

Review Article

Fahad Alsidrani, Adem Kılıçman*, and Norazak Senu

A comprehensive review of the recent numerical methods for solving FPDEs

<https://doi.org/10.1515/math-2024-0036>

received December 26, 2023; accepted July 1, 2024

Abstract: Fractional partial differential equations (FPDEs) have gained significant attention in various scientific and engineering fields due to their ability to describe complex phenomena with memory and long-range interactions. Solving FPDEs analytically can be challenging, leading to a growing need for efficient numerical methods. This review article presents the recent analytical and numerical methods for solving FPDEs, where the fractional derivatives are assumed in Riemann-Liouville's sense, Caputo's sense, Atangana-Baleanu's sense, and others. The primary objective of this study is to provide an overview of numerical techniques commonly used for FPDEs, focusing on appropriate choices of fractional derivatives and initial conditions. This article also briefly illustrates some FPDEs with exact solutions. It highlights various approaches utilized for solving these equations analytically and numerically, considering different fractional derivative concepts. The presented methods aim to expand the scope of analytical and numerical solutions available for time-FPDEs and improve the accuracy and efficiency of the techniques employed.

Keywords: fractional derivatives, partial differential equation, numerical methods, nonlinear equation, integral transforms

MSC 2020: 35A22, 35R11, 65N99, 65R10

1 Introduction

In classical calculus, the integer-order derivative and integral are well-established mathematical operations with a clear physical and geometric interpretation, making them highly useful for solving practical problems in the scientific fields. The meaning of derivative is the rate of change in the relationship between the dependent and independent variables, and there is a well-accepted geometrical explanation, for example, in the relationship between the position and speed of an object. Unfortunately, the situation changes when we consider fractional-order differentiation or integration. The existing literature has no entirely acceptable geometrical interpretation for the fractional-order derivatives. However, fractional derivatives (FDs) capture a system's history and memory in fractional modeling, allowing for a more accurate representation of complex phenomena. In contrast to integer-order derivatives, fractional-order derivatives are considered nonlocal operators, which implies that when computing a FD at a particular point, it relies on data in the vicinity of the current moment and information from the entire historical dataset. While this characteristic allows for the

* **Corresponding author: Adem Kılıçman**, School of Mathematical Sciences, College of Computing, Informatics and Mathematics, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia, e-mail: kilicman@uitm.edu.my

Fahad Alsidrani: Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia; Institute for Mathematical Research, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia, e-mail: f.alsedrani@qu.edu.sa

Norazak Senu: Institute for Mathematical Research, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia; Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia, e-mail: norazak@upm.edu.my

description of physical phenomena with memory features, it can pose challenges in numerical computations [1–5].

Furthermore, Podlubny [1] presented a geometric interpretation based on left-sided and right-sided Riemann-Liouville fractional integrals. Also, he suggested a physical interpretation for the Riemann-Liouville FD and for the Caputo FD. There is still a lack of geometric and physical interpretation of fractional integration and differentiation compared to simple interpretations of their integer order. Another geometrical interpretation based on the conformable FD of ψ of order μ , using the concept of fractional cords, was introduced by Khalil *et al.* [6]. This definition for $\mu \in (0, 1)$ and $\varsigma > 0$ can be defined by

$$\mathcal{D}^\mu \psi(\varsigma) = \lim_{\varepsilon \rightarrow 0} \frac{\psi(\varsigma + \varepsilon \varsigma^{1-\mu}) - \psi(\varsigma)}{\varepsilon}. \quad (1.1)$$

Let $\psi(v, \varsigma) = c$ be an equation represent some curve in the $v\varsigma$ plane with $v > 0$, then

$$\frac{\varsigma^\mu - \varsigma_0^\mu}{v^\mu - v_0^\mu} = \frac{\varsigma_0^{\mu-1} \varsigma^{(\mu)}(v_0)}{v_0^{\mu-1}}, \quad (1.2)$$

called the fractional cord of the curve $\psi(v, \varsigma) = c$ at the point (v_0, ς_0) . Here, authors argue that the conformable FD $\varsigma^\mu(v_0)$ of the function $\varsigma(v)$ in the equation $\psi(v, \varsigma) = c$ is the slope of the tangent line to the fractional cords associated with the curve $\psi(v, \varsigma) = c$ at (v_0, ς_0) .

2 Fundamentals of fractional calculus

Fractional calculus stands out in the modeling problems involving nonlocality and memory effect concepts that are not well explained by integer-order calculus. The historical development of fractional calculus dates back to several centuries and involves contributions from multiple mathematicians. The concept of FDs has more than 300 years of history, yet it is still an exciting research topic, and interested researchers are actively working on it. It can be traced back to the letter from L'Hôpital to Leibnitz in 1695, which has the meaning of the derivative of a function of order $1/2$. Later investigations and further developments were by other mathematicians, such as Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1823–1826, Liouville in 1832–1873, Riemann in 1847, Holmgren in 1865–1867, and Grunwald in 1867–1872. These are early mathematicians who explored the possibility of extending differentiation and integration to noninteger orders [4,7–9].

However, the formalization of fractional calculus took several more years to develop. Since then, many researchers tried to put a helpful definition of a FD. Most of them used an integral form for the FD. We briefly introduce the commonly used FD in literature. In 1819, the first mention of FD in a published paper by Lacroix [10]. Starting with $\psi = \varsigma^m$, where $m, n \in \mathbb{N}$, and $m \geq n$, Lacroix found the n th derivative of ψ and he further obtained the generalized form with $\Gamma(\cdot)$ the well-known Gamma function by

$$\frac{d^n \psi}{d\varsigma^n} = \frac{m! \varsigma^{m-n}}{(m-n)!} = \frac{\Gamma(m+1) \varsigma^{m-n}}{\Gamma(m-n+1)}. \quad (2.1)$$

In particular, he calculated the following derivative of fractional order, for $m = 1$ and $n = 1/2$, he obtained

$$\frac{d^{1/2} \psi}{d\varsigma^{1/2}} = \frac{2\sqrt{\varsigma}}{\sqrt{\pi}}. \quad (2.2)$$

The next stage was taken from Fourier in 1822. He gave a definition of FD through the so-called Fourier transform by

$$\mathcal{D}^\mu \psi(\varsigma) = \frac{1}{2\pi} \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \psi(\xi) v^\mu \cos\left[v(\varsigma - \xi) + \frac{\mu\pi}{2}\right] dv d\xi. \quad (2.3)$$

Abel in 1823, discussed the FD as well.

The comprehensive development and formalization of fractional calculus occurred in the 19th and 20th centuries. Riemann and Liouville established the foundation of modern fractional calculus in the mid-19th century. Riemann introduced the concept of fractional integration, and Liouville further developed the theory by introducing the FD. Liouville made the progress work in FDs and successfully applied it into potential theory. If a function ψ can be expanded into an infinite series, then its FD can be obtained by

$$\mathcal{D}^\mu \psi(\varsigma) = \sum_{k=0}^{\infty} c_k a_k^\mu e^{a_k \varsigma}, \quad (2.4)$$

and if the function ψ cannot be expanded into an infinite series, then its FD for $k > 0$ is obtained by using the Gamma function by

$$\mathcal{D}^\mu \varsigma^{-k} = \frac{(-1)^\mu}{\Gamma(k)} \int_0^\infty \eta^{k+\mu-1} e^{-\varsigma \eta} d\eta = \frac{(-1)^\mu \Gamma(k+\mu) \varsigma^{-k-\mu}}{\Gamma(k)}. \quad (2.5)$$

In 1870, the Riemann-Liouville derivative of order μ for a given power function kernel was defined by

$${}^{\text{RL}}\mathcal{D}_\varsigma^\mu \psi(\varsigma) = \frac{1}{\Gamma(1-\mu)} \frac{d}{d\varsigma} \int_a^\varsigma \psi(\xi) (\varsigma - \xi)^{-\mu} d\xi. \quad (2.6)$$

The Riemann-Liouville FD is singular when $\varsigma = \xi$. To avoid such singularity, several other definitions have been proposed. The Caputo FD was introduced by Michele Caputo in 1967. This approach is often preferred in applications due to its ability to handle initial conditions more effectively.

The μ th Caputo FD of ψ is defined by [11]

$${}^{\text{C}}\mathcal{D}_\varsigma^\mu \psi(\varsigma) = \frac{1}{\Gamma(n-\mu)} \int_a^\varsigma \psi^{(n)}(\xi) (\varsigma - \xi)^{n-\mu-1} d\xi. \quad (2.7)$$

Hadamard in 1892 proposed a nonlocal FD with singular logarithmic function kernel with memory of order μ defined by [12]

$${}^{\text{H}}\mathcal{D}_\varsigma^\mu \psi(\varsigma) = \left(\varsigma \frac{d}{d\varsigma} \right)^n \frac{1}{\Gamma(n-\mu)} \int_a^\varsigma \left(\log \frac{\varsigma}{\xi} \right)^{n-\mu-1} \frac{\psi(\xi)}{\xi} d\xi. \quad (2.8)$$

The FD with a logarithmic function kernel of order $n-1 < \mu < n$ for $n \in \mathbb{N}$ has been proposed again by Beghin et al. in 2014, which is defined by [13]

$$\mathcal{D}_\varsigma^\mu \psi(\varsigma) = \frac{1}{\Gamma(n-\mu)} \int_{\frac{1-a}{b}}^\varsigma \log \left(\frac{a+b\varsigma}{a+b\xi} \right)^{n-\mu-1} \left(\left(\frac{a}{b} + \xi \right) \frac{d}{d\xi} \right)^n \psi(\xi) \frac{b}{a+b\xi} d\xi. \quad (2.9)$$

If $a = 0$ and $b = 1$, it is also known as the Hadamard FD. Caputo and Fabrizio in 2015 proposed a FD with an exponential function kernel [14]. By changing the kernel $(\varsigma - \xi)^{-\mu}$, where $0 < \mu < 1$ with the function $\exp\left(\frac{\mu(\varsigma - \xi)}{\mu - 1}\right)$ and $\frac{1}{\Gamma(1-\mu)}$ with $\frac{\mathcal{P}(\mu)}{1-\mu}$, we obtain the following new definition of fractional time derivative defined by

$${}^{\text{CF}}\mathcal{D}_\varsigma^\mu \psi(\varsigma) = \frac{\mathcal{P}(\mu)}{1-\mu} \int_a^\varsigma \psi'(\xi) \exp\left(\frac{\mu(\varsigma - \xi)}{\mu - 1}\right) d\xi, \quad (2.10)$$

where $\mathcal{P}(\mu)$ denotes a normalization function such that $\mathcal{P}(0) = \mathcal{P}(1) = 1$.

The main recent development in the field of fractional calculus was in 2016. Atangana and Baleanu announced a new definition of the FD with a nonlocal and no-singular kernel to overcome the weakness of the existence of the singular kernels involved in the previous fractional operators [15]. They presented some useful properties of the Atangana-Baleanu derivative and applied them to solve a fractional heat transfer

model. The AB operator in the Caputo sense and Riemann-Liouville sense of order $0 < \mu \leq 1$ and $\psi \in \mathbb{L}^1(a, b)$, $a < b$ are defined, respectively, by

$${}^{ABC}\mathcal{D}_\zeta^\mu \psi(\zeta) = \frac{\mathcal{P}(\mu)}{1-\mu} \int_a^\zeta \psi'(\xi) E_\mu \left[\frac{\mu(\zeta-\xi)^\mu}{\mu-1} \right] d\xi \quad (2.11)$$

and

$${}^{ABR}\mathcal{D}_\zeta^\mu \psi(\zeta) = \frac{\mathcal{P}(\mu)}{1-\mu} \frac{d}{d\zeta} \int_a^\zeta \psi(\xi) E_\mu \left[\frac{\mu(\zeta-\xi)^\mu}{\mu-1} \right] d\xi, \quad (2.12)$$

where $E_\mu(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(k\mu+1)}$ represents the Mittag-Leffler function.

Remark 2.1. Some significant advantages of the FDs

(1) The relation between the AB-Caputo and AB-Riemann-Liouville operators is given by

$${}^{ABC}\mathcal{D}_\zeta^\mu \psi(v, \zeta) = {}^{ABR}\mathcal{D}_\zeta^\mu \psi(v, \zeta) - \frac{\mathcal{P}(\mu)}{1-\mu} \psi(v, a) E_\mu \left[\frac{\mu(\zeta-a)^\mu}{\mu-1} \right]. \quad (2.13)$$

- (2) Both Caputo and Riemann-Liouville FDs have singular kernels.
- (3) Both ABC and ABR FDs have nonsingular and nonlocal kernels.
- (4) Both Caputo and ABC FDs of a constant function are zero.
- (5) Both Riemann-Liouville and ABR FDs of a constant function do not equal zero.

Proposition 2.1. For $\psi(v, \zeta)$ defined in $[a, b]$, $\zeta > 0$, $n-1 < \mu \leq n$, and $n \in \mathbb{N}$. The Atangana-Baleanu fractional integral operator ${}^{AB}\mathcal{I}_\zeta^\mu$ satisfies the following properties were verified in [16].

- (1) ${}^{ABC}\mathcal{D}_\zeta^\mu \psi(v, \zeta) = 0$, if $\psi(v, \zeta)$ is a constant function.
- (2) ${}^{ABR}\mathcal{D}_\zeta^\mu [{}^{AB}\mathcal{I}_\zeta^\mu \psi(v, \zeta)] = \psi(v, \zeta)$.
- (3) ${}^{AB}\mathcal{I}_\zeta^\mu [{}^{ABC}\mathcal{D}_\zeta^\mu \psi(v, \zeta)] = \psi(v, \zeta) - \sum_{k=0}^n \frac{(\zeta-a)^k}{k!} \frac{\partial^k \psi(v, a)}{\partial \zeta^k}$.
- (4) ${}^{AB}\mathcal{I}_\zeta^\mu [{}^{ABR}\mathcal{D}_\zeta^\mu \psi(v, \zeta)] = \psi(v, \zeta) - \sum_{k=0}^{n-1} \frac{(\zeta-a)^k}{k!} \frac{\partial^k \psi(v, a)}{\partial \zeta^k}$.

Mathematical models established using FDs overlap better with experimental data than models based on integer-order derivatives. FD provides a mathematical tool to describe and analyze systems that exhibit memory or long-term dependence, which provides an understanding of real-world processes. These derivatives have been developed to deal with the solutions of significant equations in various scientific fields. The definitions of the fractional order derivative are not unique and can yield different behaviors for the FD, mainly when the order is noninteger. One practical reason for the different FD formulations in literature is that due to the necessity of using a model describing the behavior of viscoelastic materials, thermal media, or electromagnetic systems, which cannot be defined very well in the old approach. Moreover, each formulation in Tables 1 and 2 captures specific aspects of the viscoelastic response, such as memory effects, relaxation processes, or singularities, allowing researchers to choose the most appropriate derivative. Consequently, each application area may require a specific FD formulation. The choice of initial and boundary conditions can influence the definition of FDs depending on the problem. For example, in modeling anomalous diffusion phenomena behavior in physics, the Riemann-Liouville FD is commonly used in which the derivative captures the memory effect in the diffusion process. On the other hand, if the initial conditions are essential in control systems, anomalous diffusion phenomena, or wave equations, then the Caputo derivative is more suitable and often preferred. It is important to note that when the order is an integer, the behavior of FDs is the same as that of classical derivatives. However, their behavior can differ when the order is noninteger. For instance, in the case of noninteger orders, the Caputo derivative of a constant is zero, unlike the Riemann-Liouville derivative, which does not yield zero. The development of new derivatives, including FDs based on nonlocal kernel functions, arises from the need to enhance the modeling capabilities of various systems. The nonlocality of

the new kernel allows a better description of the memory within structure and media with different scales [15]. The use of fractional partial differential equations (FPDEs) presents a more accurate description of diffusion processes in diverse scientific domains and is increasingly crucial in the representation of real-world situations.

Over the past few years, researchers have demonstrated that mathematical models can successfully describe many phenomena in engineering, physics, biology, and chemistry using fractional calculus's mathematical tools and solving different types of noninteger order partial differential equations. Indeed, determining an exact solution for a nonlinear fractional partial differential equation (FPDE) is challenging and tough to compute for some problems. Consequently, obtaining exact solutions for nonlinear FPDEs becomes crucial for evaluating and validating the accuracy and efficiency of numerical methods. Time-fractional differential equations involving FDs can be solved numerically using various methods. In the mathematical literature, there are many mathematical approaches of fractional order have been implemented to obtain approximate solutions for TFPDEs. These methods include the variational iteration method (VIM), adomian decomposition method (ADM), homotopy analysis method (HAM), homotopy perturbation method (HPM), Hermite wavelet method (HWM), optimal homotopy asymptotic method (OHAM), Shehu decomposition method (SDM), direct power series method (DPSM), and others. In modified numerical methods, an integral transform is used in the FD operators to enhance the performance and accuracy of the numerical methods. It is worth mentioning that there are different ways to define FDs, and thus using any of these definitions can be viewed as a fractional calculus.

3 Special functions of fractional calculus

The specialized mathematical functions in the field of mathematical physics have occurred to address the demands of applied sciences, offering solutions to integer-order differential equations derived from mathematical physics models. In this section, we provide some basic information on the most important functions in the theory of differentiation of arbitrary order and the theory of fractional differential equations including the Gamma and Beta functions and the Mittag-Leffler functions, and the readers may refer to [4,9,17,18].

3.1 Gamma function

The Gamma function $\Gamma(\cdot)$ plays a crucial role in various areas of mathematics, including complex analysis, number theory, and probability theory.

Let u be a complex number with $\text{Re}(u) > 0$, then the integral

$$\Gamma(u) = \int_0^{\infty} e^{-\eta} \eta^{u-1} d\eta, \quad (3.1)$$

is known as the Gamma function or the Euler integral of the second kind and converges absolutely. It is defined for all complex numbers except for the nonpositive integers. This function is a generalization of the factorial in the form defined by $\Gamma(u) = (u-1)!$.

It is obvious that by using integration by parts, we obtain the fundamental property of the Gamma function as follows:

$$\int_0^{\infty} e^{-\eta} \eta^u d\eta = -e^{-\eta} \eta^u \Big|_{\eta=0}^{\eta=\infty} + u \int_0^{\infty} e^{-\eta} \eta^{u-1} d\eta. \quad (3.2)$$

Proposition 3.1. *Some important properties of the Gamma function include*

- (1) $\Gamma(u+1) = u\Gamma(u)$.
- (2) $\Gamma(u+1) = u!$, $\forall u \in \mathbb{N}$.
- (3) $\Gamma(1/2) = \sqrt{\pi}$.

3.2 Beta function

The Beta function is a special function defined as follows:

$$\beta(u, v) = \int_0^1 \eta^{u-1} (1 - \eta)^{v-1} d\eta, \quad (3.3)$$

where $\operatorname{Re}(u) > 0$ and $\operatorname{Re}(v) > 0$, which is also called the Euler integral of the first kind.

Corollary 3.1. *The Beta function can be written in the form of Gamma function as follows:*

$$\beta(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \quad (3.4)$$

3.3 Mittag-Leffler function

The Mittag-Leffler function with two-parameters is a special function defined by

$$E_{u,v}(\eta) = \sum_{m=0}^{\infty} \frac{\eta^m}{\Gamma(mu + v)}, \quad \operatorname{Re}(u) > 0. \quad (3.5)$$

For $v = 1$, we obtain the so-called one-parameter Mittag-Leffler function:

$$E_{u,1}(\eta) = \sum_{m=0}^{\infty} \frac{\eta^m}{\Gamma(mu + 1)} \equiv E_u(\eta). \quad (3.6)$$

Corollary 3.2. *The most known relationships for the Mittag-Leffler can be mentioned as follows [19]:*

$$E_{1,1}(\eta) = e^\eta, \quad E_{1,2}(\eta) = \frac{e^\eta - 1}{\eta},$$

$$E_{2,1}(\eta) = \cosh(\sqrt{\eta}), \quad E_{2,1}(-\eta^2) = \cos(\eta).$$

3.4 Integral transforms

Integral transforms have proven to be powerful tools in addressing FPDEs. In this exploration, we focus on employing a range of transforms, such as the Laplace transform, Sumudu transform, Elzaki transform, or Shehu transform to convert FPDEs into ordinary differential equations. This transformation simplifies the solution process, making it possible to apply well-established techniques. The modified methods depend on combining an analytical or numerical approach with an appropriate transformation operator, making it easier to achieve analytical or numerical solutions for FPDEs that involve fractional-order derivatives. In the following, we provide the most employed integral transforms in fractional calculus.

Defined a set of function:

$$\mathbb{A} = \{\psi(\zeta) : \exists m, n_1, n_2 > 0, |\psi(\zeta)| \leq m e^{\frac{|\zeta|}{n_k}}, \quad \text{if } \zeta \in (-1)^k \times [0, \infty)\}.$$

(1) If $\psi(\zeta)$ is defined over interval $[0, \infty)$, then Laplace integral transform of a function $\psi(\zeta)$ is given by [20]

$$\mathbb{L}_\zeta[\psi(\zeta); \eta] = \mathcal{F}(\eta) = \int_{-\infty}^{\infty} e^{-\eta\zeta} \psi(\zeta) d\zeta, \quad (3.7)$$

and the inverse Laplace transform of $\mathcal{F}(\eta)$ is defined by the following complex integral

$$\psi(\zeta) = \mathbb{L}_\zeta^{-1}[\mathcal{F}(\eta); \zeta] = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow \infty} \int_{\gamma - i\varepsilon}^{\gamma + i\varepsilon} e^{\eta\zeta} \mathcal{F}(\eta) d\eta, \quad (3.8)$$

where the integration is done along the vertical line $\operatorname{Re}(\eta) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $\mathcal{F}(\eta)$.

- (2) The Sumudu integral transform of a function $\psi(\zeta) \in \mathbb{A}$ is defined by [21]

$$\mathbb{S}_{\zeta}[\psi(\zeta); \eta] = \mathcal{S}(\eta) = \frac{1}{\eta} \int_0^{\infty} e^{-\frac{\zeta}{\eta}} \psi(\zeta) d\zeta, \quad (3.9)$$

and the inverse of Sumudu integral transform is defined as:

$$\psi(\zeta) = \mathbb{S}_{\zeta}^{-1}[\mathcal{S}(\eta); \zeta] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\eta\zeta} \mathcal{S}\left(\frac{1}{\eta}\right) \frac{d\eta}{\eta}, \quad (3.10)$$

where $\operatorname{Re}(\eta) = \gamma$.

- (3) The Shehu integral transform of a function $\psi(\zeta) \in \mathbb{A}$ of exponential order is defined by [21,22]

$$\mathbb{SH}_{\zeta}[\psi(\zeta); \eta] = \mathcal{U}(w, \eta) = \int_0^{\infty} e^{-\frac{\eta\zeta}{w}} \psi(\zeta) d\zeta, \quad w, \eta > 0. \quad (3.11)$$

- (4) The Elzaki integral transform of a function $\psi(\zeta) \in \mathbb{A}$ is closely related with the Laplace transform and Sumudu transform, which is defined by [23]

$$\mathbb{E}_{\zeta}[\psi(\zeta); \eta] = \mathcal{V}(\eta) = \eta \int_0^{\infty} e^{-\frac{\zeta}{\eta}} \psi(\zeta) d\zeta. \quad (3.12)$$

Proposition 3.2. *The integral transforms have many important properties. Let $\psi(\zeta)$ and $\phi(\zeta)$ be two functions of $\zeta \geq 0$, where $\mathcal{F}(\eta)$ denotes the Laplace transform of $\psi(\zeta)$ and $\mathcal{G}(\eta)$ denotes the Laplace transform of $\phi(\zeta)$.*

- (1) *Linearity property: The Laplace transform with respect to ζ of sum functions $\psi(\zeta)$ and $\phi(\zeta)$ is given as follows:*

$$\mathbb{L}_{\zeta}[(\alpha\psi(\zeta) + \gamma\phi(\zeta)); \eta] = \alpha\mathbb{L}_{\zeta}[\psi(\zeta); \eta] + \gamma\mathbb{L}_{\zeta}[\phi(\zeta); \eta] = \alpha\mathcal{F}(\eta) + \gamma\mathcal{G}(\eta), \quad (3.13)$$

where $\alpha, \gamma \in \mathbb{R}$.

- (2) *Convolution property: The convolution property of the Laplace transform is given as follows:*

$$\mathbb{L}_{\zeta}[(\psi(\zeta) * \phi(\zeta)); \eta] = \mathbb{L}_{\zeta}\left[\int_0^{\zeta} \psi(\xi)\phi(\zeta - \xi) d\xi\right] = \mathbb{L}_{\zeta}[\psi(\zeta); \eta] \mathbb{L}_{\zeta}[\phi(\zeta); \eta] = \mathcal{F}(\eta)\mathcal{G}(\eta), \quad (3.14)$$

where the convolution of two functions $\psi(\zeta)$ and $\phi(\zeta)$ is defined as follows:

$$(\psi * \phi)(\zeta) = \int_0^{\zeta} \psi(\xi)\phi(\zeta - \xi) d\xi = \int_0^{\zeta} \psi(\zeta - \xi)\phi(\xi) d\xi. \quad (3.15)$$

Proof.

$$\begin{aligned} \mathbb{L}_{\zeta}[(\psi * \phi)(\zeta); \eta] &= \mathbb{L}_{\zeta}\left[\int_0^{\zeta} \psi(\xi)\phi(\zeta - \xi) d\xi\right] \\ &= \int_0^{\infty} \int_0^{\infty} \psi(\xi)\phi(\zeta - \xi) d\xi \left[e^{-\eta\zeta} d\zeta \right] \\ &= \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(\zeta - \xi) e^{-\eta\zeta} d\zeta \right] d\xi. \end{aligned}$$

By setting $u = \varsigma - \xi \Rightarrow du = d\varsigma$, we obtain

$$\begin{aligned}\mathbb{L}_{\varsigma}[(\psi * \phi)(\varsigma); \eta] &= \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\eta(u+\xi)} du \right] d\xi \\ &= \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\eta u} e^{-\eta \xi} du \right] d\xi \\ &= \int_0^{\infty} e^{-\eta \xi} \psi(\xi) \left[\int_0^{\infty} e^{-\eta u} \phi(u) du \right] d\xi \\ &= \int_0^{\infty} e^{-\eta \xi} \psi(\xi) d\xi \mathcal{G}(\eta) \\ &= \mathcal{F}(\eta) \mathcal{G}(\eta).\end{aligned}$$

□

Proposition 3.3. [24] *The Elzaki transform meets the following properties.*

(1) *Linearity property.* For $\alpha, \gamma \in \mathbb{R}$, then

$$\mathbb{E}_{\varsigma}[(\alpha\psi(\varsigma) + \gamma\phi(\varsigma)); \eta] = \alpha\mathbb{E}_{\varsigma}[\psi(\varsigma); \eta] + \gamma\mathbb{E}_{\varsigma}[\phi(\varsigma); \eta] = \alpha\mathcal{V}(\eta) + \gamma\mathcal{K}(\eta). \quad (3.16)$$

(2) *Convolution property.*

$$\mathbb{E}_{\varsigma}[(\psi * \phi)(\varsigma); \eta] = \mathbb{E}_{\varsigma} \left[\int_0^{\varsigma} \psi(\xi) \phi(\varsigma - \xi) d\xi \right] = \frac{1}{\eta} \mathbb{E}_{\varsigma}[\psi(\varsigma); \eta] \mathbb{E}_{\varsigma}[\phi(\varsigma); \eta] = \frac{1}{\eta} \mathcal{V}(\eta) \mathcal{K}(\eta). \quad (3.17)$$

Proof.

$$\begin{aligned}\mathbb{E}_{\varsigma}[(\psi * \phi)(\varsigma); \eta] &= \mathbb{E}_{\varsigma} \left[\int_0^{\infty} \psi(\xi) \phi(\varsigma - \xi) d\xi \right] \\ &= \eta \int_0^{\infty} \left[\int_0^{\infty} \psi(\xi) \phi(\varsigma - \xi) d\xi \right] e^{-\frac{\varsigma}{\eta}} d\varsigma \\ &= \eta \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(\varsigma - \xi) e^{-\frac{\varsigma}{\eta}} d\varsigma \right] d\xi.\end{aligned}$$

By setting $u = \varsigma - \xi \Rightarrow du = d\varsigma$, we obtain

$$\begin{aligned}\mathbb{E}_{\varsigma}[(\psi * \phi)(\varsigma); \eta] &= \eta \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\frac{(u+\xi)}{\eta}} du \right] d\xi \\ &= \eta \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\frac{u}{\eta}} e^{-\frac{\xi}{\eta}} du \right] d\xi \\ &= \eta \int_0^{\infty} e^{-\frac{\xi}{\eta}} \psi(\xi) \left[\int_0^{\infty} e^{-\frac{u}{\eta}} \phi(u) du \right] d\xi \\ &= \eta \int_0^{\infty} e^{-\frac{\xi}{\eta}} \psi(\xi) d\xi \frac{1}{\eta} \mathcal{K}(\eta) \\ &= \frac{1}{\eta} \mathcal{V}(\eta) \mathcal{K}(\eta),\end{aligned}$$

where $\mathcal{V}(\eta)$ and $\mathcal{K}(\eta)$ are the Elzaki transform of the functions $\psi(\varsigma)$ and $\phi(\varsigma)$, respectively.

□

Proposition 3.4. Let $\gamma\psi(\varsigma)$ and $\alpha\phi(\varsigma)$ be in a set $\mathbb{A}$, then $\gamma\psi(\varsigma) + \alpha\phi(\varsigma) \in \mathbb{A}$, where $\gamma, \alpha \in \mathbb{R} \setminus \{0\}$, then we have the following:

(1) The Shehu transform is a linear operator.

$$\mathbb{S}\mathbb{H}[(\gamma\psi(\varsigma) + \alpha\phi(\varsigma)); \eta] = \gamma\mathbb{S}\mathbb{H}[\psi(\varsigma); \eta] + \alpha\mathbb{S}\mathbb{H}[\phi(\varsigma); \eta]. \quad (3.18)$$

Proof.

$$\begin{aligned} \mathbb{S}\mathbb{H}[(\gamma\psi(\varsigma) + \alpha\phi(\varsigma)); \eta] &= \int_0^{\infty} e^{-\frac{\eta\varsigma}{w}} (\gamma\psi(\varsigma) + \alpha\phi(\varsigma)) d\varsigma \\ &= \int_0^{\infty} e^{-\frac{\eta\varsigma}{w}} \gamma\psi(\varsigma) d\varsigma + \int_0^{\infty} e^{-\frac{\eta\varsigma}{w}} \alpha\phi(\varsigma) d\varsigma \\ &= \gamma \int_0^{\infty} e^{-\frac{\eta\varsigma}{w}} \psi(\varsigma) d\varsigma + \alpha \int_0^{\infty} e^{-\frac{\eta\varsigma}{w}} \phi(\varsigma) d\varsigma \\ &= \gamma\mathbb{S}\mathbb{H}[\psi(\varsigma); \eta] + \alpha\mathbb{S}\mathbb{H}[\phi(\varsigma); \eta], \end{aligned}$$

where $\mathbb{S}\mathbb{H}[\cdot]$ is the Shehu transformation. □

(2) The Shehu transform meets the following convolution equality.

$$\mathbb{S}\mathbb{H}[(\psi(\varsigma) * \phi(\varsigma)); \eta] = \mathcal{U}(w, \eta) \mathcal{V}(w, \eta). \quad (3.19)$$

Proof.

$$\begin{aligned} \mathbb{S}\mathbb{H}_{\varsigma}[(\psi * \phi)(\varsigma); \eta] &= \mathbb{S}\mathbb{H}_{\varsigma} \left[\int_0^{\infty} \psi(\xi) \phi(\varsigma - \xi) d\xi \right] \\ &= \int_0^{\infty} \int_0^{\infty} \psi(\xi) \phi(\varsigma - \xi) d\xi \left[e^{-\frac{\eta\varsigma}{w}} d\varsigma \right] \\ &= \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(\varsigma - \xi) e^{-\frac{\eta\varsigma}{w}} d\varsigma \right] d\xi. \end{aligned}$$

By setting $u = \varsigma - \xi \Rightarrow du = d\varsigma$, we obtain

$$\begin{aligned} \mathbb{S}\mathbb{H}_{\varsigma}[(\psi * \phi)(\varsigma); \eta] &= \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\frac{\eta(u+\xi)}{w}} du \right] d\xi \\ &= \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\frac{\eta u}{w}} e^{-\frac{\eta \xi}{w}} du \right] d\xi \\ &= \int_0^{\infty} e^{-\frac{\eta \xi}{w}} \psi(\xi) \left[\int_0^{\infty} e^{-\frac{\eta u}{w}} \phi(u) du \right] d\xi \\ &= \int_0^{\infty} e^{-\frac{\eta \xi}{w}} \psi(\xi) d\xi \mathcal{V}(w, \eta) \\ &= \mathcal{U}(w, \eta) \mathcal{V}(w, \eta), \end{aligned}$$

where $\mathcal{U}(w, \eta), \mathcal{V}(w, \eta)$ are Shehu transforms of $\psi(\varsigma)$ and $\phi(\varsigma)$, respectively, and both defined in the set $\mathbb{A}$. □

(3) The Sumudu transform of convolution is given by

$$\mathbb{S}_{\varsigma}[(\psi(\varsigma) * \phi(\varsigma)); \eta] = \eta \mathbb{S}_{\varsigma}[\psi(\varsigma); \eta] \mathbb{S}_{\varsigma}[\phi(\varsigma); \eta] = \eta \mathcal{U}(\eta) \mathcal{V}(\eta). \quad (3.20)$$

Proof.

$$\begin{aligned}\mathbb{S}_{\zeta}[(\psi * \phi)(\zeta); \eta] &= \mathbb{S}_{\zeta} \left[\int_0^{\infty} \psi(\xi) \phi(\zeta - \xi) d\xi \right] \\ &= \frac{1}{\eta} \int_0^{\infty} \int_0^{\infty} \psi(\xi) \phi(\zeta - \xi) d\xi \left[e^{-\frac{\zeta}{\eta}} d\zeta \right] \\ &= \frac{1}{\eta} \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(\zeta - \xi) e^{-\frac{\zeta}{\eta}} d\zeta \right] d\xi.\end{aligned}$$

By setting $u = \zeta - \xi \Rightarrow du = d\zeta$, we obtain

$$\begin{aligned}\mathbb{S}_{\zeta}[(\psi * \phi)(\zeta); \eta] &= \frac{1}{\eta} \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\frac{(u+\xi)}{\eta}} du \right] d\xi \\ &= \frac{1}{\eta} \int_0^{\infty} \psi(\xi) \left[\int_0^{\infty} \phi(u) e^{-\frac{u}{\eta}} e^{-\frac{\xi}{\eta}} du \right] d\xi \\ &= \frac{1}{\eta} \int_0^{\infty} e^{-\frac{\xi}{\eta}} \psi(\xi) \left[\int_0^{\infty} e^{-\frac{u}{\eta}} \phi(u) du \right] d\xi \\ &= \frac{1}{\eta} \int_0^{\infty} e^{-\frac{\xi}{\eta}} \psi(\xi) d\xi \int_0^{\infty} e^{-\frac{u}{\eta}} \phi(u) du \\ &= \eta \mathcal{U}(\eta) \mathcal{V}(\eta),\end{aligned}$$

where $\mathcal{U}(\eta), \mathcal{V}(\eta)$ are the Sumudu transforms of $\psi(\zeta)$ and $\phi(\zeta)$, respectively. $\square$

Lemma 3.1. [25,26] For $\operatorname{Re}(u), \operatorname{Re}(v) > 0$, and $\lambda \in \mathbb{R}$, then the integral transforms of the two-parameter Mittag-Leffler function $E_{u,v}(\lambda \zeta^u)$ are given as follows:

(1) Laplace transform of $\zeta^{v-1} E_{u,v}(\lambda \zeta^u)$ is

$$\mathbb{L}_{\zeta}[\zeta^{v-1} E_{u,v}(\lambda \zeta^u); \eta] = \frac{\eta^{u-v}}{\eta^u - \lambda}. \quad (3.21)$$

Proof.

$$\begin{aligned}\mathbb{L}_{\zeta}[\zeta^{v-1} E_{u,v}(\lambda \zeta^u); \eta] &= \int_0^{\zeta} \zeta^{v-1} \sum_{k=0}^{\infty} \frac{(\lambda \zeta^u)^k}{\Gamma(ku + v)} e^{-\eta \zeta} d\zeta \\ &= \sum_{k=0}^{\infty} \int_0^{\zeta} \frac{\zeta^{v-1} \lambda^k \zeta^{ku}}{\Gamma(ku + v)} e^{-\eta \zeta} d\zeta \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \int_0^{\zeta} \zeta^{v+ku-1} e^{-\eta \zeta} d\zeta \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \mathbb{L}[\zeta^{v+ku-1}; \eta] \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \frac{\Gamma(ku + v)}{\eta^{v+ku}} \\ &= \frac{1}{\eta^v} \sum_{k=0}^{\infty} \frac{\lambda^k}{\eta^{ku}} \\ &= \frac{1}{\eta^v} \left[1 + \left(\frac{\lambda}{\eta^u} \right) + \left(\frac{\lambda}{\eta^u} \right)^2 + \left(\frac{\lambda}{\eta^u} \right)^3 + \left(\frac{\lambda}{\eta^u} \right)^4 + \dots \right]\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\eta^v} \frac{1}{1 - \frac{\lambda}{\eta^u}} \\
 &= \frac{\eta^{u-v}}{\eta^u - \lambda}.
 \end{aligned}
 \quad \square$$

(2) Sumudu transform of $\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u)$ is

$$\mathbb{S}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u); \eta] = \frac{\eta^{v-1}}{1 - \lambda\eta^u}. \quad (3.22)$$

Proof.

$$\begin{aligned}
 \mathbb{S}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u); \eta] &= \frac{1}{\eta} \int_0^{\varsigma} \varsigma^{v-1} \sum_{k=0}^{\infty} \frac{(\lambda\varsigma^u)^k}{\Gamma(ku + v)} e^{-\frac{\varsigma}{\eta}} d\varsigma \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \frac{1}{\eta} \int_0^{\varsigma} \varsigma^{v+ku-1} e^{-\frac{\varsigma}{\eta}} d\varsigma \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \mathbb{E}[\varsigma^{v+ku-1}; \eta] \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \Gamma(ku + v) \eta^{v+ku-1} \\
 &= \eta^{v-1} \sum_{k=0}^{\infty} \lambda^k \eta^{ku} \\
 &= \eta^{v-1} [1 + (\lambda\eta^u) + (\lambda\eta^u)^2 + (\lambda\eta^u)^3 + (\lambda\eta^u)^4 + \dots] \\
 &= \frac{\eta^{v-1}}{1 - \lambda\eta^u}.
 \end{aligned}
 \quad \square$$

(3) Shehu transform of $\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u)$ is

$$\mathbb{SH}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u); \eta] = \frac{\left(\frac{w}{\eta}\right)^v}{1 - \lambda\left(\frac{w}{\eta}\right)^u}. \quad (3.23)$$

Proof.

$$\begin{aligned}
 \mathbb{SH}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u); \eta] &= \int_0^{\varsigma} \varsigma^{v-1} \sum_{k=0}^{\infty} \frac{(\lambda\varsigma^u)^k}{\Gamma(ku + v)} e^{-\frac{\eta\varsigma}{w}} d\varsigma \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \int_0^{\varsigma} \varsigma^{v+ku-1} e^{-\frac{\eta\varsigma}{w}} d\varsigma \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \mathbb{SH}[\varsigma^{v+ku-1}; \eta] \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \Gamma(ku + v) \left(\frac{w}{\eta}\right)^{v+ku} \\
 &= \left(\frac{w}{\eta}\right)^v \sum_{k=0}^{\infty} \lambda^k \left(\frac{w}{\eta}\right)^{ku} \\
 &= \left(\frac{w}{\eta}\right)^v \left[1 + \lambda\left(\frac{w}{\eta}\right)^u + \lambda^2\left(\frac{w}{\eta}\right)^{2u} + \lambda^3\left(\frac{w}{\eta}\right)^{3u} + \dots \right]
 \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{w}{\eta}\right)^v \left[1 + \left(\lambda \left(\frac{w}{\eta}\right)^u\right) + \left(\lambda \left(\frac{w}{\eta}\right)^u\right)^2 + \left(\lambda \left(\frac{w}{\eta}\right)^u\right)^3 + \dots \right] \\
&= \frac{\left(\frac{w}{\eta}\right)^v}{1 - \lambda \left(\frac{w}{\eta}\right)^u}.
\end{aligned}$$

□

(4) Elzaki transform of $\zeta^{v-1}E_{u,v}(\lambda\zeta^u)$ is

$$\mathbb{E}_{\zeta}[\zeta^{v-1}E_{u,v}(\lambda\zeta^u); \eta] = \frac{\eta^{v+1}}{1 - \lambda\eta^u}. \quad (3.24)$$

Proof.

$$\begin{aligned}
\mathbb{E}_{\zeta}[\zeta^{v-1}E_{u,v}(\lambda\zeta^u); \eta] &= \eta \int_0^{\zeta} \zeta^{v-1} \sum_{k=0}^{\infty} \frac{(\lambda\zeta^u)^k}{\Gamma(ku + v)} e^{-\frac{\zeta}{\eta}} d\zeta \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \eta \int_0^{\zeta} \zeta^{v+ku-1} e^{-\frac{\zeta}{\eta}} d\zeta \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \mathbb{E}[\zeta^{v+ku-1}; \eta] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ku + v)} \Gamma(ku + v) \eta^{v+ku+1} \\
&= \eta^{v+1} \sum_{k=0}^{\infty} \lambda^k \eta^{ku} \\
&= \eta^{v+1} [1 + \lambda\eta^u + \lambda^2\eta^{2u} + \lambda^3\eta^{3u} + \lambda^4\eta^{4u} + \dots] \\
&= \eta^{v+1} [1 + (\lambda\eta^u) + (\lambda\eta^u)^2 + (\lambda\eta^u)^3 + (\lambda\eta^u)^4 + \dots] \\
&= \frac{\eta^{v+1}}{1 - \lambda\eta^u}.
\end{aligned}$$

□

Table 1: The classical FDs

Name	Definition	Domain
Riemann-Liouville left-sided derivative	${}^{\text{RL}}\mathcal{D}_a^{\mu}\psi(\zeta) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{d\zeta^n} \int_a^{\zeta} \psi(\xi)(\zeta - \xi)^{n-\mu-1} d\xi$	$\zeta \geq a$
Riemann-Liouville right-sided derivative	${}^{\text{RL}}\mathcal{D}_b^{\mu}\psi(\zeta) = \frac{(-1)^n}{\Gamma(n-\mu)} \frac{d^n}{d\zeta^n} \int_{\zeta}^b \psi(\xi)(\zeta - \xi)^{n-\mu-1} d\xi$	$\zeta \geq b$
Caputo left-sided derivative	${}^{\text{C}}\mathcal{D}_a^{\mu}\psi(\zeta) = \frac{1}{\Gamma(n-\mu)} \int_a^{\zeta} \psi^{(n)}(\xi)(\zeta - \xi)^{n-\mu-1} d\xi$	$\zeta \geq a$
Caputo right-sided derivative	${}^{\text{C}}\mathcal{D}_b^{\mu}\psi(\zeta) = \frac{(-1)^n}{\Gamma(n-\mu)} \int_{\zeta}^b \psi^{(n)}(\xi)(\zeta - \xi)^{n-\mu-1} d\xi$	$\zeta \geq b$
Grünwald-Letnikov left-sided derivative	${}^{\text{GL}}\mathcal{D}_a^{\mu}\psi(\zeta) = \lim_{h \rightarrow 0} \frac{1}{h^{\mu}} \sum_{k=0}^{\left\lfloor \frac{\zeta-a}{h} \right\rfloor} (-1)^k \binom{\Gamma(\mu+1)}{\Gamma(k+1)\Gamma(\mu-k+1)} \psi(\zeta - kh)$	$\zeta \geq a$
Grünwald-Letnikov right-sided derivative	${}^{\text{GL}}\mathcal{D}_b^{\mu}\psi(\zeta) = \lim_{h \rightarrow 0} \frac{1}{h^{\mu}} \sum_{k=0}^{\left\lfloor \frac{b-\zeta}{h} \right\rfloor} (-1)^k \binom{\Gamma(\mu+1)}{\Gamma(k+1)\Gamma(\mu-k+1)} \psi(\zeta + kh)$	$\zeta \geq b$

Table 2: The commonly used FDs with nonsingular kernel and the order $\mu \in (0, 1)$ [27]

Name	Definition	Domain
Caputo-Fabrizio	${}^{\text{CF}}\mathcal{D}_{\zeta}^{\mu}\psi(\zeta) = \frac{\mathcal{P}(\mu)}{1-\mu} \int_a^{\zeta} \psi'(\xi) \exp\left(\frac{\mu(\zeta-\xi)}{\mu-1}\right) d\xi$	$\zeta \geq a$
Yang et al.	${}^{\text{Y}}\mathcal{D}_{\zeta}^{\mu}\psi(\zeta) = \frac{\mathcal{P}(\mu)}{1-\mu} \frac{d}{d\zeta} \int_a^{\zeta} \psi(\xi) \exp\left(\frac{\mu(\zeta-\xi)}{\mu-1}\right) d\xi$	$\zeta \geq a$
Atangana-Baleanu-Caputo	${}^{\text{ABC}}\mathcal{D}_{\zeta}^{\mu}\psi(\zeta) = \frac{\mathcal{P}(\mu)}{1-\mu} \int_a^{\zeta} \psi'(\xi) E_{\mu}\left[\frac{\mu(\zeta-\xi)^{\mu}}{\mu-1}\right] d\xi$	$\zeta \geq a$
Atangana-Baleanu-Riemann-Liouville	${}^{\text{ABR}}\mathcal{D}_{\zeta}^{\mu}\psi(\zeta) = \frac{\mathcal{P}(\mu)}{1-\mu} \frac{d}{d\zeta} \int_a^{\zeta} \psi(\xi) E_{\mu}\left[\frac{\mu(\zeta-\xi)^{\mu}}{\mu-1}\right] d\xi$	$\zeta \geq a$
Caputo-Fabrizio, Gaussian kernel	${}^{\text{CFG}}\mathcal{D}_{\zeta}^{\mu}\psi(\zeta) = \frac{1+\mu^2}{\sqrt{\pi\mu(1-\mu)}} \int_a^{\zeta} \psi'(\xi) \exp\left(\frac{\mu(\zeta-\xi)^2}{\mu-1}\right) d\xi$	$\zeta \geq a$
Sun-Hao-Zhang-Baleanu	${}^{\text{SHZB}}\mathcal{D}_{\zeta}^{\mu}\psi(\zeta) = \frac{\mathcal{P}(\mu)}{(1-\mu)^{\frac{1}{\mu}}} \int_a^{\zeta} \psi'(\xi) \exp\left(\frac{\mu(\zeta-\xi)^{\mu}}{\mu-1}\right) d\xi$	$\zeta \geq a$

3.5 A criterion for defining FD

How can we say whether a given operator is a FD? Determining whether an operator qualifies to be a FD involves several key criteria and mathematical principles. To establish a clear understanding of FDs, we select the wide sense criterion (WSC) and the strict sense criterion (SSC), which was proposed in [28].

An operator $\mathcal{D}^{\mu}$ is called a FD in WSC if it satisfies the following properties:

- (1) Linearity: The operator $\mathcal{D}^{\mu}$ must be linear.
- (2) Identity: The zero-order derivative of a function is the function itself, that is,

$$\lim_{\mu \rightarrow 0} \mathcal{D}^{\mu}\psi(\zeta) = \mathcal{D}^0\psi(\zeta) = \psi(\zeta). \quad (3.25)$$

- (3) The index law holds for $\mu < 0$ and $\nu < 0$, that is,

$$\mathcal{D}^{\mu}\mathcal{D}^{\nu}\psi(\zeta) = \mathcal{D}^{\mu+\nu}\psi(\zeta), \quad (3.26)$$

- (4) Backward compatibility: The FD gives the same result as the classical derivative when the order μ is an integer, that is,

$$\lim_{\mu \rightarrow 1} \mathcal{D}^{\mu}\psi(\zeta) = \mathcal{D}^1\psi(\zeta) = \psi'(\zeta) \quad \text{and} \quad \mathcal{D}'[\psi(\zeta)\phi(\zeta)] = \psi(\zeta)\phi'(\zeta) + \psi'(\zeta)\phi(\zeta). \quad (3.27)$$

- (5) The generalized Leibnitz rule is valid.

$$\mathcal{D}^{\mu}(\psi(\zeta)\phi(\zeta)) = \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(k+1)\Gamma(\mu-k+1)} \mathcal{D}^k\psi(\zeta)\mathcal{D}^{\mu-k}\phi(\zeta), \quad (3.28)$$

when $\mu = m$ for $m \in \mathbb{N}$, we observe the classical Leibnitz rule.

Remark 3.1. If the index law (3.26) modified to include the positive order, it will lead to the SSC. Therefore, the modified index law is

$$\mathcal{D}^{\mu}\mathcal{D}^{\nu}\psi(\zeta) = \mathcal{D}^{\mu+\nu}\psi(\zeta), \quad (3.29)$$

for any μ and ν .

Therefore, based on these properties, the operators Grünwald-Letnikov derivative ${}^{\text{GL}}\mathcal{D}_{\zeta}^{\mu}$, Riemann-Liouville derivative ${}^{\text{RL}}\mathcal{D}_{\zeta}^{\mu}$, and Caputo derivative ${}^{\text{C}}\mathcal{D}_{\zeta}^{\mu}$, $\mu > 0$ can be called fractional, as established in the work by Ortigueira and Machado [28]. The different definitions allow us to choose a suitable one to apply to

the problem at hand. The choice of the FD definitions depends on the specific situation and the properties one is willing to capture.

Remark 3.2.

(1) For problems involving time, the causal derivative definitions such as the Grünwald-Letnikov derivative and the Liouville derivative are commonly employed.

(a) The Liouville derivative is defined by

$$\mathcal{D}_+^\mu \psi(\varsigma) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{d\varsigma^n} \int_0^\infty \psi(\varsigma - \xi) \xi^{n-\mu-1} d\xi. \quad (3.30)$$

(b) Similarly, the Liouville-Caputo derivative can be used and is defined by

$$\mathcal{D}_+^\mu \psi(\varsigma) = \frac{1}{\Gamma(n-\mu)} \int_{-\infty}^\varsigma \psi^{(n)}(\xi) (\varsigma - \xi)^{n-\mu-1} d\xi. \quad (3.31)$$

(2) For space problems, we can use these formulas again, but with the meaning of the two-sided left and right derivatives. These derivative definitions are equivalent to a function with a Laplace transform or Fourier transform. Consequently, we can consider both singular and nonsingular kernel functions of FD definitions as tools that can be useful and effective in tackling problems and applications across different scientific fields. Their applications range from physics to signal processing, machine learning, and engineering.

(3) Ortigueira and Machado [29] suggested that Grünwald-Letnikov and regularized Liouville derivatives are the suitable derivatives for a signal-processing approach. Fractional differential equations are essential for modeling various physical phenomena, including anomalous diffusion and viscoelasticity.

4 Classical numerical approaches and their applications

4.1 Variational iteration method

The VIM [30,31] is a numerical technique that can be used to solve ordinary and partial differential equations. It has been an efficient scheme for handling analytically fractional differential equations. The main idea behind VIM is to find an appropriate correction functional, which is added to the given initial condition of the solution to generate a better approximation. The correction functional is determined by solving a linear variational equation, which is obtained by taking the variational derivative of a suitable functional with respect to the unknown solution. This method requires determining a general Lagrange multiplier λ , which can be identified by variational theory. The resulting sequence of approximations is expected to converge to the exact solution of the time-fractional differential equation. It can be used to solve both linear and nonlinear TFDEs.

To illustrate the basic idea of the VIM, consider the following differential equation:

$$\mathcal{D}_\varsigma^\mu \psi(v, \varsigma) + \mathcal{L}\psi(v, \varsigma) + \mathcal{N}\psi(v, \varsigma) = \mathcal{G}(v, \varsigma), \quad (4.1)$$

where $\mathcal{D}_\varsigma^\mu = \frac{\partial^\mu}{\partial \varsigma^\mu}$ is the FD of order $m-1 < \mu \leq m$ for $m \in \mathbb{N}$, $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively, and $\mathcal{G}(v, \varsigma)$ is a known analytical function. According to He's VIM, the correction functional for equation (4.1) can be written as follows:

$$\psi_{n+1}(v, \varsigma) = \psi_n(v, \varsigma) + \int_0^\varsigma \lambda(\xi) (\mathcal{D}_\xi^\mu \psi(v, \xi) + \mathcal{L}\tilde{\psi}_n(v, \xi) + \mathcal{N}\tilde{\psi}_n(v, \xi) - \mathcal{G}(v, \xi)) d\xi, \quad (4.2)$$

where λ is a general Lagrange multiplier, which can be identified by variational theory. It can be either a constant or a function. $\tilde{\psi}_0(v)$ represents an initial approximation, and $\tilde{\psi}_n$ is considered a restricted variation,

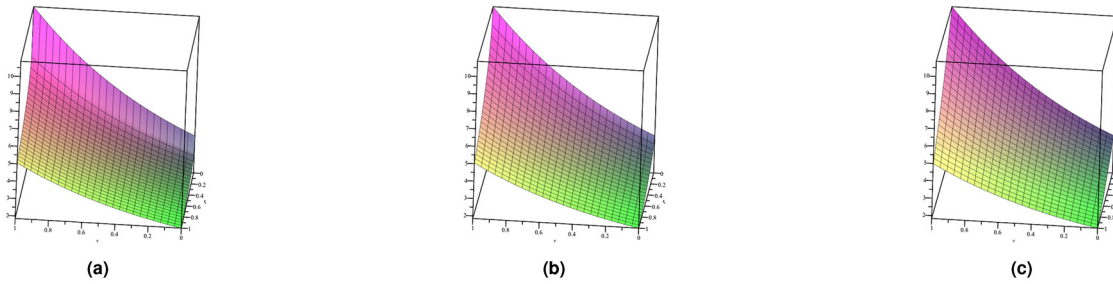


Figure 1: Three-dimensional surface for the behavior of the approximate solution $\psi(v, \zeta)$ with respect to v and ζ , when $\mu = 0.3$, $\mu = 0.7$, and $\mu = 1$, respectively, for equation (4.4). (a) ψ_{Approx} for $\mu = 0.3$, (b) ψ_{Approx} for $\mu = 0.7$, (c) ψ_{Approx} for $\mu = 1$.

implying $\delta\tilde{\psi}_0(v) = 0$. This method requires determining the Lagrange multiplier λ first. Once the Lagrange multiplier is obtained, the successive approximations ψ_{n+1} can be calculated using any initial function ψ_0 .

Consequently, the solution is given by

$$\psi(v, \zeta) = \lim_{n \rightarrow \infty} \psi_n(v, \zeta). \quad (4.3)$$

So, the corrections functional equation (4.2) will generate a sequence of approximations. The exact solution is obtained by taking the limit of these successive approximations. In [32], Prakash et al. employed the VIM to solve a fractional model of the Newell-Whitehead-Segel equation, which has the form

$$\begin{aligned} \mathcal{D}_\zeta^\mu \psi(v, \zeta) &= \psi_{vv}(v, \zeta) - 2\psi(v, \zeta), \\ \psi(v, 0) &= e^v, \end{aligned} \quad (4.4)$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. Figure 1 illustrates the graphical representation of the derived analytical solution $\psi(v, \zeta) = 4e^v - \frac{3e^v \zeta^\mu}{\Gamma(\mu+1)} + \frac{2e^v \zeta^{2\mu}}{\Gamma(2\mu+1)} - \frac{e^v \zeta^{3\mu}}{\Gamma(3\mu+1)}$ for different values of μ . When $\mu = 1$, the solution of equation (4.4) reduces to the exact solution $\psi(v, \zeta) = e^{v-\zeta}$. In [33], the VIM used to solve a time-fractional Black-Scholes equation involving Caputo FDs, which has the form

$$\begin{aligned} \mathcal{D}_\zeta^\mu \psi(v, \zeta) &= \psi_{vv}(v, \zeta) + (k-1)\psi_v(v, \zeta) - k\psi(v, \zeta), \\ \psi(v, 0) &= \max(e^v - 1, 0), \end{aligned} \quad (4.5)$$

where $0 < \mu \leq 1$ is the order of FD. The obtained analytical solution is defined as $\psi(v, \zeta) = \max(e^v, 0)(1 - E_\mu(-k\zeta^\mu)) + \max(e^v - 1, 0)E_\mu(-k\zeta^\mu)$, where E_μ is the one parameter Mittag-Leffler function.

It can be expanded in series form for $\mu = 1$ by $\psi(v, \zeta) = k\zeta + kv\zeta - \frac{k^2\zeta^2}{2} + \frac{k^3\zeta^3}{6} - \frac{k^2v\zeta^2}{2} + \frac{k^3v\zeta^3}{6} - \frac{k^2v^2\zeta^2}{4}$. When $\mu = 1$, the exact solution of equation (4.5) is $\psi(v, \zeta) = \max(e^v, 0)E_\mu(k\zeta^\mu) - \max(e^v - 1, 0)E_\mu(k\zeta^\mu)$. The VIM allows for systematically obtaining analytical approximations to fractional differential equations by leveraging correction functionals and successive approximations.

4.2 Adomian decomposition method

The ADM is a very effective approach for solving broad classes of nonlinear ordinary and partial differential equations, with essential applications in different fields of applied mathematics, engineering, physics, and biology. Adomian first introduced it to find an approximate solution to a differential equation. An advantage of the decomposition method is that it can provide an analytical approximation to a relatively wide class of nonlinear and stochastic equations without linearization, perturbation, closure approximations, or discretization methods, resulting in massive numerical computation [34,35]. With the presented decomposition method, the structure of the problem will not be changed, and no linearization will be made.

To illustrate the basic idea of the ADM, consider the differential equation:

$$\mathcal{D}_\zeta^\mu \psi(v, \zeta) + \mathcal{L}\psi(v, \zeta) + \mathcal{N}\psi(v, \zeta) = \mathcal{G}(v, \zeta), \quad (4.6)$$

where $\mathcal{D}_\zeta^\mu = \frac{\partial^\mu}{\partial \zeta^\mu}$ is the FD of order $n - 1 < \mu \leq n$ for $n \in \mathbb{N}$, $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively, and $\mathcal{G}(v, \zeta)$ is a known analytical function. Solving equation (4.6) for $\mathcal{D}_\zeta^\mu$, decomposing the nonlinear term into an infinite series of polynomials $\mathcal{N}\psi(v, \zeta) = \sum_{m=0}^{\infty} \mathcal{B}_m(v, \zeta)$ and $\mathcal{B}_m(v, \zeta)$ are the Adomian polynomials of $\psi_0, \psi_1, \dots, \psi_m$ defined by

$$\mathcal{B}_m(\psi_0, \psi_1, \dots, \psi_m) = \frac{1}{m!} \frac{d^m}{dq^m} \left[\mathcal{N} \left(\sum_{k=0}^{\infty} \psi_k q^k \right) \right]_{q=0}, \quad m = 0, 1, \dots \quad (4.7)$$

Applying the inverse operator $\mathcal{I}_\zeta^\mu$ on both sides yields a set of recursive relations given by

$$\begin{aligned} \psi_0(v, \zeta) &= \sum_{k=0}^{n-1} \frac{\zeta^k}{k!} \frac{\partial^k \psi(v, \zeta)}{\partial \zeta^k} \bigg|_{\zeta=0}, \\ \psi_{m+1}(v, \zeta) &= \mathcal{I}_\zeta^\mu [\mathcal{L}\psi_m(v, \zeta) - \mathcal{B}_m(v, \zeta)], \quad m \geq 0. \end{aligned} \quad (4.8)$$

4.3 Homotopy analysis method

In 1992, Liao introduced a novel and powerful analytical method for solving nonlinear problems, known as the HAM. This technique has proven to be a highly efficient and adaptive tool for finding analytical solutions to a wide range of nonlinear problems. One of the key advantages of the HAM is that it provides a straightforward way to adjust and control the convergence of the series solution. It offers several notable benefits. First, it is applicable even when the given nonlinear problem does not involve any small or large parameters. Second, it allows convenient adjustment and control of the convergence region and approximation rate of the series solution when necessary. Third, by selecting different sets of base functions, the HAM can be employed to efficiently approximate nonlinear problems [36,37].

To illustrate the basic idea of the HAM, consider the differential equation:

$$\mathcal{N}[\psi(v, \zeta)] = 0, \quad (4.9)$$

where $\mathcal{N}$ is a nonlinear operator, v and ζ are independent variables, $\psi(v, \zeta)$ is an unknown function. Liao [38] constructed the so-called zero-order deformation equation:

$$(1 - q)\mathcal{L}[\varphi(v, \zeta; q) - \psi_0(v, \zeta)] = q\hbar\mathcal{H}(v, \zeta)\mathcal{N}[\varphi(v, \zeta; q)], \quad (4.10)$$

where $q \in [0, 1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $\psi_0(v, \zeta)$ is an initial guess of $\psi(v, \zeta)$, $\varphi(v, \zeta; q)$ is an unknown function, and $\mathcal{H}(v, \zeta)$ is an auxiliary function. Expanding $\varphi(v, \zeta; q)$ into Taylor series with respect to q , we obtain

$$\varphi(v, \zeta; q) = \psi_0(v, \zeta) + \sum_{m=1}^{\infty} \psi_m(v, \zeta) q^m, \quad (4.11)$$

where

$$\psi_m(v, \zeta) = \frac{1}{m!} \frac{\partial^m \varphi(v, \zeta; q)}{\partial q^m} \bigg|_{q=0}. \quad (4.12)$$

Therefore, after simplification, we obtain the solution of m th-order deformation equation:

$$\psi_m(v, \zeta) = \mathcal{Q}_m \psi_{m-1}(v, \zeta) + \hbar \mathcal{L}^{-1}[\mathcal{R}_m[\vec{\psi}_{m-1}(v, \zeta)]], \quad (4.13)$$

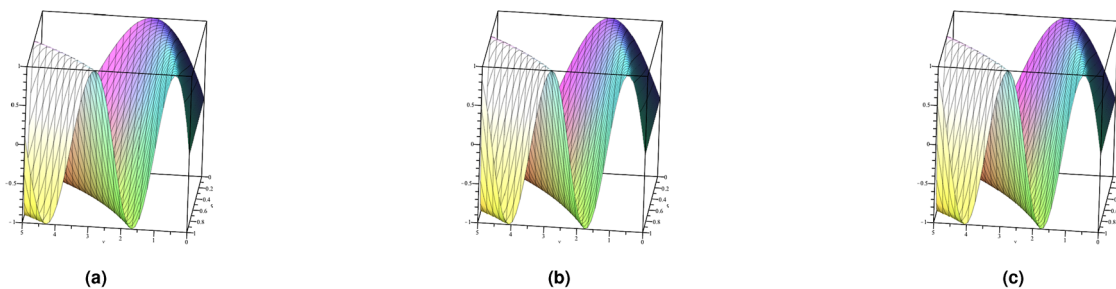


Figure 2: Three-dimensional surface for the behavior of the approximate solution when $\mu = \alpha = 0.9$, and $\mu = \alpha = 1$ and the exact solution $\psi(v, \zeta)$ with respect to v and ζ , respectively, for the model 1 given in [39]. (a) ψ_{Approx} for $\mu = \alpha = 0.9$, (b) ψ_{Approx} for $\mu = \alpha = 1$, and (c) Exact.

where

$$\mathcal{R}_m[\vec{\psi}_{m-1}] = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\varphi(v, \zeta; q)]}{\partial q^{m-1}} \bigg|_{q=0}, \quad (4.14)$$

$$Q_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \quad (4.15)$$

Recently, in the work by Qu et al. [39], the HAM was employed to solve three types of fractional-order partial differential equations: fractional Cauchy-Riemann equations, the fractional acoustic wave equation, and a two-dimensional space partial differential equation with time-fractional order. The derived analytical solution

$\psi(v, \zeta) = \sin_\beta v^\beta \left(1 + \frac{\zeta^\mu}{\Gamma(\mu+1)} + \frac{\zeta^{2\mu}}{\Gamma(2\mu+1)} + \dots \right)$ is graphically represented in Figure 2 for different values of μ and β .

4.4 Hermite wavelet method

The Hermite polynomials $H_m(v)$ of order m are defined on the interval $[-\infty, \infty]$ and given by the following repetition formula:

$$\begin{aligned} H_0(v) &= 1, \\ H_1(v) &= 2v, \\ H_{m+1}(v) &= 2vH_m(v) - 2mH_{m-1}(v), \quad m = 1, 2, \dots \end{aligned} \quad (4.16)$$

The Hermite polynomials $H_m(v)$ are orthogonal with respect to the weight function e^{-v^2} , that is,

$$\int_{-\infty}^{\infty} e^{-v^2} H_n(v) H_m(v) dv = \begin{cases} n! 2^n \sqrt{\pi}, & m = n, \\ 0, & m \neq n. \end{cases} \quad (4.17)$$

The Hermite wavelets are defined on interval $[0, 1]$ by [40]

$$\chi_{n,m}(v) = \begin{cases} 2^{k/2} \sqrt{\frac{1}{n! 2^n \sqrt{\pi}}} H_m(2^k v - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq v < \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise,} \end{cases} \quad (4.18)$$

where $k = 1, 2, \dots$ is the level of resolution, $n = 1, 2, \dots, \hat{n} = 2n - 1$, is the translation parameter, and $m = 1, 2, \dots, M - 1$ for $M > 0$ is the order of the Hermite polynomials.

To illustrate the basic idea of the HWM of FPDEs, consider the following differential equation

$$\mathcal{D}_\zeta^\mu \psi(v, \zeta) + \mathcal{L}\psi(v, \zeta) + \mathcal{N}\psi(v, \zeta) = \mathcal{G}(v, \zeta), \quad (4.19)$$

where $\mathcal{D}_\zeta^\mu = \frac{\partial^\mu}{\partial \zeta^\mu}$ is the FD of order $s - 1 < \mu \leq s$ for $s \in \mathbb{N}$, $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively, and $\mathcal{G}(v, \zeta)$ is a known analytical function.

The two-dimensional Hermite wavelets are defined in the following formula:

$$\chi_{n,i,m,j}(v, \zeta) = \begin{cases} AH_i(2^{k_1}v - \hat{n})H_j(2^{k_2}\zeta - \hat{m}), & \frac{\hat{n}-1}{2^{k_1}} \leq v < \frac{\hat{n}+1}{2^{k_1}}, \frac{\hat{m}-1}{2^{k_2}} \leq \zeta < \frac{\hat{m}+1}{2^{k_2}}, \\ 0, & \text{otherwise,} \end{cases} \quad (4.20)$$

where $A = 2^{\frac{k_1+k_2}{2}} \sqrt{\frac{1}{n!2^n\sqrt{\pi}}} \sqrt{\frac{1}{m!2^m\sqrt{\pi}}}$, k_1 , and $k_2 \in \mathbb{N}$ are the level of resolution, $n = 1, 2, \dots$, $\hat{n} = 2n - 1$ and $m = 1, 2, \dots$, $\hat{m} = 2m - 1$ are the translation parameters, and i, j are the orders of the Hermite polynomials. Then, we can set the function $\psi(v, \zeta)$ in terms of Hermite wavelet series as follows:

$$\psi(v, \zeta) = \sum_{n=1}^{2^{k_1-1}} \sum_{i=0}^{M_1-1} \sum_{m=1}^{2^{k_2-1}} \sum_{j=0}^{M_2-1} a_{n,i,m,j} \chi_{n,i,m,j}(v, \zeta), \quad (4.21)$$

and the nonlinear term $\mathcal{N}\psi(v, \zeta)$

$$\mathcal{N}\psi(v, \zeta) = \sum_{n=1}^{2^{k_1-1}} \sum_{i=0}^{M_1-1} \sum_{m=1}^{2^{k_2-1}} \sum_{j=0}^{M_2-1} b_{n,i,m,j} \chi_{n,i,m,j}(v, \zeta). \quad (4.22)$$

Thus, applying the inverse operator $\mathcal{I}_\zeta^\mu$ on both sides of equation (4.19) we obtain a system of algebraic equations in $a_{n,i,m,j}$ and $b_{n,i,m,j}$. By solving the system of equations using Newton's method, the Hermite wavelet coefficients $a_{n,i,m,j}$ and $b_{n,i,m,j}$ can be obtained, and finally, by substituting the value of coefficients in equation (4.21), we obtain the approximate solution for $\psi(v, \zeta)$. In [41], Ray and Gupta obtained the approximate solutions of time-fractional modified Fornberg-Whitham equation by HWM, with FD considered in the Caputo sense. The obtained results are compared with the exact solutions, which is given by [42] as well as with OHAM.

The nonlinear time-fractional modified Fornberg-Whitham equation

$$\begin{aligned} \psi_\zeta^\mu(v, \zeta) - \psi_{vv\zeta}(v, \zeta) + \psi_v(v, \zeta) - \psi(v, \zeta)\psi_{vvv}(v, \zeta) + \psi^2(v, \zeta)\psi_v(v, \zeta) - 3\psi_v(v, \zeta)\psi_{vv}(v, \zeta) &= 0, \\ \psi(v, 0) &= \frac{3}{4}(\sqrt{15} - 5)\operatorname{sech}^2\left(\frac{1}{20}\sqrt{10(5 - \sqrt{15})}v\right), \end{aligned} \quad (4.23)$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the exact solution of equation (4.23) is $\psi(v, \zeta) = \frac{3}{4}(\sqrt{15} - 5)\operatorname{sech}^2\left(\frac{1}{20}\sqrt{10(5 - \sqrt{15})}(v - (5 - \sqrt{15})\zeta)\right)$.

5 Recent numerical approaches and their applications

5.1 Direct power series method

A novel approach called the direct power series method was proposed by Salah et al. [43] to solve fractional initial value problems. This method offers a straightforward and efficient way to obtain better approximate solutions for linear and nonlinear fractional differential equations, including both ordinary and partial differential equations. The core principle behind the power series approach is to express the solution as an infinite series and subsequently determine the coefficients of the series.

To illustrate the basic idea of the direct power series method, consider the following differential equation:

$$\mathcal{L}\psi(v, \zeta) + \mathcal{N}\psi(v, \zeta) = 0, \quad (5.1)$$

where $\psi(v, \zeta)$ is an analytical functions and $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively. This method is expressing the solution of equation (5.1) in a fractional power series representation at $\zeta = 0$ for $s - 1 < \mu \leq s$ for $s \in \mathbb{N}$ of the form

$$\psi(v, \varsigma) = \sum_{n=0}^{\infty} a_n(v) \frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}. \quad (5.2)$$

The m th derivative of the fractional power series representation is defined as follows:

$$\mathcal{D}_{\varsigma}^m \psi(v, \varsigma) = \sum_{n=0}^{\infty} a_n^{(m)}(v) \frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}. \quad (5.3)$$

Therefore, we can obtain the following:

- (1) $a_{n+k}(v)$ is the coefficient for $\frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}$ in the series expansion of $\mathcal{D}_{\varsigma}^{k\mu}(v, \varsigma)$ for any $k = 0, 1, \dots$.
- (2) $\gamma^{(n+k)\mu} a_{n+k}(v)$ is the coefficient for $\frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}$ in the series expansion of $\mathcal{D}_{\varsigma}^{n\mu}(v, \gamma\varsigma)$ for any $k = 0, 1, \dots$ and $\gamma \in \mathbb{R}$.
- (3) $\sum_{i=0}^n a_i(v) b_{n-i}(v) \frac{\Gamma(n\mu + 1)}{\Gamma(i\mu + 1)\Gamma((n-i)\mu + 1)}$ is the coefficient for $\frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}$ in the series expansion of $\psi(v, \varsigma)\chi(v, \varsigma)$, where $\chi(v, \varsigma) = \sum_{m=0}^{\infty} b_m(v) \frac{\varsigma^{m\mu}}{\Gamma(m\mu + 1)}$.
- (4) $\sum_{i=0}^n \beta^{i\mu} \gamma^{(n-i)\mu} a_i(v) b_{n-i}(v) \frac{\Gamma(n\mu + 1)}{\Gamma(i\mu + 1)\Gamma((n-i)\mu + 1)}$ is the coefficient for $\frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}$ in the series expansion of $\psi(v, \beta\varsigma)\chi(v, \gamma\varsigma)$, where $\beta, \gamma \in \mathbb{R}$.
- (5) $\sum_{i=0}^n \beta^{(i+k)\mu} \gamma^{(n-i+s)\mu} a_{i+k}(v) b_{n-i+s}(v) \frac{\Gamma(n\mu + 1)}{\Gamma(i\mu + 1)\Gamma((n-i)\mu + 1)}$ is the coefficient for $\frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}$ in the series expansion of $\mathcal{D}_{\varsigma}^{k\mu}\psi(v, \beta\varsigma)\mathcal{D}_{\varsigma}^{s\mu}\chi(v, \gamma\varsigma)$, where $\beta, \gamma \in \mathbb{R}$ and $k, s = 0, 1, \dots$

The coefficient $a_n(v) = \mathcal{D}_{\varsigma}^{n\mu}(v, 0)$ for any $n = 0, 1, \dots$, if $\mathcal{D}_{\varsigma}^{n\mu}(v, \varsigma)$ is continuous on $(0, r)$, where r is the radius of convergence. The algorithm of DPSM in solving time-FPDEs as described in [44] can be given in the following simple steps:

- (1) Replace each term of the target equation by its suitable coefficient a_n of $\frac{\varsigma^{n\mu}}{\Gamma(n\mu + 1)}$ in the series expansion.
- (2) Simplify the obtained series representations to get a general term of the coefficients.
- (3) Substitute the values of $n = 1, 2, \dots$, the number of terms of the numerical series solution.

Qazza et al. [44], proposed a novel technique, termed the DPSM, for solving two distinct types of time-FPDEs formulated using the Caputo derivatives. The authors compared the fifth approximate solution obtained through DPSM with the exact solution for the integer-order case, as provided by Akter and Akbar [45].

- (1) The temporal-fractional partial Burger equation:

$$\begin{aligned} \psi_{\varsigma}^{\mu}(v, \varsigma) - \psi_{vv}(v, \varsigma) + \psi(v, \varsigma)\psi_v(v, \varsigma) &= 0, \\ \psi(v, 0) &= 2v, \end{aligned} \quad (5.4)$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the exact solution of equation (5.4) is $\psi(v, \varsigma) = \frac{2v}{1+2\varsigma}$.

- (2) The time-fractional partial Phi-4 equation

$$\begin{aligned} \psi_{\varsigma}^{2\mu}(v, \varsigma) - \psi_{vv}(v, \varsigma) + \eta^2 \psi(v, \varsigma) + \lambda \psi^3(v, \varsigma) &= 0, \\ \psi(v, 0) &= \sqrt{\frac{-\eta^2}{\lambda}} \tanh\left[\eta v \sqrt{\frac{1}{2(\varpi^2 - 1)}}\right], \end{aligned} \quad (5.5)$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense, η and λ are real valued constants, and ϖ is the speed of the traveling wave. When $\mu = 1$, the exact solution of equation (5.5)

is $\psi(v, \varsigma) = \sqrt{\frac{-\eta^2}{\lambda}} \tanh\left[\eta(v - \varpi\varsigma) \sqrt{\frac{1}{2(\varpi^2 - 1)}}\right]$.

5.2 Galerkin method

In [46], Bin Jebreen and Cattani proposed a numerical scheme based on the Galerkin method for solving the time-FPDEs and using the operational matrix for the Caputo FD. The obtained results were compared with the radial basis functions approximation method by [47].

(1) The nonhomogeneous time-fractional partial differential Burger's equation:

$$\begin{aligned}\psi_{\zeta}^{\mu}(\nu, \zeta) + \psi_{\nu}(\nu, \zeta) - \psi_{\nu\nu}(\nu, \zeta) &= 2 \frac{\zeta^{2-\mu}}{\Gamma(3-\mu)} + 2\nu - 2, \\ \psi(\nu, 0) &= \nu^2, \quad 0 \leq \nu \leq 1, \\ \psi(0, \zeta) &= \zeta^2, \quad \psi(1, \zeta) = \zeta^2 + 1, \quad \zeta \geq 1,\end{aligned}\tag{5.6}$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the exact solution of equation (5.6) is $\psi(\nu, \zeta) = \nu^2 + \zeta^2$.

(2) The nonhomogeneous time-fractional partial differential wave equation:

$$\begin{aligned}\psi_{\zeta}^{\mu}(\nu, \zeta) + \psi_{\nu}(\nu, \zeta) &= \frac{\zeta^{1-\mu} \sin(\nu)}{\Gamma(2-\mu)} + \zeta \cos(\nu), \\ \psi(\nu, 0) &= 0, \quad 0 \leq \nu \leq 1, \\ \psi(0, \zeta) &= 0, \quad \psi(1, \zeta) = \zeta \sin(1), \quad \zeta \geq 0,\end{aligned}\tag{5.7}$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the exact solution of equation (5.7) is $\psi(\nu, \zeta) = \zeta \sin(\nu)$, which is given by [48].

5.3 Variational iteration transform method (VITM)

This modified method combines the VIM with a convenient transformation operator, including the Laplace transform, Shehu transform, Sumudu transform, or Elzaki transform, integrating suitable FDs. The goal of this integration is to achieve analytical or numerical solutions for time-FPDEs. As an illustration, Shah et al. [49], have successfully combined the Shehu transform and the VIM to obtain an approximate analytical solution for the time-fractional Fornberg-Whitham equation.

(1) The nonlinear time-fractional Fornberg-Whitham equation:

$$\begin{aligned}\psi_{\zeta}^{\mu}(\nu, \zeta) - \psi_{\nu\nu\zeta}(\nu, \zeta) + \psi_{\nu}(\nu, \zeta) - \psi(\nu, \zeta)\psi_{\nu\nu}(\nu, \zeta) + \psi(\nu, \zeta)\psi_{\nu}(\nu, \zeta) - 3\psi_{\nu}(\nu, \zeta)\psi_{\nu\nu}(\nu, \zeta) &= 0, \\ \psi(\nu, 0) &= e^{\nu/2},\end{aligned}\tag{5.8}$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$,

the exact solution of equation (5.8) is $\psi(\nu, \zeta) = e^{\left(\frac{\nu}{2} - \frac{2\zeta}{3}\right)}$, which was given by [50].

(2) The nonlinear time-fractional Fornberg-Whitham equation:

$$\begin{aligned}\psi_{\zeta}^{\mu}(\nu, \zeta) - \psi_{\nu\nu\zeta}(\nu, \zeta) + \psi_{\nu}(\nu, \zeta) - \psi(\nu, \zeta)\psi_{\nu\nu}(\nu, \zeta) + \psi(\nu, \zeta)\psi_{\nu}(\nu, \zeta) - 3\psi_{\nu}(\nu, \zeta)\psi_{\nu\nu}(\nu, \zeta) &= 0, \\ \psi(\nu, 0) &= \cosh^2\left(\frac{\nu}{4}\right),\end{aligned}\tag{5.9}$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the

exact solution of equation (5.9) is $\psi(\nu, \zeta) = \cosh^2\left(\frac{\nu}{4} - \frac{11\zeta}{24}\right)$.

In [51], Zhang et al. have implemented the VITM for investigating the solution of two types of fractional Emden-Fowler equations.

(1) The linear homogeneous time-fractional Emden-Fowler heat equation:

$$\begin{aligned}\psi_{\zeta}^{\mu}(\nu, \zeta) &= \psi_{\nu\nu}(\nu, \zeta) + \frac{5}{\nu}\psi_{\nu}(\nu, \zeta) - (12\zeta^2 - 2\zeta\nu^2 + 4\nu^2\zeta^4)\psi(\nu, \zeta), \\ \psi(\nu, 0) &= 1,\end{aligned}\quad (5.10)$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the exact solution of equation (5.10) is $\psi(\nu, \zeta) = e^{\nu^2\zeta^2}$.

(2) The linear nonhomogeneous time-fractional Emden-Fowler heat equation:

$$\begin{aligned}\psi_{\zeta}^{\mu}(\nu, \zeta) &= \psi_{\nu\nu}(\nu, \zeta) + \frac{2}{\nu}\psi_{\nu}(\nu, \zeta) - (5 + 4\nu^2)\psi(\nu, \zeta) - (6 - 5\nu^2 - 4\nu^4), \\ \psi(\nu, 0) &= \nu^2 + e^{\nu^2},\end{aligned}\quad (5.11)$$

where $0 < \mu \leq 1$ describes the order of the time-FD, which has been taken in Caputo sense. When $\mu = 1$, the exact solution of equation (5.11) is $\psi(\nu, \zeta) = \nu^2 + e^{\nu^2+\zeta}$. Figure 3 illustrates the graphical representation of the derived analytical solution $\psi(\nu, \zeta) = \nu^2 + e^{\nu^2} + \left(\frac{4e^{\nu^2}}{\nu} - 3e^{\nu^2} - 8\nu^4 - 10\nu^2\right)\frac{\zeta^{\mu}}{\Gamma(\mu+1)} + \left(32\nu e^{\nu^2} - \frac{8e^{\nu^2}}{\nu} + 24\nu^2 e^{\nu^2} + 32e^{\nu^2} - 110\nu^2 + 80\nu^4 + 32\nu^6 - 20\right)\frac{\zeta^{2\mu}}{\Gamma(2\mu+1)} - (6 - 5\nu^2 - 4\nu^4)\frac{\zeta^{\mu}}{\Gamma(\mu+1)}$ for different values of μ .

In this method, the Lagrange multiplier $\lambda(\eta)$ can be defined as follows:

$$\lambda(\eta) = -\frac{u^{\mu}}{\eta^{\mu}}. \quad (5.12)$$

Therefore, the iteration formula for $m \geq 0$ can be constructed as follows:

$$\psi_{m+1}(\nu, \zeta) = \mathbb{S}_{\zeta}^{-1}\left[\frac{\psi_m(\nu, \zeta)}{\eta}\right] - \mathbb{S}_{\zeta}^{-1}\left[\frac{u^{\mu}}{\eta^{\mu}}\mathbb{S}_{\zeta}[\mathcal{L}\psi_m(\nu, \zeta) + \mathcal{N}\psi_m(\nu, \zeta)]\right]. \quad (5.13)$$

In [52], Iqbal et al. have successfully applied the modified VIM involving Atangana-Baleanu derivative of fractional-order to investigate the approximate solutions of fractional Fornberg-Whitham equations (5.8) and (5.9). This method is a mix of variational iteration technique and the Laplace transform approach. The LVIM approach has been successfully applied to a wide range of fractional differential equations, including those arising in physics, engineering, and other fields. In this method, the Lagrange multiplier $\lambda(\eta)$ can be defined as follows:

$$\lambda(\eta) = -\frac{\eta^{\mu}(1-\mu) + \mu}{\eta^{\mu}}. \quad (5.14)$$

Therefore, the iteration formula for $m \geq 0$ can be constructed as follows:

$$\psi_{m+1}(\nu, \zeta) = \mathbb{L}_{\zeta}^{-1}\left[\frac{\psi_m(\nu, \zeta)}{\eta}\right] - \mathbb{L}_{\zeta}^{-1}\left[\frac{\eta^{\mu}(1-\mu) + \mu}{\eta^{\mu}}\mathbb{L}_{\zeta}[\mathcal{L}\psi_m(\nu, \zeta) + \mathcal{N}\psi_m(\nu, \zeta)]\right]. \quad (5.15)$$

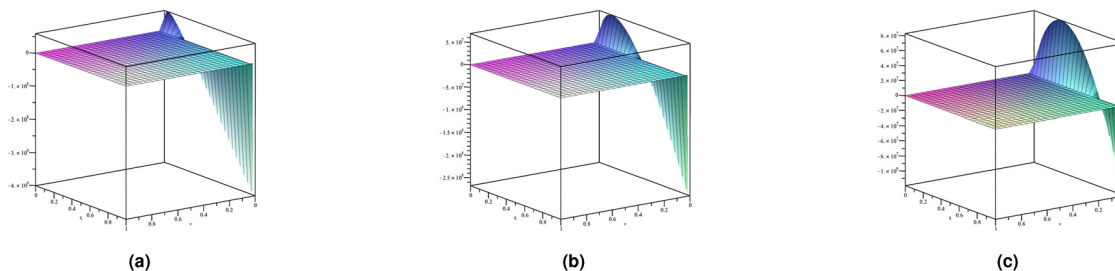


Figure 3: Three-dimensional surface for the behavior of the approximate solution $\psi(\nu, \zeta)$ when $\mu = 0.4, 0.6$ and 0.8 with respect to ν and ζ , respectively, for equation (5.11) given in [51]. (a) ψ_{Approx} for $\mu = 0.4$, (b) ψ_{Approx} for $\mu = 0.6$, and (c) ψ_{Approx} for $\mu = 0.8$.

In [24], Haroon et al. implemented the variational iteration transform involving fractional-order derivatives with the Atangana-Baleanu derivative. The Elzaki transformation is used in the Atangana-Baleanu-Caputo derivative to find the solution to the Fornberg-Whitham Equations (5.8) and (5.9). In this method, the Lagrange multiplier $\lambda(\eta)$ can be defined as follows:

$$\lambda(\eta) = -\frac{\mu\eta^\mu + 1 - \mu}{\mathcal{P}(\mu)}, \quad (5.16)$$

where $\mathcal{P}(\mu)$ denotes a normalization function such that $\mathcal{P}(0) = \mathcal{P}(1) = 1$.

Therefore, the iteration formula for $m \geq 0$ can be constructed as follows:

$$\psi_{m+1}(v, \zeta) = \psi_m(v, \zeta) - \mathbb{E}_\zeta^{-1} \left[\frac{\mu\eta^\mu + 1 - \mu}{\mathcal{P}(\mu)} \mathbb{E}_\zeta [\mathcal{L}\psi_m(v, \zeta) + \mathcal{N}\psi_m(v, \zeta)] \right]. \quad (5.17)$$

5.4 Adomian decomposition transform method

This modified method is based on a combination of the ADM with an appropriate transform operator such as Laplace transform, Shehu transform, Sumudu transform, or Elzaki transform with suitable FDs to achieve an analytical or numerical solutions of the time-FPDEs. In [24], Haroon et al. implemented the Adomian decomposition transform involving fractional-order derivatives with the Atangana-Baleanu-Caputo derivative. The Elzaki transformation is used in the Atangana-Baleanu-Caputo derivative to find the solution to the FW equations (5.8) and (5.9).

By using ADM procedure and Elzaki transform on the Caputo FD, we obtain the following recursive relations for $m \geq 0$

$$\begin{aligned} \psi_0(v, \zeta) &= \mathbb{E}_\zeta^{-1} [\eta^2 \psi(v, 0)], \\ \psi_{m+1}(v, \zeta) &= -\mathbb{E}_\zeta^{-1} \left[\frac{\mu\eta^\mu + 1 - \mu}{\mathcal{P}(\mu)} \mathbb{E}_\zeta [\mathcal{L}(\psi_m(v, \zeta)) + \mathcal{B}_m(v, \zeta)] \right]. \end{aligned} \quad (5.18)$$

In [49], Shah et al. implemented modified techniques, namely, the SDM to achieve an approximate analytical solution for the fractional Fornberg-Whitham equations (5.8) and (5.9). The FD is considered in the Caputo sense. Figure 4 and Table 3, illustrate the graphical representation and numerical values of the derived

analytical solution $\psi(v, \zeta) = \cosh^2(\frac{v}{4}) - \frac{2,717 \sinh(\frac{v}{4})\zeta^\mu}{3,584\Gamma(\mu+1)} + \frac{363 \cosh(\frac{v}{4})\zeta^{2\mu}}{2,048\Gamma(2\mu+1)} - \frac{1,331 \sinh(\frac{v}{4})\zeta^{3\mu}}{49,152\Gamma(3\mu+1)}$ for different values of μ .

By using ADM procedure and Shehu transform on the Caputo FD, we obtain the following recursive relations for $m \geq 0$

$$\begin{aligned} \psi_0(v, \zeta) &= \psi(v, 0), \\ \psi_{m+1}(v, \zeta) &= -\mathbb{S}_\zeta^{-1} \left[\frac{u^\mu}{\eta^\mu} \mathbb{S}_\zeta [\mathcal{L}(\psi_m(v, \zeta)) + \mathcal{B}_m(v, \zeta)] \right]. \end{aligned} \quad (5.19)$$

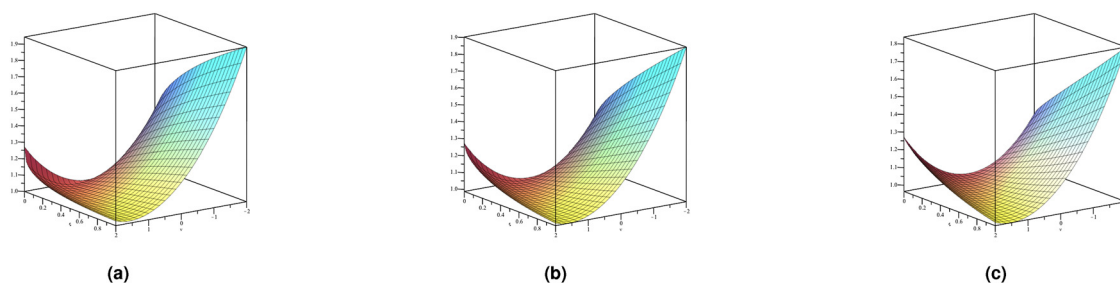


Figure 4: Three-dimensional surface for the behavior of the approximate solution $\psi(v, \zeta)$ when $\mu = 0.4, 0.6$ and 0.8 with respect to v and ζ , respectively, for equation (5.9) given in [49]. (a) ψ_{Approx} for $\mu = 0.4$, (b) ψ_{Approx} for $\mu = 0.6$, and (c) ψ_{Approx} for $\mu = 0.8$.

Table 3: This table exhibits the absolute errors $|\psi_{\text{Exact}} - \psi_{\text{SADM}}|$ of equation (5.9) at $\varsigma = 0.1$ for different fractional-order μ given in [49]

ν	$\mu = 0.4$	$\mu = 0.6$	$\mu = 0.8$	$\mu = 1$
0	0.028059	0.008048	0.001012	0.001216
0.3	0.009379	0.001055	0.001779	0.000018
0.6	0.009167	0.010082	0.004500	0.001261
0.9	0.027565	0.018968	0.007046	0.002746
1.2	0.045797	0.027637	0.009310	0.004568
1.5	0.063834	0.036011	0.011175	0.006869
1.8	0.081638	0.043994	0.012509	0.009800
2.1	0.099156	0.051480	0.013169	0.013533
2.4	0.116315	0.058338	0.012986	0.018258
2.7	0.133021	0.064418	0.011768	0.024193
3	0.149152	0.069536	0.009294	0.031588

In [52], Iqbal et al. have successfully applied the modified decomposition method involving Atangana-Baleanu derivative of fractional-order to investigate the approximate solutions of fractional Fornberg-Whitham equations (5.8) and (5.9). This method is a mix of Adomian decomposition technique and the Laplace transform approach. In addition, in [53], Kumar et al. solved equation (5.8), involving the Atangana-Baleanu FD by using the Adomian decomposition technique and the Laplace transform approach.

By using ADM procedure and Laplace transform on the Atangana-Baleanu derivative, we obtain the following recursive relations for $m \geq 0$

$$\begin{aligned}\psi_0(\nu, \varsigma) &= \psi(\nu, 0), \\ \psi_{m+1}(\nu, \varsigma) &= -\mathbb{L}_{\varsigma}^{-1} \left[\frac{\eta^{\mu}(1-\mu) + \mu}{\eta^{\mu}} \mathbb{L}_{\varsigma} [\mathcal{L}(\psi_m(\nu, \varsigma)) + \mathcal{B}_m(\nu, \varsigma)] \right].\end{aligned}\quad (5.20)$$

Recently in [54], Ganie et al. implemented an efficient analytical technique EADM within the Caputo operator to investigate the solutions of some FPDEs. The method is based on a combination of ADM and Elzaki transformation. The transformation of Elzaki is a very useful and powerful method for solving the integral equation that cannot be solved by the Sumudu transformation method.

Consider the following equations:

(1) The nonlinear fractional differential equation:

$$\begin{aligned}\psi_{\varsigma}^{\mu}(\nu, \varsigma) + \psi^2(\nu, \varsigma) &= 2\psi(\nu, \varsigma) + 1, \\ \psi(\nu, 0) &= 0,\end{aligned}\quad (5.21)$$

where $0 < \mu \leq 1$ describes the order of the time-FD. When $\mu = 1$, the exact solution of equation (5.21)

is $\psi(\nu, \varsigma) = 1 + \sqrt{2} \tanh \left[\sqrt{2} \varsigma + \frac{1}{2} \log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right]$.

(2) The diffusion fractional differential equation:

$$\begin{aligned}\psi_{\varsigma}^{\mu}(\nu, \varsigma) &= \psi_{\nu\nu}(\nu, \varsigma) + \psi(\nu, \varsigma), \\ \psi(\nu, 0) &= \cos(\pi\nu),\end{aligned}\quad (5.22)$$

where $0 < \mu \leq 1$ describes the order of the time-FD. When $\mu = 1$, the exact solution of equation (5.22) is $\psi(\nu, \varsigma) = \cos(\pi\nu)e^{(1-\pi^2)\varsigma}$.

By using ADM procedure and Elzaki transform on the Caputo FD, we obtain the following recursive relations for $m \geq 0$:

$$\begin{aligned}\psi_0(\nu, \varsigma) &= \psi(\nu, 0) + \mathbb{E}_{\varsigma}^{-1} [\eta^{\mu} \mathbb{E}_{\varsigma} [\mathcal{G}(\nu, \varsigma)]], \\ \psi_{m+1}(\nu, \varsigma) &= -\mathbb{E}_{\varsigma}^{-1} [\eta^{\mu} \mathbb{E}_{\varsigma} [\mathcal{L}(\psi_m(\nu, \varsigma)) + \mathcal{B}_m(\nu, \varsigma)]].\end{aligned}\quad (5.23)$$

Furthermore, in [55], Shah et al. developed a novel technique known as LADM, to derive a series type approximate semi-analytical solution for the Benney equation in the context of Caputo-Febrizio FDs.

Consider the following nonlinear equations.

- (1) The second-order fractional Benney equation with the CF derivative

$$\begin{aligned} {}^{\text{CF}}\mathcal{D}_\varsigma^\mu \psi(\nu, \varsigma) + 3\psi(\nu, \varsigma)\psi_\nu(\nu, \varsigma) + \psi_{\nu\nu}(\nu, \varsigma) + \chi\psi_{\nu\nu\nu}(\nu, \varsigma) + \psi_{\nu\nu\nu}(\nu, \varsigma) &= 0, \\ \psi(\nu, 0) &= \frac{1}{\nu}. \end{aligned} \quad (5.24)$$

- (2) The fractional third-order Benney equation with the CF derivative

$$\begin{aligned} {}^{\text{CF}}\mathcal{D}_\varsigma^\mu \psi(\nu, \varsigma) + 3\psi^2(\nu, \varsigma)\psi_\nu(\nu, \varsigma) + \psi_{\nu\nu}(\nu, \varsigma) + \chi\psi_{\nu\nu\nu}(\nu, \varsigma) + \psi_{\nu\nu\nu}(\nu, \varsigma) &= 0, \\ \psi(\nu, 0) &= \nu^3, \end{aligned} \quad (5.25)$$

where $\chi > 0$ and $0 < \mu \leq 1$ describes the order of the time-FD.

By using ADM procedure and Laplace transform on the Caputo-Febrizio FDs, we obtain the following recursive relations for $m \geq 0$:

$$\begin{aligned} \psi_0(\nu, \varsigma) &= \psi(\nu, 0), \\ \psi_{m+1}(\nu, \varsigma) &= \mathbb{L}_\varsigma^{-1} \left[\frac{\eta + \mu(1 - \eta)}{\mathcal{P}(\mu)} \frac{1}{\eta^{m+1}} \mathbb{L}_\varsigma [\mathcal{L}(\psi_m(\nu, \varsigma)) + \mathcal{B}_m(\nu, \varsigma)] \right]. \end{aligned} \quad (5.26)$$

5.5 Homotopy analysis transform method

This modified method is based on a combination of the HAM with an appropriate transform operator in order to achieve an analytical or numerical solutions of the time-FPDEs. In [56], Sunitha et al. used the q-homotopy analysis method combined with the Elzaki transform to investigate the two-dimensional advection-dispersion problem. These equations are mainly used to describe the fate of pollutants in aquifers. Recently, in [57], Shah et al. have obtained approximate solutions for the nonlinear partial differential Benney equation by applying a combination of the HAM and Laplace transformation, with the consideration given to the fractional-order Atangana-Baleanu derivative in the Riemann-Liouville sense. The Benney equation applications arise in different areas of physics and engineering, including dynamics and fluid mechanics.

Consider the following nonlinear equations:

- (1) The fractional order Benney equation with the ABR derivative

$$\begin{aligned} {}^{\text{ABR}}\mathcal{D}_\varsigma^\mu \psi(\nu, \varsigma) + 3\psi^2(\nu, \varsigma)\psi_\nu(\nu, \varsigma) + \psi_{\nu\nu}(\nu, \varsigma) + \chi\psi_{\nu\nu\nu}(\nu, \varsigma) + \psi_{\nu\nu\nu}(\nu, \varsigma) &= 0, \\ \psi(\nu, 0) &= \nu^2, \end{aligned} \quad (5.27)$$

- (2) The fractional order Benney equation with the ABR derivative

$$\begin{aligned} {}^{\text{ABR}}\mathcal{D}_\varsigma^\mu \psi(\nu, \varsigma) + 3\psi^2(\nu, \varsigma)\psi_\nu(\nu, \varsigma) + \psi_{\nu\nu}(\nu, \varsigma) + \chi\psi_{\nu\nu\nu}(\nu, \varsigma) + \psi_{\nu\nu\nu}(\nu, \varsigma) &= 0, \\ \psi(\nu, 0) &= \cos(\nu), \end{aligned} \quad (5.28)$$

where $\chi > 0$ and $0 < \mu \leq 1$ describes the order of the time-FD.

The solution of the m th order deformation for equations (5.18) and (5.28) is given by

$$\psi_m(\nu, \varsigma) = Q_m \psi_{m-1}(\nu, \varsigma) + \hbar \mathbb{L}^{-1} [\mathcal{R}_m[\vec{\psi}_{m-1}(\nu, \varsigma)]], \quad (5.29)$$

where

$$\begin{aligned} \mathcal{R}_m[\vec{\psi}_{m-1}] = & \mathbb{L}[\psi(v, \varsigma)] - \frac{\psi(v, 0)}{s} + \frac{\left(1 - \mu + \frac{\mu}{s^\mu}\right)}{\mathcal{P}(\mu)} \mathbb{L} \\ & \times \left[3 \sum_{i=0}^{m-1} \left(\sum_{j=0}^i \psi_j(v, \varsigma) \psi_{i-j}(v, \varsigma) \right) \frac{\partial}{\partial v} \psi_{m-i-1}(v, \varsigma) + \psi_{vv}(v, \varsigma) + \beta \psi_{vvv}(v, \varsigma) + \psi_{vvv}(v, \varsigma) \right] \end{aligned} \quad (5.30)$$

$$Q_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \quad (5.31)$$

6 Computational challenges

The review provides analytical methods for solving FPDEs, including the ability to obtain exact solutions, gain an understanding of the underlying physical phenomena, and reduce computational costs compared to numerical methods. Solving FPDEs poses several challenges that are not present in classical PDEs. These challenges include nonlocality, singularities, memory effects, and long-range interactions. However, in the following, we address some of the challenges and limitations of numerical or analytical methods when applied to FPDEs.

- (1) Dealing with fractional nonlinear partial differential equations presents a significant limitation due to their complex nature and the absence of readily available solutions in closed forms. Analytical techniques are confined to specific systems, frequently necessitating the use of approximation methods or numerical simulations. In contrast, numerical methods offer broader applicability but demand substantial computational resources and strict validation to ensure accuracy and stability.
- (2) FPDEs often have nonlocal properties, where the FD at a point depends on the function values at all other points in the domain. Classical methods, which mostly depend on local information, struggle to capture this nonlocal behavior efficiently.
- (3) Due to the nonlocality of FDs, numerical methods for FPDEs may require keeping a large amount of historical data, making them memory-intensive and unusable for problems with long-time histories.
- (4) Many standard numerical software libraries and packages are designed for integer-order PDEs. Adapting them for FPDEs can be challenging, and dedicated software tools and libraries for FPDEs are less prevalent.

7 Conclusion

This comprehensive review explored various techniques and approaches revolving around fractional calculus, integral transforms, and fractional differential operators. We highlighted their applications and advantages in deriving solutions for FPDEs. These equations extend conventional partial differential equations by incorporating FDs, enabling the modeling of memory and nonlocal effects in diverse fields such as physics, biology, and engineering. Solving FPDEs analytically poses significant challenges due to the nonlocal nature of FDs. However, recent years have witnessed remarkable progress in analytical methods for tackling these equations, providing valuable tools for understanding complex phenomena involving memory and nonlocal effects. By combining semi-analytical techniques with appropriate transformations, researchers can expedite the process of obtaining solutions for nonlinear partial differential equations that model real-world phenomena. As FPDEs continue to gain importance in modeling complex phenomena, developing efficient numerical approaches remains crucial. With the ongoing research and exploration, the future of solving FPDEs analytically looks promising, paving the way for potential applications across diverse scientific and engineering disciplines.

Acknowledgement: The researchers would like to thank the Deanship of Scientific Research, Qassim University, for funding the publication of this project.

Author contributions: All authors have read and agreed to the published version of the manuscript. Methodology: FA; investigation: FA; validation: AK; writing-original draft: FA; writing-review and editing: FA and AK; supervision: AK and NS.

Conflict of interest: The authors declare no conflict of interest.

References

- [1] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal. **5** (2001), no. 4, 367–386, DOI: <https://doi.org/10.48550/arXiv.math/0110241>.
- [2] R. E. Gutierrez, J. M. Rosário, and J. Tenreiro Machado, *Fractional order calculus: basic concepts and engineering applications*, Math. Probl. Eng. **2010** (2010), 375858, DOI: <https://doi.org/10.1155/2010/375858>.
- [3] M. H. Tavassoli, A. Tavassoli, and M. O. Rahimi, *The geometric and physical interpretation of fractional order derivatives of polynomial functions*, Differ. Geom. Dyn. Syst. **15** (2013), 93–104, <http://www.mathem.pub.ro/dgds/v15/D15-ta.pdf>.
- [4] B. Guo, X. Pu, and F. Huang, *Fractional Partial Differential Equations and their Numerical Solutions*, World Scientific, Singapore, 2015.
- [5] A. Ahmadi, A. M. Amani, and F. A. Boroujeni, *Fractional-order IMC-PID controller design for fractional-order time delay processes*, 2020 28th Iranian Conference on Electrical Engineering (ICEE), Math. Probl. Eng., 2020, pp. 1–6, DOI: <https://doi.org/10.1109/ICEE50131.2020.9261041>.
- [6] R. Khalil, M. Al Horani, and M. A. Hammad, *Geometric meaning of conformable derivative via fractional cords*, J. Math. Comput. Sci. **19** (2019), no. 4, 241–245, DOI: <http://doi.org/10.22436/jmcs.019.04.03>.
- [7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [8] V. Daftardar-Gejji, *Fractional Calculus and Fractional Differential Equations*, Springer, Berlin, 2019.
- [9] I. Podlubny, *Fractional: An Introduction to Fractional Derivatives, Fractional, to Methods of their Solution and Some of Their Applications*, Elsevier, San Diego, 1998.
- [10] B. Ross, *Fractional Calculus and its Applications; Proceedings of the International Conference Held at the University of New Haven, June 1974*, Springer-Verlag, Berlin, 2006.
- [11] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophys. J. Int. **13** (1967), no. 5, 529–539, DOI: <https://doi.org/10.1111/j.1365-246X.1967.tb02303.x>.
- [12] F. Jarad, T. Abdeljawad, and D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Differential Equations **2012** (2012), 1–8, DOI: <https://doi.org/10.1186/1687-1847-2012-142>.
- [13] L. Beghin, R. Garra, and C. Macci, *Correlated fractional counting processes on a finite-time interval*, J. Appl. Probab. **52** (2015), no. 4, 1045–1061, DOI: <https://doi.org/10.1239/jap/1450802752>.
- [14] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl. **1** (2015), no. 2, 73–85, <https://www.naturalspublishing.com/files/published/0gb83k287mo759.pdf>.
- [15] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Therm. Sci. **20** (2016), 763–769, DOI: <https://doi.org/10.48550/arXiv.1602.03408>.
- [16] T. Abdeljawad and A. Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, J. Inequal. Appl. **2017** (2017), 1–11, DOI: <http://doi.org/10.1186/s13660-017-1400-5>.
- [17] R. Gorenflo, Y. Luchko, and F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. **2** (2007), no. 4, 383–414, DOI: <https://doi.org/10.48550/arXiv.math-ph/0701069>.
- [18] F. Mainardi, A. Mura, and G. Pagnini, *The M-Wright function in time-fractional diffusion processes: a tutorial survey*, Int. J. Differential Equations **2010** (2010), 104505, <https://arxiv.org/pdf/1004.2950.pdf>.
- [19] I. Petráš, *Fractional Derivatives, Fractional Integrals, and Fractional Differential Equations in Matlab*, IntechOpen, 2011.
- [20] M. R. Spiegel, *Schaumas Outline of Theory and Problems of Laplace Transforms*, Schaums Outline Series, McGraw-Hill, New York, 1965.
- [21] S. Maitama and W. Zhao, *New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations*, Int. J. Anal. Appl. **17** (2019), no. 2, 167–190, DOI: <https://doi.org/10.28924/2291-8639-17-2019-167>.
- [22] G. K. Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, Int. J. Math. Educ. Sci. Technol. **24** (1993), no. 1, 35–43, DOI: <https://doi.org/10.1080/0020739930240105>.
- [23] T. M. Elzaki, *The new integral transform Elzaki transform*, Glob. J. Pure Appl. Math. **7** (2011), no. 1, 57–64.
- [24] F. Haroon, S. Mukhtar, and R. Shah, *Fractional view analysis of Fornberg-Whitham equations by using Elzaki transform*, Symmetry **14** (2022), no. 10, 2118, DOI: <http://doi.org/10.3390/sym14102118>.
- [25] R. Belgacem, D. Baleanu, and A. Bokhari, *Shehu transform and applications to Caputo-fractional differential equations*, Int. J. Anal. Appl. **17** (2019), no. 6, 917–927, DOI: <https://doi.org/10.28924/2291-8639-17-2019-917>.
- [26] M. Meddahi, J. Hossein, and M. N. Ncube, *New general integral transform via Atangana-Baleanu derivatives*, Adv. Differential Equations **2021** (2021), no. 385, 1–14, DOI: <https://doi.org/10.1186/s13662-021-03540-4>.
- [27] D. Valério, M. D. Ortigueira, and A. M. Lopes, *How many fractional derivatives are there?*, Mathematics **10** (2022), no. 5, 737, DOI: <https://doi.org/10.3390/math10050737>.

- [28] M. D. Ortigueira and J. T. Machado, *What is a fractional derivative?*, J. Comput. Phys. **293** (2015), 4–13, DOI: <https://doi.org/10.1016/j.jcp.2014.07.019>.
- [29] M. Ortigueira and J. Machado, *Which derivative?*, Fractal Fract. **1** (2017), no. 1, 3, DOI: <https://doi.org/10.3390/fractalfract1010003>.
- [30] J. He, *A new approach to nonlinear partial differential equations*, Commun. Nonlinear Sci. Numer. Simul. **2** (1997), no. 4, 230–235, DOI: [https://doi.org/10.1016/S1007-5704\(97\)90007-1](https://doi.org/10.1016/S1007-5704(97)90007-1).
- [31] J. H. He, *Variational iteration method—a kind of non-linear analytical technique: some examples*, Int. J. Non-Linear Mech. **34** (1999), no. 4, 699–708, DOI: [http://doi.org/10.1016/S0020-7462\(98\)00048-1](http://doi.org/10.1016/S0020-7462(98)00048-1).
- [32] A. Prakash, M. Goyal, and S. Gupta, *Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation*, Nonlinear Eng. **8** (2019), no. 1, 164–171, DOI: <https://doi.org/10.1515/nleng-2018-0001>.
- [33] S. Gupta, *Numerical simulation of time-fractional Black-Scholes equation using fractional variational iteration method*, J. Comput. Math. Sci. **9** (2019), no. 9, 1101–1110.
- [34] G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl. **135** (1988), no. 2, 501–544, DOI: [http://doi.org/10.1016/0022-247X\(88\)90170-9](http://doi.org/10.1016/0022-247X(88)90170-9).
- [35] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Springer Science & Business Media, New York, 2013.
- [36] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PhD thesis, Shanghai Jiao Tong University, Shanghai, 1992.
- [37] S. Liao, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput. **147** (2004), no. 2, 499–513, DOI: [http://doi.org/10.1016/S0096-3003\(02\)00790-7](http://doi.org/10.1016/S0096-3003(02)00790-7).
- [38] S. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, CRC Press, Boca Raton, 2003.
- [39] H. Qu, Z. She, and X. Liu, *Homotopy analysis method for three types of fractional partial differential equations*, Complexity **2020** (2020), 7232907, DOI: <https://doi.org/10.1155/2020/7232907>.
- [40] U. Saeed and M. ur Rehman, *Hermite wavelet method for fractional delay differential equations*, J. Differential Equations **2014** (2014), 359093, DOI: <https://doi.org/10.1155/2014/359093>.
- [41] S. S. Ray and A. K. Gupta, *A numerical investigation of time-fractional modified Fornberg-Whitham equation for analyzing the behavior of water waves*, Appl. Math. Comput. **266** (2015), 135–148, DOI: <https://doi.org/10.1016/j.amc.2015.05.045>.
- [42] B. He, Q. Meng, and S. Li, *Explicit peakon and solitary wave solutions for the modified Fornberg-Whitham equation*, Appl. Math. Comput. **217** (2010), no. 5, 1976–1982, DOI: <https://doi.org/10.1016/j.amc.2010.06.055>.
- [43] E. Salah, R. Saadeh, A. Qazza, and R. Hatamleh, *Direct power series approach for solving nonlinear initial value problems*, Axioms **12** (2023), no. 2, 111, DOI: <https://doi.org/10.3390/axioms12020111>.
- [44] A. Qazza, R. Saadeh, and E. Salah, *Solving fractional partial differential equations via a new scheme*, AIMS Math. **8** (2023), no. 3, 5318–5337, DOI: <https://doi.org/10.3934/math.2023267>.
- [45] J. Akter and M. A. Akbar, *Exact solutions to the Benney-Luke equation and the Phi-4 equations by using modified simple equation method*, Results in Phys. **5** (2015), 125–130, DOI: <https://doi.org/10.1016/j.rinp.2015.01.008>.
- [46] H. Bin Jebreen and C. Cattani, *Solving time-fractional partial differential equation using Chebyshev cardinal functions*, Axioms **11** (2022), no. 11, 642, DOI: <https://doi.org/10.3390/axioms11110642>.
- [47] M. Uddin and S. Haq, *RBFs approximation method for time fractional partial differential equations*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011), no. 11, 4208–4214, DOI: <https://doi.org/10.1016/j.cnsns.2011.03.021>.
- [48] Z. Odibat and S. Momani, *The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics*, Comput. Math. Appl. **58** (2009), no. 11–12, 2199–2208, DOI: <https://doi.org/10.1016/j.camwa.2009.03.009>.
- [49] N. A. Shah, I. Dassios, E. R. El-Zahar, J. D. Chung, and S. Taherifar, *The variational iteration transform method for solving the time-fractional Fornberg-Whitham equation and comparison with decomposition transform method*, Mathematics **9** (2021), no. 2, 141, DOI: <http://doi.org/10.3390/math9020141>.
- [50] B. Fornberg and G. B. Whitham, *A numerical and theoretical study of certain nonlinear wave phenomena*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci. **289** (1978), no. 1361, 373–404, DOI: <https://doi.org/10.1098/rsta.1978.0064>.
- [51] R. Zhang, N. A. Shah, E. R. El-Zahar, A. Akgül, and J. D. Chung, *Numerical analysis of fractional-order Emden-Fowler equations using modified variational iteration method*, Fractals **31** (2023), no. 2, 2340028, DOI: <https://doi.org/10.1142/S0218348X23400285>.
- [52] N. Iqbal, H. Yasmin, A. Ali, A. Bariq, M. M. Al-Sawalha, and W. W. Mohammed, *Numerical methods for fractional-order Fornberg-Whitham equations in the sense of Atangana-Baleanu derivative*, J. Funct. Spaces **2021** (2021), 2197247, DOI: <http://dx.doi.org/10.1155/2021/2197247>.
- [53] D. Kumar, J. Singh, and D. Baleanu, *A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel*, Eur. Phys. J. Plus **133** (2018), 70, DOI: <http://doi.org/10.1140/epjp/i2018-11934-y>.
- [54] A. H. Ganie, M. M. AlBaidani, and A. Khan, *A comparative study of the fractional partial differential equations via novel transform*, Symmetry **15** (2023), no. 5, 1101, DOI: <http://doi.org/10.3390/sym15051101>.
- [55] K. Shah, A. R. Seadawy, and A. B. Mahmoud, *On theoretical analysis of nonlinear fractional order partial Benney equations under nonsingular kernel*, Open Phys. **20** (2022), no. 1, 587–595, DOI: <https://doi.org/10.1515/phys-2022-0046>.
- [56] M. Sunitha, F. Gamaoun, A. Abdulrahman, N. S. Malagi, S. Singh, R. J. Gowda, et al., *An efficient analytical approach with novel integral transform to study the two-dimensional solute transport problem*, Ain Shams Eng. J. **14** (2023), no. 3, 101878, DOI: <http://doi.org/10.1016/j.asej.2022.101878>.
- [57] R. Shah, Y. Alkhezi, and K. Alhamad, *An analytical approach to solve the fractional Benney equation using the q-homotopy analysis transform method*, Symmetry **15** (2023), no. 3, 669, DOI: <http://doi.org/10.3390/sym15030669>.