stability is the method's ability to control the propagation of errors introduced in initial values as the step number increases. A method is zero-stable if small perturbations in the initial conditions do not lead to unbounded growth in the numerical solution. The focus of zero stability is on the behavior of the method near the limit of zero step size. For example, if a method is zero-stable, it means that the errors do not grow uncontrollably as the grid gets finer. It is different with linear stability that refers to the behavior of a numerical method when applied to a linear problem, often in the context of time-stepping algorithms for solving ODEs or PDEs. Linear stability examines the growth or decay of perturbations in the solution using a linearized analysis, often through the linear test equation. The stability region, typically derived from a linear stability test, is plotted in the complex plane and illustrates the values of λ (or eigenvalues) for which the numerical method remains stable. This region helps in determining the step sizes for which the method will effectively solve the problem without introducing excessive errors. Functions of the stability region include guiding the choice of appropriate time step sizes for time-stepping schemes and ensuring that the method maintains controlled growth of errors during computation.

Linear stability analysis focuses on the evolution of numerical errors over time, while zero stability ensures that the method behaves appropriately as the step size approaches zero. Both are crucial for the accuracy and reliability of numerical methods in solving differential equations. In this study, the linear stability analysis begins by deriving the first characteristic polynomial associated with the proposed method, using an appropriate test problem. The roots of this polynomial are then examined to investigate the zero-stability property. Additionally, the eigenvalues of the first characteristic polynomial are analyzed to construct the stability region of the TFTDLMM5 method, providing insight into its stability performance.

To construct the first characteristic polynomial, we use the following second-order linear test problem:

$$u'' = -\lambda^2 u. \quad (3.26)$$

Apply TFTDLMM5 method into the test problem and substitute $v = \lambda h$, we obtain

$$u_{n+\frac{3}{2}} = \left(-\frac{1}{2} - K_1 v^2\right) u_n + \left(\frac{3}{2} - \frac{9}{32} v^2\right) u_{n+1} + (-K_2 v^2) hu'_n + \left(-\frac{1}{128} v^2\right) hu'_{n+1}, \quad (3.27)$$

$$u_{n+2} = \left(-1 - \frac{1}{10} v^2\right) u_n + (2 - K_3 v^2) u_{n+1} + \left(-\frac{1}{90} v^2\right) hu'_n + (-K_4 v^2) hu'_{n+\frac{3}{2}}, \quad (3.28)$$

$$hu'_{n+\frac{3}{2}} = (-1 - L_1 v^2) u_n + \left(1 - \frac{111}{160} v^2\right) u_{n+1} + (-L_2 v^2) hu'_n + \left(-\frac{167}{960} v^2\right) hu'_{n+1}, \quad (3.29)$$

$$hu'_{n+2} = \left(-1 - \frac{1}{100} v^2\right) u_n + (1 - L_3 v^2) u_{n+1} + \left(\frac{19}{900} v^2\right) hu'_n + (-L_4 v^2) hu'_{n+\frac{3}{2}}. \quad (3.30)$$

Differentiating (3.27) and multiplying both sides by h , we get

$$hu'_{n+\frac{3}{2}} = \left(-\frac{1}{2} - K_1 v^2\right) hu'_n + \left(\frac{3}{2} + \frac{9}{32} v^2\right) hu'_{n+1} + (K_2 v^4) u_n + \left(\frac{1}{128} v^4\right) u_{n+1} \quad (3.31)$$

Next, substitute (3.31) into (3.28),

$$\begin{aligned}
u_{n+2} &= \left(-1 - \frac{1}{10} v^2 - K_4 K_2 v^6\right) u_n + \left(2 - K_3 v^2 - \frac{1}{128} K_4 v^4\right) u_{n+1} \\
&\quad + \left[-\frac{1}{90} v^2 - K_4 v^2 \left(-\frac{1}{2} - K_1 v^2\right)\right] hu'_n + \left[-K_4 v^2 \left(\frac{3}{2} + \frac{9}{32} v^2\right)\right] hu'_{n+1}.
\end{aligned} \quad (3.32)$$

Same goes to the derivative of u in TFTDLMM5 method, we substitute (3.29) into (3.30), we obtain

$$\begin{aligned} hu'_{n+2} = & \left[-1 - \frac{1}{100}v^2 - L_4v^2(-1 - L_1v^2) \right] u_n + \left[1 - L_3v^2 - L_4v^2 \left(1 - \frac{111}{160}v^2 \right) \right] u_{n+1} \\ & + \left(\frac{19}{900}v^2 + L_2L_4v^4 \right) hu'_n + \left(\frac{1}{128}L_4v^4 \right) hu'_{n+1}. \end{aligned} \quad (3.33)$$

We can summarize (3.32) and (3.33) into

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{n+2} \\ hu'_{n+2} \end{pmatrix} = K_1(v) \begin{pmatrix} u_{n+1} \\ hu'_{n+1} \end{pmatrix} + K_2(v) \begin{pmatrix} u_n \\ hu'_n \end{pmatrix}, \quad (3.34)$$

where

$$\begin{aligned} K_1(v) &= \begin{pmatrix} 2 - K_3v^2 - \frac{1}{128}K_4v^4 & -K_4v^2 \left(\frac{3}{2} + \frac{9}{32}v^2 \right) \\ 1 - L_3v^2 - L_4v^2 \left(1 - \frac{111}{160}v^2 \right) & \frac{1}{128}L_4v^4 \end{pmatrix}, \\ K_2(v) &= \begin{pmatrix} -1 - \frac{1}{10}v^2 - K_4K_2v^6 & -\frac{1}{90}v^2 - K_4v^2 \left(-\frac{1}{2} - K_1v^2 \right) \\ -1 - \frac{1}{100}v^2 - L_4v^2(-1 - L_1v^2) & \frac{19}{900}v^2 + L_2L_4v^4 \end{pmatrix}. \end{aligned}$$

Then, we substitute $\begin{pmatrix} u_{n+2} \\ hu'_{n+2} \end{pmatrix} = R^2$, $\begin{pmatrix} u_{n+1} \\ hu'_{n+1} \end{pmatrix} = R$ and $\begin{pmatrix} u_n \\ hu'_n \end{pmatrix} = 1$, we yield the following first characteristic polynomial

$$\rho(R, v) = R^2 - K_1(v)R - K_2(v). \quad (3.35)$$

Determine the determinant of first characteristic polynomial and set $v = 0$, we get the stability polynomial as follows:

$$R^4 - 2R^3 + R^2 = 0. \quad (3.36)$$

Hence, the roots of stability polynomial are 0,0,1,1. All of the roots have modulus less or equal to one, which satisfied the zero stable conditions given in Definition 1. Thus, we conclude that TFTDLMM5 is zero stable.

Next, by setting $\rho(R, v) = 0$ and solving for R in terms of v within the matrix, we obtain the 2×2 matrix $P(v)$. The stability region in complex plane of TFTDLMM5 method, S_R can then be defined as

$$S_R = \{v : |\lambda_i(P(v))| < 1, i = 1, 2\}, \quad (3.37)$$

where λ_i are eigenvalues of $P(v)$. The stability region of TFTDLMM5 method is shown in Fig. 1.

Additionally, we introduce some definitions related to consistency, order and convergence for the linear multistep method as follows:

Definition 2 (Consistency). The linear multistep method acquires consistency if it has the order of $\rho \geq 1$ [44].

Definition 3 (Order). The linear multistep method associated with the linear difference operator is said to be of order ρ if all the error constants with order less than ρ are equal to zero [46].

Definition 4 (Convergence). According to the Lax Equivalence Theorem, a linear multistep method is convergent if and only if it possesses both zero stability and consistency [47].

To determine the error constant, $C_n, n = 0, 1, \dots, \rho + 1$, we set $v = 0$ and rearrange TFTDLMM5 method into:

$$\alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_{\frac{3}{2}} u_{n+\frac{3}{2}} + \alpha_2 u_{n+2} = h \left(\beta_{\frac{3}{2}} u'_{n+\frac{3}{2}} + \beta_2 u'_{n+2} \right) + h^2 (\gamma_0 f_n + \gamma_1 f_{n+1}) + h^3 \left(\zeta_0 g_n + \zeta_1 g_{n+1} + \zeta_{\frac{3}{2}} g_{n+\frac{3}{2}} \right), \quad (3.38)$$

where

$$\begin{aligned} \alpha_0 &= \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} -3/2 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \quad \alpha_{\frac{3}{2}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_{\frac{3}{2}} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \\ \gamma_0 &= \begin{pmatrix} 3/32 \\ 1/10 \\ 49/160 \\ 1/100 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 9/32 \\ 9/10 \\ 111/160 \\ 149/100 \end{pmatrix}, \quad \zeta_0 = \begin{pmatrix} 3/128 \\ 1/90 \\ 29/320 \\ -19/900 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 1/128 \\ 0 \\ 167/960 \\ 0 \end{pmatrix}, \quad \zeta_{\frac{3}{2}} = \begin{pmatrix} 0 \\ 4/45 \\ 0 \\ 82/225 \end{pmatrix}. \end{aligned}$$

Referring to [46], the linear difference operator L associated with TFTDLMM5 method is

$$L[u(t), h] = C_0 u(t) + C_1 h u'(t) + C_2 h^2 u''(t) + \dots + C_q h^q u^{(q)}(t), \quad (3.39)$$

where $q = 0, 1, \dots, \rho + 1$ and $C_n, n = 0, 1, \dots, q$ is error constant.

Adapted to the TFTDLMM5 method, the error constants are as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_{\frac{3}{2}} + \alpha_2 = \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -3/2 \\ -2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C_1 = 0(\alpha_0) + 1(\alpha_1) + \frac{3}{2}(\alpha_{\frac{3}{2}}) + 2\alpha_2 - \left(\beta_{\frac{3}{2}} + \beta_2\right) = 0 \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -3/2 \\ -2 \\ -1 \\ -1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.40)$$

$$C_2 = \frac{1}{2} \left[0^2(\alpha_0) + 1^2(\alpha_1) + \left(\frac{3}{2}\right)^2(\alpha_{\frac{3}{2}}) + 2^2(\alpha_2) \right] - \left[\frac{3}{2}(\beta_{\frac{3}{2}}) + 2(\beta_2) \right] - (\gamma_0 + \gamma_1)$$

$$= \frac{1}{2} \left[0^2 \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1^2 \begin{pmatrix} -3/2 \\ -2 \\ -1 \\ -1 \end{pmatrix} + \left(\frac{3}{2}\right)^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] - \frac{3}{2} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 3/32 \\ 1/10 \\ 49/160 \\ 1/100 \end{pmatrix} - \begin{pmatrix} 9/32 \\ 9/10 \\ 111/160 \\ 149/100 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C_3 = \frac{1}{3} \left[0^3(\alpha_0) + 1^3(\alpha_1) + \left(\frac{3}{2}\right)^3(\alpha_{\frac{3}{2}}) + 2^3(\alpha_2) \right] - \frac{1}{2} \left[\left(\frac{3}{2}\right)^2(\beta_{\frac{3}{2}}) + 2^2(\beta_2) \right] - [0(\gamma_0) + 1(\gamma_1)] - \left(\zeta_0 + \zeta_1 + \zeta_{\frac{3}{2}}\right)$$

$$= \frac{1}{6} \left[0^3 \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1^3 \begin{pmatrix} -3/2 \\ -2 \\ -1 \\ -1 \end{pmatrix} + \left(\frac{3}{2}\right)^3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2^3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] - \frac{1}{2} \left[\left(\frac{3}{2}\right)^2 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + 2^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right] - 0 \begin{pmatrix} 3/32 \\ 1/10 \\ 49/160 \\ 1/100 \end{pmatrix} - 1 \begin{pmatrix} 9/32 \\ 9/10 \\ 111/160 \\ 149/100 \end{pmatrix}$$

$$- \begin{pmatrix} 3/128 \\ 1/90 \\ 29/320 \\ -19/900 \end{pmatrix} - \begin{pmatrix} 1/128 \\ 0 \\ 167/960 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4/45 \\ 0 \\ 82/225 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C_4 = \frac{1}{4} \left[0^4(\alpha_0) + 1^4(\alpha_1) + \left(\frac{3}{2}\right)^4(\alpha_{\frac{3}{2}}) + 2^4(\alpha_2) \right] - \frac{1}{3} \left[\left(\frac{3}{2}\right)^3(\beta_{\frac{3}{2}}) + 2^3(\beta_2) \right] - \frac{1}{2} [0^2(\gamma_0) + 1^2(\gamma_1)]$$

$$- \left[0(\zeta_0) + 1(\zeta_1) + \frac{3}{2}(\zeta_{\frac{3}{2}}) \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.41)$$

$$C_5 = \frac{1}{5} \left[0^5(\alpha_0) + 1^5(\alpha_1) + \left(\frac{3}{2}\right)^5(\alpha_{\frac{3}{2}}) + 2^5(\alpha_2) \right] - \frac{1}{4} \left[\left(\frac{3}{2}\right)^4(\beta_{\frac{3}{2}}) + 2^4(\beta_2) \right] - \frac{1}{3} [0^3(\gamma_0) + 1^3(\gamma_1)]$$

$$- \frac{1}{2} \left[0^2(\zeta_0) + 1^2(\zeta_1) + \left(\frac{3}{2}\right)^2(\zeta_{\frac{3}{2}}) \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C_6 = \frac{1}{6} \left[0^6(\alpha_0) + 1^6(\alpha_1) + \left(\frac{3}{2}\right)^6(\alpha_{\frac{3}{2}}) + 2^6(\alpha_2) \right] - \frac{1}{5} \left[\left(\frac{3}{2}\right)^5(\beta_{\frac{3}{2}}) + 2^5(\beta_2) \right] - \frac{1}{4} [0^4(\gamma_0) + 1^4(\gamma_1)]$$

$$- \frac{1}{3} \left[0^3(\zeta_0) + 1^3(\zeta_1) + \left(\frac{3}{2}\right)^3(\zeta_{\frac{3}{2}}) \right] = \begin{pmatrix} 11/15360 \\ -1/720 \\ 23/5760 \\ -13/7200 \end{pmatrix}.$$

Since $C_0 = C_1 = \dots = C_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $C_6 \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, thus, TFTDLMM5 is proven to have an order of 5. Also, TFTDLMM5 is of order,

$\rho = 5 \geq 1$, it is consistency by definition. The error constant of the TFTDLMM5 method up to order 6 is equal to C_6 mentioned above.

Since TFTDLMM5 method is zero-stable and consistent, referring to Lax equivalence theorem, TFTDLMM5 method is convergent.

By the fundamental theorem of numerical analysis, if a method is both consistent and stable, the global error, E_n tends to zero as $n \rightarrow \infty$ and $h \rightarrow 0$. In the analysis above, we have

$$E_n = O(h^{p+1}), \quad (3.42)$$

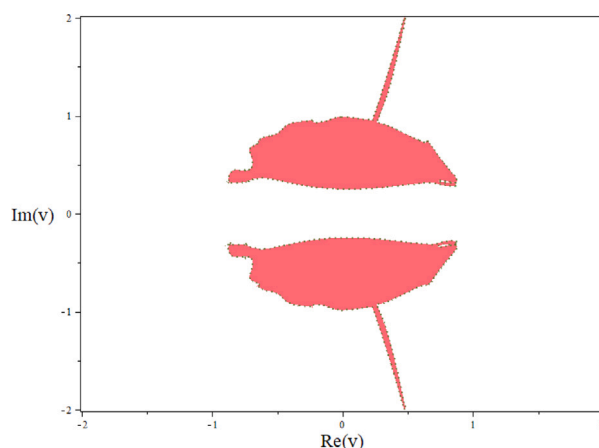


Fig. 1. The stability region of TFTDLMM5 method.

where $p = 5$ is the order of TFTDLMM5 method. This implies that the global error goes to zero as the step size decreases, establishing the convergence of the method.

4. Numerical tests and results

In this section, we apply the TFTDLMM5 method to solve second-order ordinary differential equations of the form $u'' = f(t, u(t))$ and an application problem featuring a periodic solution. The efficiency of the proposed method in the literature is demonstrated by comparing it with various existing linear multistep methods, including classical-type and fitted techniques. The following methods have been selected for numerical comparison.

- **TFTDLMM5** - Trigonometrically-fitted two derivative linear multistep method with fifth-order proposed in this paper.
- **TFLMMA4** - Trigonometrically-fitted extra derivative multistep method with fourth-order, proposed by Ahmad et al. [48]
- **LFDMM6** - Linear finite difference method with sixth-order, proposed by Jator [49]
- **TFTDLMM(S3)** - Trigonometrically-fitted block multistep methods in predictor–corrector mode with three step number, $k = 3$, proposed by Mansor et al. [38]
- **TFTDLMM(S5)** - Trigonometrically-fitted block multistep methods in predictor–corrector mode with five step number, $k = 5$, proposed by Mansor et al. [38]

In each second-order oscillatory initial value problem (IVP), there is a specific fitted frequency, λ , that can be identified from the analytical solutions. The value of $v = \lambda h$, where h is the step size, will be determined and then substituted into the frequency-dependent parameters for all selected trigonometrically-fitted methods, including the proposed method. Six numerical problems are selected, including some application problems, two-body problem and Duffing problem. Five selected methods are utilized to solve all the problems with different step-size and endpoints, b . Then the numerical approximation generated by all selected methods will be compared with analytical solution to calculate maximum global error. For the Duffing problem, since there is no analytical solution, the classical fourth-order Runge–Kutta method with an extremely low step size, $h = 10^{-6}$ will be used to obtain the estimated maximum global error for the selected methods.

Problem 1. Homogeneous linear problem studied by [50]

$$u'' = -25u(t),$$

$$u(0) = 1, \quad u'(0) = 1, \quad t \in [0, 1000],$$

with analytical solution, $u(t) = \frac{1}{5} \sin(5t) + \cos(5t)$.

The fitted frequency, $\lambda = 5$.

Problem 2. Inhomogeneous linear problem studied by [50]

$$u'' = -10000u(t) + (\cos(t))^2,$$

$$u(0) = 1, \quad u'(0) = 1, \quad t \in [0, 100],$$

with analytical solution $u(t) = \frac{1}{100} \sin(100t) + \frac{24985001}{49980000} \cos(100t) + \frac{1}{19992} \cos(2t) + \frac{1}{20000}$.

The fitted frequency, $\lambda = 100$.

Problem 3. Homogeneous linear system [50]

$$u_1''(t) = 2498u_1(t) + 4998u_2(t), \quad u_2''(t) = -2499u_1(t) + 4999u_2(t),$$

$$u_1(0) = 2, \quad u_1'(0) = 0, \quad u_2(0) = -1, \quad u_2'(0) = 0, \quad t \in [0, 100],$$

with analytical solution, $u_1(t) = 2\cos(t)$ and $u_2(t) = -\cos(t)$.

The fitted frequency, $\lambda = 1$.

Problem 4. Two-body problem (first type) studied by [51]

$$u_1''(t) = \frac{-u_1(t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^3}, \quad u_2''(t) = \frac{-u_2(t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^3},$$

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_2(0) = 0, \quad u_2'(0) = 1,$$

(4.43)

with exact solution $u_1(t) = \cos(t)$ and $u_2(t) = \sin(t)$.

The fitted frequency, $\lambda = 1$.

Problem 5. Two-body problem with nonlinear orbital property (second type) studied by [52]

$$u_1''(t) = \frac{2u_1(t)u_2(t) - \sin(2\lambda t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^3}, \quad u_2''(t) = \frac{u_1^2(t) - u_2^2(t) - \cos(2\lambda t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^3},$$

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_2(0) = 0, \quad u_2'(0) = \lambda,$$

(4.44)

with exact solution is $u_1(t) = \cos(10t)$ and $u_2(t) = \sin(10t)$.

The fitted frequency, $\lambda = 10$.

Problem 6. Nonlinear perturbed Kepler problem with orbital property studied by [53]

$$u_1''(t) = -\frac{u_1(t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^3} - \frac{(2\epsilon + \epsilon^2)u_1(t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^5}, \quad u_2''(t) = -\frac{u_2(t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^3} - \frac{(2\epsilon + \epsilon^2)u_2(t)}{\left(\sqrt{u_1^2(t) + u_2^2(t)}\right)^5},$$

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_2(0) = 0, \quad u_2'(0) = 1 + \epsilon,$$

(4.45)

with exact solution is $u_1(t) = \cos[(1 + \epsilon)t]$ and $u_2(t) = \sin[(1 + \epsilon)t]$.

The fitted frequency, $\lambda = 1 + \epsilon$.

In this study, we use a perturbed value of $\epsilon = 0.001$.

Problem 7. Stiff application problem of second-order ODEs - Duffing problem

The Duffing equation (or Duffing oscillator), named after Georg Duffing (1861–1944), is a nonlinear second-order differential equation that models certain types of damped and driven oscillators, as well as chaotic behavior of oscillator [54]. The equation is expressed as follows:

$$u''(t) + \delta u'(t) + \alpha u(t) + \beta t^3 = \gamma \cos(\lambda t), \quad u(t_0) = u_0, \quad u'(t_0) = u'_0, \quad (4.46)$$

where $u(t)$ indicates the displacement of oscillator at time t , δ is the parameter to regulate the level of damping, α is the parameter to regulate the linear stiffness, β is the degree of non-linearity in the restoring force, γ is the magnitude of the periodic driving force and λ is the angular frequency of oscillating driving force. The Duffing equation can be interpreted as describing the oscillations of a mass attached to linear damper and nonlinear spring. The total restoring force imposed by the spring is $\alpha u(t) + \beta t^3$ [55].

If both α and β are greater than 0, the spring is categorized as hardening spring. When $\alpha > 0$ and $\beta < 0$, it is categorized as softening spring. When $\beta = \delta = 0$, the equation can be reduced into simple harmonic motion of elastic pendulum. The behavior of the Duffing equation's solution varies significantly based on the initial conditions and the parameters. Additionally, the Duffing problem showcases the jump resonance phenomenon in its frequency response, characterized by a type of frequency hysteresis behavior.

In the literature, we focus on undamped driven oscillator, which is a mechanical system where an external force drives the oscillations, but there is no damping force to dissipate the energy from the system. There are numerous cases where different initial conditions and parameters can be considered. Three simulations with various conditions are displayed in Figs. 2–4 using classical fourth-order Runge–Kutta method with $h = 10^{-6}$.

In numerical test, we focus on this kind of Duffing problem:

$$u'' = -u(t) - 0.001u(t)^3 + 0.002\cos(35t),$$

$$u(0) = 0, \quad u'(0) = 0, \quad t \in [0, 50],$$

with fitted frequency, $\lambda = 35$.

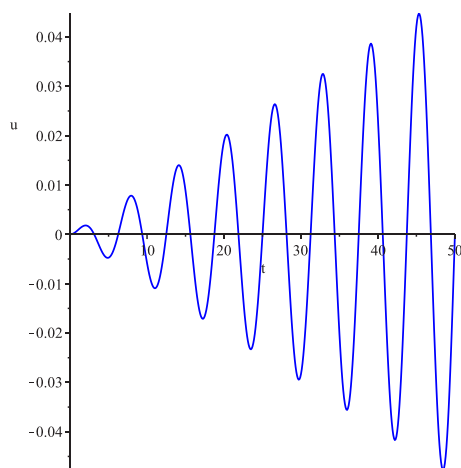


Fig. 2. Numerical simulation for Duffing problem, $\delta = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.002$, $\lambda = 1.01$, $u(0) = u'(0) = 0$ and $t \in [0, 50]$.

4.1. Algorithm: Implementation of TFTDLMM5 method

The implementation of trigonometrically-fitted TDLMM involves several key steps:

1. Initialization:

- Set initial conditions t_0, u_0 and u'_0 .
- Define the frequency parameter, $v = \lambda h$.

2. Compute coefficients:

- Calculate the frequency-dependent parameters K_i and $L_i, i = 1, 2, 3, 4$ using the provided equations or precomputed values.

3. Start iteration for $n = 0$ to endpoint:

- Evaluate the function of $f_n = f(t_n, u_n)$.
- Calculate the derivative of function, $g_n = g(t_n, u_n, u'_n)$.
- Get the approximation of $u_{n+\frac{3}{2}}$ and $u'_{n+\frac{3}{2}}$.
- Use the previous grid points of t, u and u' to determine u_{n+2} and u'_{n+2} .

4. Update the approximations:

- Set u_{n+2} as the final value for solution at t_{n+2}
- Set u'_{n+2} as the final value for the derivative at t_{n+2}

5. Output:

- Repeat Step 3 and 4 until the endpoint of t is reached.
- Return the solution u_n and u'_n for all time steps n
- Compute the computational time and relative error at every step.
- Evaluate the maximum global error.

This algorithm provides a framework for implementing the TFTDLMM5 method, allowing for efficient and accurate solutions to second-order ODEs with oscillatory characteristics.

The numerical data are presented in Tables 1–6 with different step-size, h in particular endpoints, b . The tables contain the maximum global error (ERROR) and the time of computation in seconds (TIME), where TIME refers to the CPU time. The error in the form of 2.324(–8) represents 2.324×10^{-8} . These results were obtained using Maple software, which was employed to implement the algorithms and perform the calculations.

Figs. 1–7 demonstrate the numerical performance of proposed method and other selected methods in term of maximum global truncation error against computational time. The model of computer used in computing the numerical results is Lenovo ideapad 330 Intel Core i5-8050U (1.8 GHz).

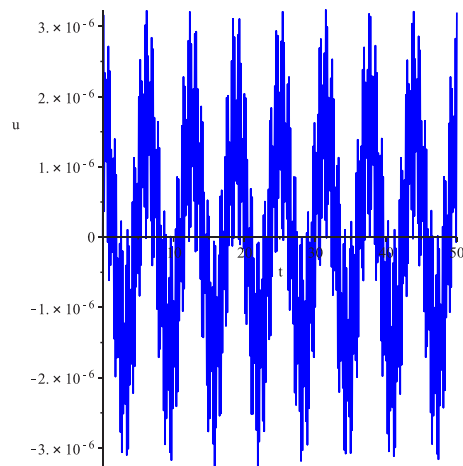


Fig. 3. Numerical simulation for Duffing problem, $\delta = 0$, $\alpha = 1$, $\beta = 0.001$, $\gamma = 0.002$, $\lambda = 35$, $u(0) = u'(0) = 0$ and $t \in [0, 50]$.

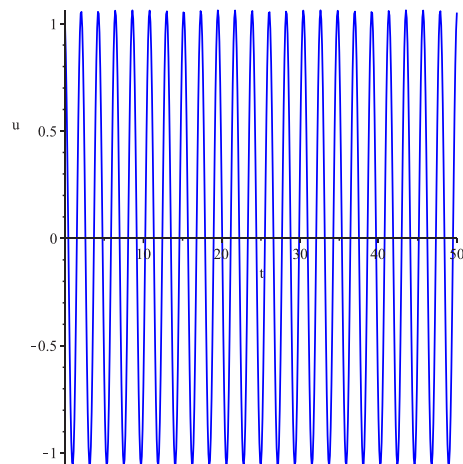


Fig. 4. Numerical simulation for Duffing problem, $\delta = 0$, $\alpha = 10$, $\beta = -2$, $\gamma = 0.002$, $\lambda = 1.01$, $u(0) = 1, u'(0) = -1$ and $t \in [0, 50]$.

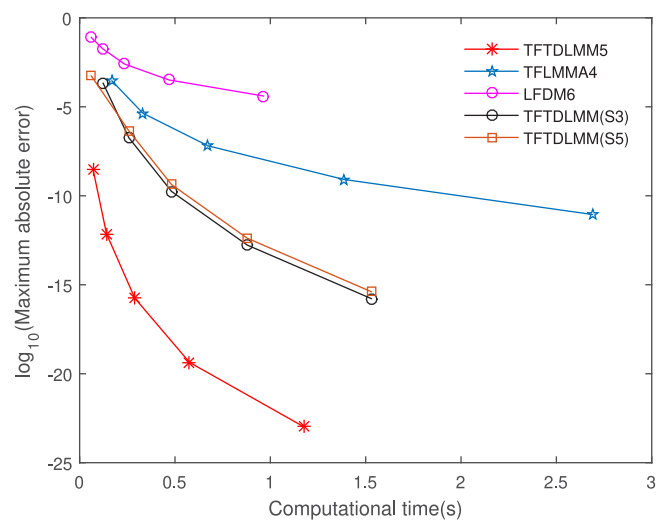


Fig. 5. Numerical curves of selected methods for problem 1 with $b = 100$ and $h = \frac{0.1}{2^i}$, $i = 0, 1, \dots, 4$.

Table 1

Numerical comparison between TFTDLMM5 method with existing methods for problem 1.

h	Methods	$b = 100$		$b = 1000$	
		ERROR	TIME	ERROR	TIME
0.1	TFTDLMM5	3.080822(−9)	0.070	3.084562(−8)	1.044
	TFLMMA4	2.879908(−4)	0.169	2.879908(−4)	2.511
	LFDM6	7.940765(−2)	0.061	5.834896(−1)	0.964
	TFTDLMM(S3)	2.122583(−4)	0.123	2.163089(−3)	1.790
	TFTDLMM(S5)	5.449040(−4)	0.140	5.497057(−3)	2.088
0.05	TFTDLMM5	7.321837(−13)	0.141	7.359522(−12)	2.076
	TFLMMA4	4.331341(−6)	0.332	4.339723(−6)	4.983
	LFDM6	1.858558(−2)	0.120	1.722294(−1)	1.880
	TFTDLMM(S3)	1.834283(−7)	0.258	1.853044(−6)	3.066
	TFTDLMM(S5)	4.454862(−7)	0.275	4.489033(−6)	3.566
0.025	TFTDLMM5	1.784523(−16)	0.289	1.791698(−15)	4.157
	TFLMMA4	6.435404(−8)	0.673	6.739007(−8)	9.418
	LFDM6	2.544737(−3)	0.235	2.530715(−2)	3.710
	TFTDLMM(S3)	1.749708(−10)	0.482	1.755904(−9)	5.117
	TFTDLMM(S5)	4.249776(−10)	0.523	4.260681(−9)	6.078
0.0125	TFTDLMM5	4.355671(−20)	0.576	4.370149(−19)	8.170
	TFLMMA4	8.307822(−10)	1.385	1.051535(−9)	18.876
	LFDM6	3.238462(−4)	0.468	3.255817(−3)	7.833
	TFTDLMM(S3)	1.696834(−13)	0.879	1.703211(−12)	9.759
	TFTDLMM(S5)	4.120322(−13)	0.957	4.132831(−12)	11.802
0.00625	TFTDLMM5	1.063306(−23)	1.174	1.067033(−22)	16.297
	TFLMMA4	8.898683(−12)	2.689	1.641794(−11)	35.986
	LFDM6	4.050655(−5)	0.964	4.092068(−4)	15.893
	TFTDLMM(S3)	1.654418(−16)	1.530	1.660466(−15)	17.538
	TFTDLMM(S5)	4.041456(−16)	1.660	4.030619(−15)	21.622

Table 2

Numerical comparison between TFTDLMM5 method with existing methods for problem 2.

h	Methods	$b = 10$		$b = 100$	
		ERROR	TIME	ERROR	TIME
0.005	TFTDLMM5	6.056155(−9)	0.876	6.062388(−8)	8.410
	TFLMMA4	8.804463(−3)	1.986	8.804463(−3)	18.059
	LFDM6	1.518060(−1)	0.786	8.325815(−1)	7.622
	TFTDLMM(S3)	4.202813(−4)	1.104	4.994741(−2)	9.948
	TFTDLMM(S5)	1.075435(−3)	1.236	1.082991(−2)	13.559
0.0025	TFTDLMM5	1.566963(−12)	1.812	1.448237(−11)	16.790
	TFLMMA4	1.088900(−3)	3.940	1.088900(−3)	35.985
	LFDM6	3.614375(−2)	1.679	3.099210(−1)	15.017
	TFTDLMM(S3)	3.622192(−7)	2.070	3.634861(−6)	19.497
	TFTDLMM(S5)	8.788966(−7)	2.493	8.808807(−6)	25.421
0.00125	TFTDLMM5	2.728844(−14)	3.643	2.739838(−14)	33.544
	TFLMMA4	1.354475(−4)	6.910	1.354475(−4)	68.575
	LFDM6	5.007624(−3)	3.305	4.902641(−2)	29.839
	TFTDLMM(S3)	3.433103(−10)	4.149	3.445535(−9)	37.932
	TFTDLMM(S5)	8.333796(−10)	4.998	8.360669(−9)	48.746
0.000625	TFTDLMM5	1.699032(−15)	7.188	1.699056(−15)	65.683
	TFLMMA4	1.620214(−5)	14.441	1.620214(−5)	135.481
	LFDM6	6.401012(−4)	6.584	6.380343(−3)	59.764
	TFTDLMM(S3)	3.362081(−13)	8.210	3.343162(−12)	76.034
	TFTDLMM(S5)	7.906715(−15)	10.876	8.109941(−12)	93.292
0.0003125	TFTDLMM5	1.060843(−16)	14.305	1.060843(−16)	131.478
	TFLMMA4	2.118873(−6)	28.796	2.118873(−6)	277.583
	LFDM6	8.038350(−5)	12.941	8.032162(−4)	114.879
	TFTDLMM(S3)	1.326901(−15)	16.325	3.886620(−15)	151.326
	TFTDLMM(S5)	7.901624(−16)	21.569	7.906715(−15)	189.648

5. Discussion and conclusions

In this paper, a novel fifth-order two-derivative linear multistep method with trigonometrically-fitting technique, denoted as TFTDLMM5 is constructed based on the Gegenbauer polynomial as basis function is derived for solving $u''(t) = f(t, u(t))$ with oscillatory solution. At first, general formulation of TDLMM method, with second-derivative, f -evaluations and third-derivative,

Table 3

Numerical comparison between TFTDLMM5 method with existing methods for problem 3.

h	Methods	$b = 5$		$b = 50$	
		ERROR	TIME	ERROR	TIME
0.5	TFTDLMM5	5.641905(−7)	0.005	4.720604(−6)	0.062
	TFLMMA4	2.935044(−5)	0.009	1.953935(−4)	0.095
	LFDM6	5.342877(−5)	0.009	4.776093(−4)	0.078
	TFTDLMM(S3)	6.895303(−6)	0.007	7.170154(−5)	0.071
	TFTDLMM(S5)	3.481899(−5)	0.008	1.061880(+7)	0.073
0.4	TFTDLMM5	1.809876(−7)	0.007	1.891834(−6)	0.074
	TFLMMA4	6.278201(−6)	0.012	4.536311(−5)	0.123
	LFDM6	1.189272(−5)	0.013	1.382468(−4)	0.099
	TFTDLMM(S3)	1.878760(−6)	0.009	2.353384(−5)	0.082
	TFTDLMM(S5)	1.784879(−5)	0.010	1.132666(+4)	0.085
0.3	TFTDLMM5	5.276895(−8)	0.009	5.823086(−7)	0.100
	TFLMMA4	8.988203(−7)	0.015	6.893348(−6)	0.164
	LFDM6	2.341808(−6)	0.016	2.611099(−5)	0.142
	TFTDLMM(S3)	5.791658(−7)	0.011	6.838890(−6)	0.106
	TFTDLMM(S5)	3.735978(−6)	0.012	5.959620(+0)	0.108
0.2	TFTDLMM5	1.049511(−8)	0.011	1.130206(−7)	0.153
	TFLMMA4	7.499951(−8)	0.018	6.175574(−7)	0.241
	LFDM6	2.226380(−7)	0.019	2.424289(−6)	0.201
	TFTDLMM(S3)	1.174565(−7)	0.014	1.342521(−6)	0.155
	TFTDLMM(S5)	4.890288(−7)	0.016	1.035996(−3)	0.159
0.1	TFTDLMM5	6.601377(−10)	0.022	6.988465(−9)	0.304
	TFLMMA4	3.107346(−9)	0.035	3.088265(−8)	0.479
	LFDM6	3.659208(−9)	0.039	3.837896(−8)	0.400
	TFTDLMM(S3)	7.677380(−9)	0.028	8.265949(−8)	0.305
	TFTDLMM(S5)	1.651870(−8)	0.031	1.946905(−7)	0.317

Table 4

Numerical comparison between TFTDLMM5 method with existing methods for problem 4.

h	Methods	$b = 10$		$b = 100$	
		ERROR	TIME	ERROR	TIME
0.1	TFTDLMM5	7.977340(−19)	0.067	3.572508(−17)	0.513
	TFLMMA4	2.938062(−8)	0.126	3.176734(−4)	0.972
	LFDM6	2.998295(−8)	0.151	3.195565(−7)	0.890
	TFTDLMM(S3)	2.237790(−12)	0.044	1.222159(−10)	0.420
	TFTDLMM(S5)	1.092689(−16)	0.046	3.402128(−16)	0.431
0.05	TFTDLMM5	2.063606(−22)	0.132	1.920456(−21)	1.072
	TFLMMA4	1.981663(−10)	0.250	1.688161(−7)	1.995
	LFDM6	4.599714(−10)	0.286	4.815364(−9)	1.796
	TFTDLMM(S3)	1.780272(−15)	0.086	6.678577(−14)	0.839
	TFTDLMM(S5)	2.763311(−20)	0.090	2.634101(−19)	0.864
0.025	TFTDLMM5	5.227559(−26)	0.262	4.334490(−25)	2.297
	TFLMMA4	1.433911(−12)	0.495	4.168009(−10)	3.813
	LFDM6	7.041945(−12)	0.547	7.561881(−11)	3.863
	TFTDLMM(S3)	1.534396(−18)	0.172	3.960639(−17)	1.688
	TFTDLMM(S5)	6.788675(−24)	0.176	7.156790(−23)	1.703
0.0125	TFTDLMM5	1.302537(−29)	0.524	1.159116(−28)	4.751
	TFLMMA4	1.077299(−14)	0.974	1.944607(−12)	7.578
	LFDM6	9.625602(−12)	1.040	9.480182(−10)	7.852
	TFTDLMM(S3)	1.396694(−21)	0.340	2.616662(−20)	3.412
	TFTDLMM(S5)	1.664261(−27)	0.346	1.765264(−26)	3.849
0.00625	TFTDLMM5	3.213882(−33)	1.023	3.069561(−32)	9.501
	TFLMMA4	8.252017(−17)	1.851	1.192627(−14)	14.854
	LFDM6	2.936344(−13)	2.079	2.894594(−11)	15.863
	TFTDLMM(S3)	1.313810(−24)	0.713	1.944082(−23)	7.015
	TFTDLMM(S5)	4.074683(−31)	0.721	4.313005(−30)	7.781

g -evaluations is proposed. Interpolation and collocation technique in various grid-points are applied to derive this method. Then, the trigonometrical-fitted technique is implemented into the proposed method whereby several parameters are chosen to be formulated into frequency and step-size dependent coefficients. In numerical analysis, our proposed method has been shown to exhibit stability, consistency and convergence properties. The analysis confirms that the proposed method offers reliability and accuracy in solving the equation in the form of $u''(t) = f(t, u(t))$.

Table 5
Numerical comparison between TFTDLMM5 method with existing methods for problem 5.

h	Methods	$b = 5$		$b = 50$	
		ERROR	TIME	ERROR	TIME
0.005	TFTDLMM5	2.883465(−22)	1.212	2.942271(−21)	10.796
	TFLMMA4	3.072194(−11)	2.579	1.935640(−10)	24.536
	LFDM6	5.998222(−10)	2.402	6.116481(−9)	20.594
	TFTDLMM(S3)	1.749946(−15)	0.860	1.790669(−14)	9.461
	TFTDLMM(S5)	3.646129(−20)	0.884	1.899662(−11)	10.523
0.0025	TFTDLMM5	7.044740(−26)	2.364	7.183239(−25)	20.950
	TFLMMA4	2.473917(−13)	5.112	1.968380(−12)	46.469
	LFDM6	9.218070(−12)	4.856	9.571915(−11)	41.895
	TFTDLMM(S3)	1.710569(−18)	1.722	1.746507(−17)	20.979
	TFTDLMM(S5)	8.867717(−24)	1.806	5.880075(−22)	20.716
0.00125	TFTDLMM5	1.720582(−29)	4.526	1.753613(−28)	42.543
	TFLMMA4	1.963414(−15)	11.199	1.775162(−14)	96.874
	LFDM6	7.336177(−12)	9.797	7.518863(−11)	84.957
	TFTDLMM(S3)	1.671770(−21)	3.488	1.704967(−20)	38.654
	TFTDLMM(S5)	2.164417(−27)	3.603	2.208382(−26)	41.750
0.000625	TFTDLMM5	4.201504(−33)	9.132	4.281238(−32)	85.982
	TFLMMA4	1.551217(−17)	22.696	1.494554(−16)	212.350
	LFDM6	2.319059(−13)	19.814	2.351709(−12)	180.847
	TFTDLMM(S3)	1.633289(−24)	6.999	1.664825(−23)	78.692
	TFTDLMM(S5)	5.285847(−31)	7.838	5.389048(−30)	82.266
0.0003125	TFTDLMM5	1.025757(−36)	19.177	1.045212(−35)	173.565
	TFLMMA4	1.218641(−19)	39.187	1.213034(−18)	422.875
	LFDM6	7.307939(−15)	37.931	7.351022(−14)	353.310
	TFTDLMM(S3)	1.595385(−27)	14.626	1.625744(−26)	154.059
	TFTDLMM(S5)	1.290839(−34)	15.439	1.315544(−33)	165.232

Table 6
Numerical comparison between TFTDLMM5 method with existing methods for problem 6.

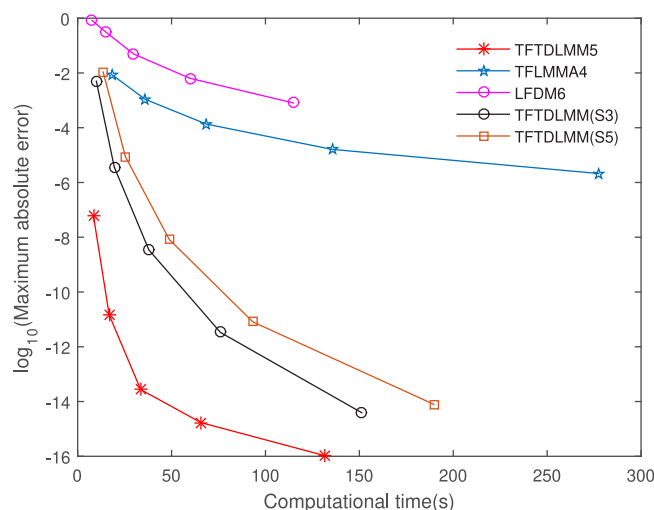
h	Methods	$b = 10$		$b = 100$	
		ERROR	TIME	ERROR	TIME
0.1	TFTDLMM5	1.108454(−18)	0.130	3.252864(−17)	1.239
	TFLMMA4	2.504817(−8)	0.203	3.201751(−6)	2.048
	LFDM6	3.023221(−8)	0.185	3.225787(−7)	1.851
	TFTDLMM(S3)	2.268773(−12)	0.098	1.241867(−10)	0.945
	TFTDLMM(S5)	1.107823(−16)	0.108	3.433128(−16)	1.007
0.05	TFTDLMM5	2.488077(−22)	0.235	5.179904(−21)	2.441
	TFLMMA4	1.837986(−10)	0.402	3.588304(−8)	3.987
	LFDM6	4.635080(−10)	0.361	4.772839(−9)	3.682
	TFTDLMM(S3)	1.803891(−15)	0.195	6.784257(−14)	1.894
	TFTDLMM(S5)	2.801710(−20)	0.216	2.672955(−19)	2.021
0.025	TFTDLMM5	5.803054(−26)	0.503	9.248865(−25)	4.876
	TFLMMA4	1.388878(−12)	0.801	2.247937(−10)	7.685
	LFDM6	5.233038(−12)	0.709	5.233038(−12)	7.204
	TFTDLMM(S3)	1.553750(−18)	0.385	4.022125(−17)	3.780
	TFTDLMM(S5)	6.884213(−24)	0.421	7.269413(−23)	4.053
0.0125	TFTDLMM5	1.383244(−29)	0.921	1.842230(−28)	9.013
	TFLMMA4	1.066655(−14)	1.565	1.488750(−12)	15.204
	LFDM6	9.307832(−12)	1.421	9.195546(−10)	14.219
	TFTDLMM(S3)	1.413996(−21)	0.762	2.656149(−20)	7.552
	TFTDLMM(S5)	1.687701(−27)	0.838	1.793263(−26)	8.091
0.00625	TFTDLMM5	3.336016(−33)	1.839	3.989720(−32)	17.471
	TFLMMA4	8.261090(−17)	3.097	1.058974(−14)	29.879
	LFDM6	2.900328(−13)	2.814	2.823020(−11)	27.635
	TFTDLMM(S3)	1.329904(−24)	1.490	1.972659(−23)	14.875
	TFTDLMM(S5)	4.132141(−31)	1.680	4.381653(−30)	15.625

In the numerical test, seven different types of problems in the form of $u''(t) = f(t, u(t))$ with periodic solutions have been selected to assess the numerical performance of all chosen methods. TFTDLMM5 method is compared to the existing trigonometrical-fitted block multistep methods in predictor–corrector mode, TFTDLMM(S3) and TFTDLMM(S5) methods with three step and five step respectively, fourth-order extra derivative multistep method with trigonometrically-fitting technique, denoted as TFLMMA4 method and sixth-order linear finite difference method, denoted as LFDM6 method. The numerical performance is assessed based on time

Table 7

Numerical comparison between TFTDLMM5 method with existing methods for problem 7.

h	Methods	$b = 5$		$b = 50$	
		ERROR	TIME	ERROR	TIME
0.06	TFTDLMM5	9.070204(−8)	0.010	6.913959(−7)	0.096
	TFLMMA4	1.942953(−3)	0.023	+∞	–
	LFDM6	1.008236(−5)	0.030	1.047901(−5)	0.293
	TFTDLMM(S3)	1.243813(−4)	0.016	1.905735(−4)	0.167
	TFTDLMM(S5)	3.484310(−4)	0.020	3.484310(−4)	0.190
0.05	TFTDLMM5	4.156115(−8)	0.012	3.981147(−7)	0.115
	TFLMMA4	2.012324(−6)	0.027	4.587678(−6)	0.281
	LFDM6	1.278385(−5)	0.033	1.318964(−5)	0.329
	TFTDLMM(S3)	1.622511(−5)	0.019	1.739963(−5)	0.189
	TFTDLMM(S5)	2.624433(−4)	0.024	2.624433(−4)	0.231
0.04	TFTDLMM5	1.812304(−8)	0.016	1.826894(−7)	0.147
	TFLMMA4	1.274522(−6)	0.036	3.390203(−6)	0.324
	LFDM6	2.892861(−6)	0.046	2.915024(−6)	0.472
	TFTDLMM(S3)	1.206657(−6)	0.023	1.231387(−6)	0.229
	TFTDLMM(S5)	1.767325(−5)	0.030	1.891734(−5)	0.281
0.03	TFTDLMM5	6.122474(−9)	0.019	6.214625(−8)	0.195
	TFLMMA4	2.020570(−7)	0.046	7.249179(−7)	0.452
	LFDM6	1.136894(−7)	0.059	1.137089(−7)	0.581
	TFTDLMM(S3)	8.012362(−8)	0.027	1.208352(−7)	0.260
	TFTDLMM(S5)	9.464944(−8)	0.038	1.179918(−7)	0.389
0.02	TFTDLMM5	1.265660(−9)	0.031	1.284038(−8)	0.325
	TFLMMA4	1.645368(−8)	0.071	9.031406(−8)	0.650
	LFDM6	4.751012(−9)	0.090	4.767446(−9)	0.948
	TFTDLMM(S3)	9.149944(−9)	0.044	8.106083(−8)	0.434
	TFTDLMM(S5)	1.515671(−9)	0.057	7.101003(−9)	0.587

**Fig. 6.** Numerical curves of selected methods for problem 2 with $b = 100$ and $h = \frac{0.005}{2^i}$, $i = 0, 1, \dots, 4$.

of computation and maximum global error generated by all selected methods. The results are displayed in [Tables 1–7](#) and [Figs. 5–11](#). The two-derivative term offers significant advantages in accuracy, particularly when coupled with the trigonometric-fitting technique. This combination ensures that the error generated at every stage is exceptionally small, particularly when the step-size is sufficiently small. Also, the complexity of the TFTDLMM5 method is relatively low compared to others. It does not involve a corrector method to rectify the approximations at every stage, resulting in lower computational time compared to TFTDLMM(S3) and TFTDLMM(S5) with corrector methods. Besides, TFTDLMM5 method generates the least maximum global error among all methods with different step-sizes and endpoints. The results indicate that TFTDLMM5 method outperforms other existing methods.

From the results, the TFTDLMM5 method has proven to be proficient in solving various types of problems, including homogeneous, inhomogeneous, linear, and nonlinear problems. Nonlinear application problems with periodic properties, such as the two-body problem and Duffing problem, have been selected for numerical tests. The nonlinearity in the two-body problem and

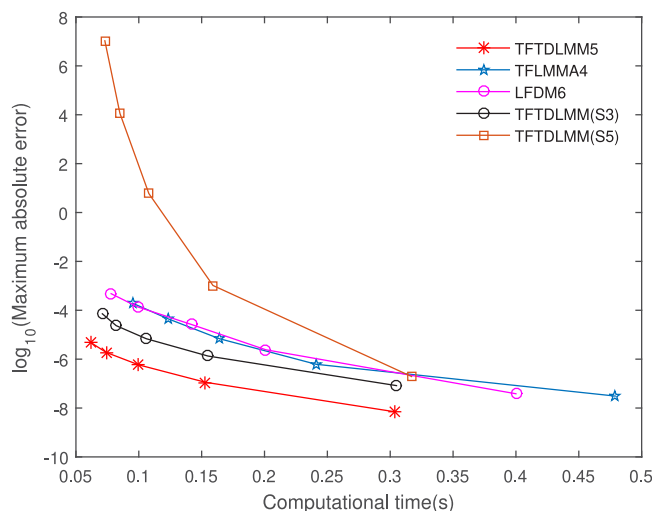


Fig. 7. Numerical curves of selected methods for problem 3 with $b = 50$ and $h = 0.5 - 0.1i, i = 0, 1, \dots, 4$.

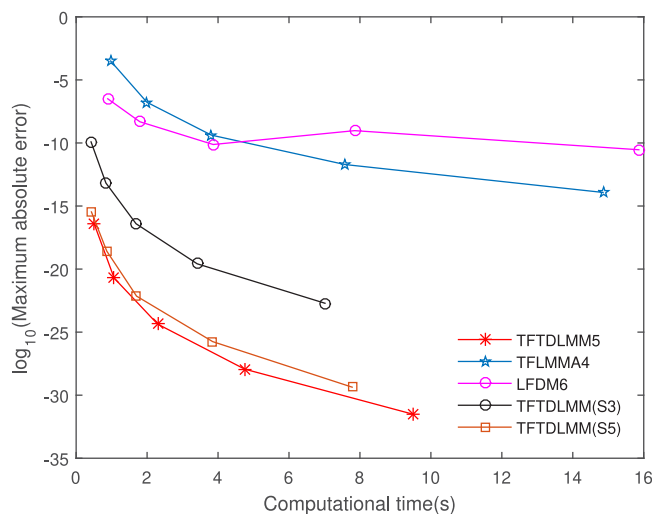


Fig. 8. Numerical curves of selected methods for problem 4 with $b = 100$ and $h = \frac{0.1}{2^i}, i = 0, 1, \dots, 4$.

Duffing problem causes some inaccuracy for certain methods, leading to relatively high global error compared to others, especially for the method without trigonometrical-fitted technique. Frequency-dependent coefficients are tailored to match the oscillatory nature of the problem being solved. By fitting these coefficients to the specific frequency of the solution, the numerical method can more accurately capture the behavior of the system, reducing the local truncation error. These coefficients help ensure that the numerical method remains stable over a wide range of step sizes. For oscillatory problems, maintaining stability is critical to prevent the solution from diverging or exhibiting unphysical behavior. In oscillatory problems, phase error can accumulate over time, leading to significant discrepancies between the numerical and exact solutions. Frequency-dependent coefficients help minimize phase error, ensuring that the numerical solution stays in phase with the true solution over long time intervals.

There are a few topics that can be explored for future research. TFTDLMM method can be extended to solve general type of second-order ODEs in the form of $u''(t) = f(t, u(t), u'(t))$ with periodic solution. Besides, two-derivative multistep method can be adapted to solve delay differential equations (DDEs), which involve delays in their formulation, can be a significant research area. This involves modifying the existing methods to handle the delayed terms effectively while maintaining stability and accuracy. Also,

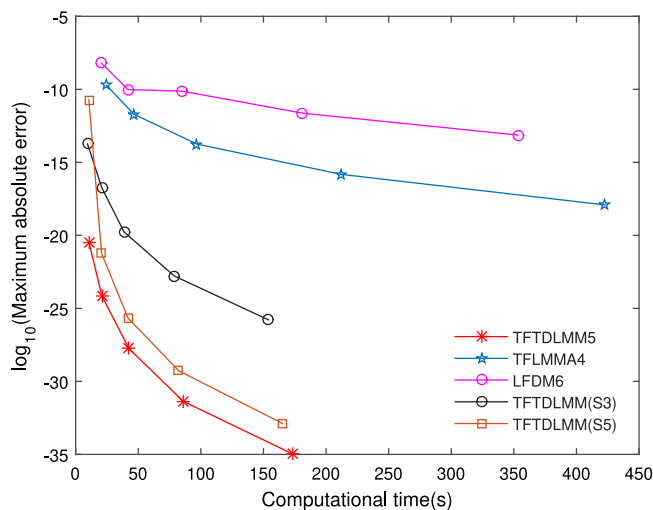


Fig. 9. Numerical curves of selected methods for problem 5 with $b = 50$ and $h = \frac{0.005}{2^i}$, $i = 0, 1, \dots, 4$.

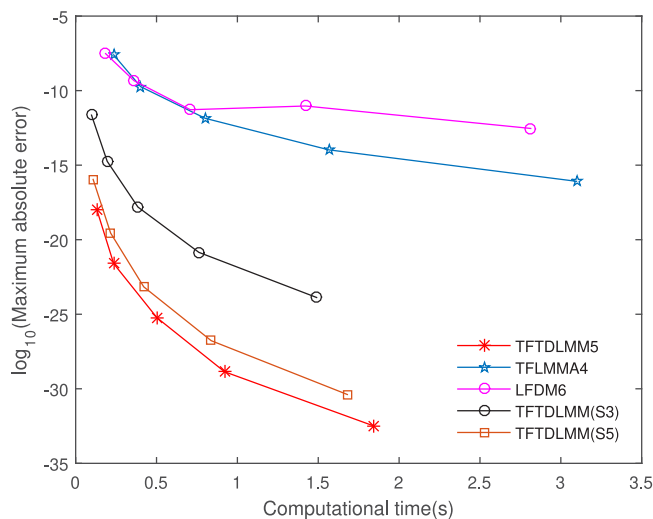


Fig. 10. Numerical curves of selected methods for problem 6 with $b = 10$ and $h = \frac{0.1}{2^i}$, $i = 0, 1, \dots, 4$.

we can adapt two-derivative linear multistep methods with trigonometric-fitting techniques to solve partial differential equations (PDEs). The proposed method can indeed be extended to partial differential equations (PDEs), particularly when the PDEs exhibit certain properties such as periodicity or separability in space and time. To apply our method to PDEs, we would typically need to discretize the spatial and temporal domains. For example, in the case of a time-dependent PDE, such as a heat equation or wave equation, the temporal discretization could be handled using our two-derivative linear multistep method, while the spatial derivatives could be approximated using methods like finite differences, finite elements, or spectral methods. The trigonometrical-fitting technique could also be adapted to improve the accuracy of spatial or temporal discretization, ensuring accuracy for problems with oscillatory behavior in both dimensions. Moreover, for nonlinear PDEs, the method can still be applicable if the nonlinearity is separable or if the problem can be treated using an iterative approach. In such cases, the method would be applied to the linearized form of the PDE in each iteration, with updates to the solution at each time step. Other than block multistep method, we could combine trigonometric-fitting technique with other efficient methods, such as finite element methods or spectral methods, to create hybrid approaches that leverage the strengths of each method for solving complex partial differential equations.

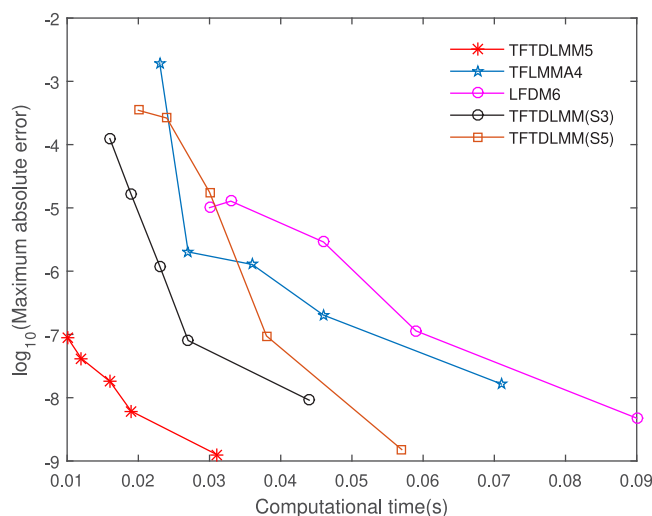


Fig. 11. Numerical curves of selected methods for Duffing problem with $b = 5$ and $h = 0.06 - 0.01i, i = 0, 1, \dots, 4$.

CRedit authorship contribution statement

K.C. Lee: Writing – review & editing, Writing – original draft, Visualization, Software, Project administration, Methodology, Funding acquisition, Formal analysis, Data curation. **I. Hashim:** Writing – review & editing, Validation, Supervision, Conceptualization. **M.N. Mohd Aris:** Writing – original draft, Visualization, Methodology, Conceptualization. **N. Senu:** Validation, Supervision, Methodology, Investigation, Conceptualization.

Declaration of competing interest

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