UNIVERSITI PUTRA MALAYSIA

NUMERICAL SOLUTIONS OF CAUCHY TYPE SINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND USING POLYNOMIAL APPROXIMATIONS

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By

MOHAMMAD ABDULKAWI MAHIUB

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirement for the Degree of Doctor of Philosophy

January 2010

This thesis is dedicated to all my family members

especially my father

Abdulkawi Mahiub Abdalmoghny

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chairman: Zainidin Eshkuvatov, PhD

Faculty : Science

In this thesis, the exact solutions of the characteristic singular integral equation of Cauchy type

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}\, dt = f(x), \quad -1 < x < 1, \tag{0.1}$$

are described, where $f(x)$ is a given real valued function belonging to the Hölder class and $\varphi(t)$ is to be determined.

We also described the exact solutions of Cauchy type singular integral equations of the form

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}\, dt + \int_{-1}^{1} K(x,t)\, \varphi(t)\, dt = f(x), \quad -1 < x < 1, \tag{0.2}$$

where $K(x,t)$ and $f(x)$ are given real valued functions, belonging to the Hölder class, by applying the exact solutions of characteristic integral equation (0.1) and the theory of Fredholm integral equations.

This thesis considers the characteristic singular integral equation (0.1) and Cauchy type singular integral equation (0.2) for the following four cases:

Case I. $\varphi(x)$ is unbounded at both end-points $x = \pm 1$,

Case II. $\varphi(x)$ is bounded at both end-points $x = \pm 1$,

Case III. $\varphi(x)$ is bounded at $x = -1$ and unbounded at $x = 1$,

Case IV. $\varphi(x)$ is bounded at $x = 1$ and unbounded at $x = -1$.

The complete numerical solutions of (0.1) and (0.2) are obtained using polynomial approximations with Chebyshev polynomials of the first kind $T_n(x)$, second kind $U_n(x)$, third kind $V_n(x)$ and fourth kind $W_n(x)$ corresponding to the weight functions $\omega_1(x) = (1-x^2)^{-\frac{1}{2}}$, $\omega_2(x) = (1-x^2)^{\frac{1}{2}}$, $\omega_3(x) = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ and $\omega_4(x) = (1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$, respectively.

The exactness of the numerical solutions of equation (0.1), when the force function $f(x)$ is a polynomial of degree n, is proved for all cases.

The exactness of the numerical solutions of equation (0.2), for some given example functions $K(x,t)$ and $f(x)$ are shown.

The estimation of errors for the numerical solutions of equations (0.1) and (0.2), for the above four cases are investigated in the classes of functions L_{2,ω_i}, $i = 1,2,3,4$, which are defined as

$$L_{2,\omega_i} = \left\{ \varphi(x) | \int_{-1}^{1} \omega_i(x) \, |\varphi(x)|^2 \, dx < \infty \right\},$$

with the corresponding norms

$$\|\varphi\|^2_{2,\omega_i} = \int_{-1}^{1} \omega_i(x) \, |\varphi(x)|^2 \, dx.$$

The linearity and boundedness of singular operators $A_i : L_{2,\omega_i} \to L_{2,\frac{1}{\omega_i}}$, and non-singular operators $B_i : L_{2,\omega_i} \to L_{2,\frac{1}{\omega_i}}$, $i = 1, 2, 3, 4$, where

$$(A_i\,\varphi)(x) = \int_{-1}^{1} \omega_i(t) \frac{\varphi(t)}{t - x}\, dt, \quad -1 < x < 1, \tag{0.3}$$

and

$$(B_i\,\varphi)(x) = \int_{-1}^{1} \omega_i(t)\, K(x,t)\, \varphi(t)\, dt, \quad -1 < x < 1, \tag{0.4}$$

are discussed.

The rate of convergence of the numerical solutions of equations (0.1) and (0.2), for the above four cases are shown.

FORTRAN codes are developed to obtain all the numerical results for different functions $K(x,t)$ and $f(x)$. Numerical experiments assert the theoretical results.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENYELESAIAN BERANGKA PERSAMAAN KAMIRAN SINGULAR JENIS CAUCHY JENIS PERTAMA MENGUNAKAN PENGHAMPIRAN POLINOMIAL

Oleh

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Dalam tesis ini, penyelesaian tepat bagi persamaan kamiran singular cirian jenis Cauchy

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}\, dt = f(x), \quad -1 < x < 1, \tag{0.5}$$

digambarkan, dengan $f(x)$ ialah fungsi bernilai nyata yang telah diberikan berada dalam kelas Hölder dan $\varphi(t)$ akan ditentukan.

Kami juga menggambarkan penyelesaian tepat bagi persamaan kamiran singular jenis Cauchy berbentuk

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}\, dt + \int_{-1}^{1} K(x,t)\, \varphi(t)\, dt = f(x), \quad -1 < x < 1, \tag{0.6}$$

dengan $K(x,t)$ dan $f(x)$ adalah fungsi nilai nyata diberi, yang tergolong dalam kelas Hölder, dengan menggunakan penyelesaian tepat bagi persamaan kamiran cirian (0.5) dan teori persamaan kamiran Fredholm.

Thesis ini mempertimbangkan persamaan kamiran singular cirian (0.5) dan persamaan kamiran singular jenis Cauchy (0.6) bagi empat kes berikut:

Kes I. $\varphi(x)$ adalah tak terbatas di kedua-dua titik hujung $x = \pm 1$,

Kes II. $\varphi(x)$ adalah terbatas di kedua-dua titik hujung $x = \pm 1$,

Kes III. $\varphi(x)$ adalah terbatas di $x = -1$ dan tak terbatas di $x = 1$,

Kes IV. $\varphi(x)$ adalah terbatas di $x = 1$ dan tak terbatas di $x = -1$.

Penyelesaian berangka lengkap bagi (0.5) and (0.6) bagi empat kes diatas telah diperolehi dengan penggunaan penghampiran polinomial dengan polinomial Chebyshev jenis pertama $T_n(x)$, kedua $U_n(x)$, ketiga $V_n(x)$ dan keempat $W_n(x)$ bersesuaian dengan fungsi pemberat $\omega_1(x) = (1-x^2)^{-\frac{1}{2}}$, $\omega_2(x) = (1-x^2)^{\frac{1}{2}}$, $\omega_3(x) = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ dan $\omega_4(x) = (1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$, masing-masingnya.

Ketepatan penyelesaian berangka bagi persamaan (0.5), apabila fungsi daya $f(x)$ ialah suatu polinomial berdarjah n, dibuktikan bagi semua kes.

Ketepatan penyelesaian berangka bagi persamaan (0.6), untuk beberapa fungsi contoh $K(x,t)$ dan $f(x)$ yang diberi ditunjukkan.

Anggaran ralat untuk penyelesaian berangka persamaan (0.5) dan (0.6) untuk keempat-empat kes di atas dikaji dalam kelas fungsi L_{2,ω_i}, $i = 1,2,3,4$, ditakrifkan sebagai

$$L_{2,\omega_i} = \left\{ \varphi(x) \middle| \int_{-1}^{1} \omega_i(x)\, |\varphi(x)|^2\, dx < \infty \right\},$$

dengan norma

$$\|\varphi\|_{2,\omega_i}^2 = \int_{-1}^{1} \omega_i(x)\, |\varphi(x)|^2\, dx.$$

Kelinearan dan keterbatasan operator singular $A_i : L_{2,\omega_i} \to L_{2,\frac{1}{\omega_i}}$, dan operator tak-singular $B_i : L_{2,\omega_i} \to L_{2,\frac{1}{\omega_i}}$, $i = 1, 2, 3, 4$, dengan

$$(A_i\,\varphi)(x) = \int_{-1}^{1} \omega_i(t) \frac{\varphi(t)}{t-x}\,dt, \quad -1 < x < 1, \tag{0.7}$$

dan

$$(B_i\,\varphi)(x) = \int_{-1}^{1} \omega_i(t)\,K(x,t)\,\varphi(t)\,dt, \quad -1 < x < 1, \tag{0.8}$$

dibincangkan.

Kadar penumpuan bagi penyelesaian berangka persamaan (0.5) dan (0.6) dalam keempat-empat kes di atas ditunjukkan.

Kod FORTRAN dibangunkan bagi memperolehi semua keputusan berangka untuk fungsi berbeza $K(x,t)$ dan $f(x)$. Eksperimen berangka mengukuhkan keputusan yang diperolehi secara teori.

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I certify that a Thesis Examination Committee has met on 20 January 2010 to conduct the final examination of Mohammad Abdulkawi Mahiub on his thesis entitled "Numerical Solutions of Cauchy Type Singular Integral Equations of the First Kind using Polynomial Approximations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institutions.

MOHAMMAD ABDULKAWI MAHIUB

Date: 8 March 2010

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LIST OF ABBREVIATIONS

SI	Singular integral
SIE	Singular integral equation
CSIE	Singular integral equation with Cauchy kernel
$\fint$	Cauchy principal value integral
$H(\alpha)$	Class of Hölder, $0 < \alpha \leq 1$
Eq.	Equation
$\Gamma(x)$	Gamma function $= \displaystyle\int_0^\infty t^{x-1}\, e^{-t}\, dt$
$B(x, y)$	Beta function $= \displaystyle\int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$
$\Vert x\Vert$	Norm or length of a vector x
$\Vert f\Vert_\infty$	Uniform-norm of a function f
$\Vert f\Vert_1$	L_1 -norm of a function f
$\Vert f\Vert_2$	L_2 -norm of a function f
$\Vert f\Vert_p$	L_p -norm of a function f
$(A, \Vert.\Vert)$	A norm space where A is a vector space
l^p	$\left\{x = \{x_n\}_{n\geq 1} \,:\, \displaystyle\sum_{k=1}^{\infty} \lvert x_k\rvert^p < \infty\right\}$, $(1 \leq p < \infty)$
$L^p([a, b])$	$\left\{f \,:\, \displaystyle\int_a^b \lvert f(x)\rvert^p\, dx < \infty\right\}$
$\Vert f\Vert_{p,\omega}$	L_p -norm of a function f with respect to the weight function ω
$C([a, b])$	Space of continuous real valued functions on the interval $[a, b]$
P_n	Family of all polynomials of degree at most n
$\rho(f; [a, b]; \delta)$	Modulus of continuity of a function f on interval $[a, b]$
$\omega(x)$	Weight function
$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{F}$	Field $\mathbb{R}$ or $\mathbb{C}$

$\langle .,. \rangle$	An inner product
$(A, \langle .,. \rangle)$	An inner product space
$\hat{P}_m(x)$	Legendre polynomials of degree n
$P_m^{(\alpha,\beta)}(x)$	Jacobi polynomials of degree n
$L_m^{(0)}(x)$	Laguerre polynomials of degree n
$L_m^{(\alpha)}(x)$	Generalized Laguerre polynomials of degree n
$H_m(x)$	Hermite polynomials of degree n
$T_n(x)$	Chebyshev polynomials of the first kind of degree n
$U_n(x)$	Chebyshev polynomials of the second kind of degree n
$V_n(x)$	Chebyshev polynomials of the third kind kind of degree n
$W_n(x)$	Chebyshev polynomials of the fourth kind kind of degree n
$\binom{m}{n}$	Binomial coefficients $= \dfrac{m!}{n!(m-n)!}$
$\lfloor \ldots \rfloor$	Largest integer $\leq \ldots$
$L_{n,k}(x)$	Lagrange polynomials of degree n
$det\ A$	Determinant of the matrix A
$Q[f]$	Numerical integration or quadrature formula
δ_{ij}	Kronecker delta
$\times\!\!\!\!\!\!\int \equiv =\!\!\!\!\!\!\int$	Hadamard finite-part integral
$B_{i,n}(x)$	Bernstein polynomials of degree n
$\sum{}'$	Finite or infinite summation with first term halved, $\displaystyle\sum_{k=0}^{n}{}' a_k\, T_k = \frac{1}{2}\, a_0\, T_0 + a_1\, T_1 + a_2\, T_2 + \ldots$
$\sum{}''$	Finite summation with first and last terms halved, $\displaystyle\sum_{k=0}^{n}{}'' a_k\, T_k = \frac{1}{2}\, a_0\, T_0 + a_1\, T_1 + \ldots + a_{n-1}\, T_{n-1} + \frac{1}{2}\, a_n\, T_n$
$\mathbb{B}\,(V, W)$	Space of bounded linear operators $T\ :\ V \to W$
$x \perp B$	x is perpendicular with the set B i.e. $\langle x, y \rangle = 0$, $\forall\, y \in B$
P_{kn}	Projection operator
$\lVert . \rVert_{H^\alpha}$	Norm of Hölder

CHAPTER 1

INTRODUCTION

1.1 Preliminary

The theory of integral equations have a close contacts with broad areas of mathematics. Foremost among these are differential equations and operator theory. Many problems in the fields of ordinary and partial differential equations can be reduced to integral equations. Existence and uniqueness of the solution then can be derived from the corresponding integral equations. Many problems of science and engineering can be stated in the form of integral equations. It is sufficient to say that there is almost no area of applied mathematics and mathematical physics where integral equations do not play a role (Hochstadt, 1973).

Integral equation containing integrals, in the sense of Cauchy principle value, with integrands having a singularity in the domain of integration is called Cauchy singular integral equations (Kanwal, 1997).

In this research we will consider one-dimensional singular integral equations (SIEs) that occurs in varieties of mixed boundary value problems of mathematical physics and engineering such as, isotropic elastic bodies involving cracks, aerodynamic, hydrodynamic, elasticity and other related areas. The investigations of these SIEs with Cauchy Kernels (CSIEs) by Gakhov, Muskhelishvili, Vekua, and others give a great impact on the further development of the general theory of SIEs. For a comprehensive study of CSIEs we refer to Muskhelishvili (1953), Gakhov (1963) and Ladopoulos (2000).

For the purpose of investigation of CSIEs, we first need to introduce the singular integral and Cauchy principle value.

1.2 Cauchy Singular Integral

Definition 1.1. *Let x be a point on contour L outside its nodes. Consider a circle with center x and small radius $\epsilon > 0$ that intersects L at two points t' and t'' . Denote by ℓ the arc $t't'' \subset L$. If the integral $(Belotserkovskii\ and\ Lifanov,$ 1993)*

$$\int_{L/\ell} \frac{f(t)}{t-x}\, dt,$$

has a finite limit $F(x)$ as $\epsilon \to 0$, this limit is called the Cauchy principal value of the singular integral,

$$F(x) = \lim_{\epsilon \to 0} \int_{L/\ell} \frac{f(t)}{t-x}\, dt, \quad x \in L \tag{1.1}$$

and it is denoted by $(Kanwal,$ 1997)

$$\int_L^* \frac{f(t)}{t-x}\, dt. \quad or \quad P \int_L \frac{f(t)}{t-x}\, dt$$

or by $(Kythe\, and\, Schaferkotter,$ 2005)

$$⨍_L \frac{f(t)}{t-x}\, dt. \tag{1.2}$$

Definition 1.2. *A function $f(x)$ defined on a set D is said to satisfy the Hölder condition with exponent α if for any $x_1,\ x_2 \in D$, the inequality*

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|^\alpha$$

holds with constants $K > 0$ and $0 < \alpha \leq 1$. These constants are respectively called the coefficient and the exponent in the Hölder condition $(Kanwal,$ 1997).

We simply say that the function $f(x)$ satisfies the H-condition or belongs to the class H on the set D. Such a function $f(x)$ is also said to be Hölder continuous. We usually write $f(x) \in H(\alpha)$ or $f(x) \in H^\alpha(K, D)$.

Definition 1.3. *A function $\varphi(t)$ belongs to the class H^* on a piecewise smooth curve L, if*

$$\varphi(t) = \frac{\varphi^*(t)}{P_L^{\nu}(t)}, \quad P_L^{\nu}(t) = \Pi_{k=1}^{p}|t - c_k|^{\nu_k},$$

where $\varphi^*(t) \in H_o$ on L, i.e., it belongs to the class H on every smooth piece of the curve L; $0 \le \nu_k < 1$; and c_k, $k = 1, \ldots, p$, are the nodes of the curve L. Without loss of generality, we can assume that $\varphi^*(t) \in H$ on L (Lifanov et al., 2004).

Now, we need to investigate the existence of the singular integral

$$\int_L \frac{f(t)}{t - x}\, dt,$$

where L is the single arc ab. Then, formula (1.1) reads

$$⨍_a^b \frac{f(t)}{t - x}\, dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{f(t)}{t - x}\, dt + \int_{x+\varepsilon}^{b} \frac{f(t)}{t - x}\, dt \right\}, \quad a < x < b. \tag{1.3}$$

The limit in (1.3) may not exist when the density function $f(x)$ is only integrable or even continuous. On the other hand, the existence of the limit in (1.3) is ensured when the function $f(x)$ satisfies the Hölder condition in a certain neighborhood of an internal point t on the arc L, i.e., when it satisfies the following inequality

$$|f(t) - f(\tau)| < C|t - \tau|^{\alpha} \quad (0 < \alpha \le 1), \tag{1.4}$$

where τ is an arbitrary point of the arc L in a given neighborhood of the point t and C is a positive constant coefficient. Denote by ℓ_ϵ the part of the arc L cut out by the circle with center at t whose radius $\epsilon > 0$ and take the integral over the remaining arc L/ℓ_ϵ outside the circle

$$\int_{L/\ell_\epsilon} \frac{f(t)}{\tau - t}\, d\tau = \int_{L/\ell_\epsilon} \frac{f(\tau) - f(t)}{\tau - t}\, d\tau + f(t) \int_{L/\ell_\epsilon} \frac{d\tau}{\tau - t}. \tag{1.5}$$

On the basis of the condition (1.4), the function $f(t)$ in the first integral of the right-hand side of Eq. (1.5) satisfies the inequality

$$\left|\frac{f(\tau)-f(t)}{\tau-t}\right| < \frac{C}{|\tau-t|^{1-\alpha}},$$

and it thus has a weak singularity for $\tau \to t$. Therefore, we are assured of the existence of the improper integral

$$\int_{L/\ell_\epsilon} \frac{f(\tau)-f(t)}{\tau-t}\, d\tau.$$

The second integral on the right-hand side of Eq. (1.5) can be expressed as follows

$$\begin{aligned}\int_{L/\ell_\epsilon} \frac{d\tau}{\tau-t} &= [\ln(\tau-t)]_a^{t'} + [\ln(\tau-t)]_{t''}^{b} \\ &= \ln(b-t) - \ln(a-t) - [\ln(t''-t) - \ln(t'-t)], \qquad (1.6)\end{aligned}$$

where $\ln(\tau-t)$ on each of the arcs at and tb is a branch which changes continuously on this arc. For definiteness, these branches will be connected by the following condition: the value $\ln(t''-t)$ is obtained from the value $\ln(t'-t)$ by means of a continuous change of $\ln(\tau-t)$, while t varies on the arc of an infinitesimal circle, with center at t, so that it passes the point t on the left, with respect to L, (Muskhelishvili, 1953).

Rewriting (1.6) as

$$\int_{L/\ell_\epsilon} \frac{d\tau}{\tau-t} = \ln\frac{b-t}{a-t} + \ln\frac{t'-t}{t''-t}. \qquad (1.7)$$

It is obvious that

$$\ln\frac{t'-t}{t''-t} = \ln\left|\frac{t'-t}{t''-t}\right| + i[\arg(t'-t) - \arg(t''-t)]. \qquad (1.8)$$

By the condition

$$|t'-t| = |t''-t| = \epsilon,$$

one has

$$\lim_{\epsilon\to 0}[\arg(t'-t) - \arg(t''-t)] = \pi. \qquad (1.9)$$

Due to (1.8) and (1.9) we have

$$\lim_{\epsilon \to 0} \left[\ln \frac{t' - t}{t'' - t} \right] = i\pi,$$

and consequently

$$\int\!\!\!\!\!\!-_L \frac{d\tau}{\tau - t} = \ln \frac{b - t}{a - t} + i\pi. \tag{1.10}$$

The integral in (1.10) can also be represented in the form

$$\int\!\!\!\!\!\!-_L \frac{d\tau}{\tau - t} = \ln \frac{b - t}{t - a}.$$

Taking the limit on both sides of (1.5) yields

$$\int\!\!\!\!\!\!-_L \frac{f(t)}{\tau - t}\, d\tau = \int\!\!\!\!\!\!-_L \frac{f(\tau) - f(t)}{\tau - t}\, d\tau + f(t) \ln \frac{b - t}{t - a}.$$

If L is closed, then

$$\int\!\!\!\!\!\!-_L \frac{d\tau}{\tau - t} = i\pi,$$

and so

$$\int\!\!\!\!\!\!-_L \frac{f(t)}{\tau - t}\, d\tau = \int\!\!\!\!\!\!-_L \frac{f(\tau) - f(t)}{\tau - t}\, d\tau + \pi i f(t).$$

Therefore, as the conclusion, we can say that the singular integral (1.2) exists if the function f satisfies the Hölder condition (1.4) (Polyanin and Manzhirov, 1998; Davis and Rabinowitz, 1984; Pogorzelski, 1966).

1.3 Exact solutions of Cauchy type singular integral equations of the first kind on a finite interval

First, we will present the exact solutions of the characteristic singular integral equation of Cauchy type on a segment $[a, b]$ (Kanwal, 1997)

$$\int\!\!\!\!\!\!-_a^b \frac{\varphi(t)}{t - x}\, dt = f(x).$$

For this purpose, let us consider the singular integral equation

$$\int\!\!\!\!\!\!-_0^1 \frac{\varphi(t)}{t - x}\, dt = f(x), \quad 0 < x < 1. \tag{1.11}$$

In solving Eq. (1.11), we multiply it by x yield

$$\int_0^1 \frac{t\,\varphi(t)}{t-x}\,dt = xf\,(x) + c, \tag{1.12}$$

where

$$c = \int_0^1 \varphi(t)\,dt.$$

Next, we multiply both sides of Eq. (1.12) by $\dfrac{dx}{\sqrt{x(u-x)}}$ and integrate with respect to x from 0 to u, which gives

$$\int_0^u \frac{1}{\sqrt{x(u-x)}} \int_0^1 \frac{t\,\varphi(t)}{t-x}\,dt\,dx = \int_0^u \frac{\sqrt{x}f(x)}{\sqrt{u-x}}\,dx + c\int_0^u \frac{dx}{\sqrt{x(u-x)}}. \tag{1.13}$$

It is known that (Andrews et al., 1999; Gradshteyn and Ryzhik, 1965)

$$B(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{1.14}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = 1, \tag{1.15}$$

where $B(x,y)$ and $\Gamma(x)$ are the beta and gamma functions, respectively.

From (1.14)$-$(1.15) we obtain

$$\int_0^u \frac{dx}{\sqrt{x(u-x)}} = \pi. \tag{1.16}$$

Changing the order of integration in (1.13) and using (1.16), Eq. (1.13) becomes

$$\int_0^1 t\,\varphi(t)\,dt \int_0^u \frac{dx}{\sqrt{x(u-x)}(t-x)} = \int_0^u \frac{\sqrt{x}f(x)}{\sqrt{u-x}}\,dx + c\pi. \tag{1.17}$$

It is not difficult to verify that

$$\int_0^u \frac{dx}{\sqrt{x(u-x)}(x-t)} = \psi(t,u), \tag{1.18}$$