OPERATIONAL MATRIX BASED ON ORTHOGONAL POLYNOMIALS AND ARTIFICIAL NEURAL NETWORKS METHODS FOR SOLVING FRACTAL-FRACTIONAL DIFFERENTIAL EQUATIONS

By

SHLOOF AML MELAD ASAN

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DEDICATIONS

To my beloved family and friends
To the unknown who will be searching for the topic of my thesis, I dedicate my research, peace and greetings to you I dedicate to you my efforts, my knowledge and my work, I dedicate to you the fruit of my efforts, perhaps it will be a seed for your scientific project.

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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March 2024

Chairman: Associate Professor Norazak bin Senu, PhD
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This study provided some new methods to solve initial value problems (IVPs) and boundary value problems (BVPs) of fractal-fractional differential equations (FFDEs) using operational matrix (OM) and artificial neural networks (ANNs). This research is centered on deriving two methods and formulating two novel definitions of fractal-fractional differential and integral operators. The first part of this thesis presents a new definition of the generalized Caputo differential and integral operators with fractional order and fractal dimension. Utilizing the OM based on orthogonal polynomials (Legendre and Jacobi), a numerical method for addressing various types of FFDEs is provided. This thesis emphasizes the existence theory and numerical solutions of multi-order boundary and initial value FFDEs. In these chapters, we explore convergence, existence, and uniqueness of solutions to FFDEs, aiming to determine the existence and uniqueness of at least one solution. Additionally, an error-bound analysis is conducted to confirm the validity and convergence of the method. The OM simplifies FFDEs into algebraic systems, resulting in straightforward and easily solvable problems. Subsequently, the performance of the proposed technique in addressing real-world problems is demonstrated. In the second part of the thesis, we developed the Hilfer fractal-fractional derivative definition. Similarly, the OM with the tau method for Hilfer fractal-fractional differentiability is generalized for solving FFDEs based on orthogonal polynomials. Numerical results suggest that the proposed method is quite accurate compared to other existing methods. The Jacobi polynomial, with its two parameters, ξ and ϑ, leads to distinct collections of orthogonal polynomials. Adjusting these parameters generates different types of orthogonal polynomials, each with unique characteristics. We also investigated numerical illustrations by varying the values of fractional and fractal parameters as well as the number of terms from truncated shifted Legendre polynomials (SLPs) and shifted Jacobi polynomials (SJPs). Our OM techniques based on SLPs and SJPs require only a few terms to obtain an accurate solution. In the third part, ANNs based on a generalized power series method in the generalized Caputo fractal-fractional derivative (GCFFD) are derived to approximate solutions of linear and non-linear FFDEs. Finally, ANNs employing a combination of power series methods in the GCFFD are developed to approximate solutions of higher-order linear FFDEs with

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both constant and variable coefficients. Initially, the algorithm utilized a truncated series. The values of the unknown coefficients in this truncated power series were then determined using an optimization technique to minimize the criterion function. This discovery indicates convergence toward optimal model coefficients as the learning process advances. Compared to other traditional methods, the suggested approach has proven to be more accurate. The definitions and techniques provided surpass traditional methods in accuracy, representing a significant advancement in the field.

Keywords: Operational matrix, Fractal-fractional differential equations, Artificial neural networks, Generalized Caputo fractal-fractional derivative, Hilfer fractal-fractional derivative

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Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH OPERASI MATRIKS BERASASKAN POLINOMIAL ORTOGON DAN RANGKAIAN NEURAL BUATAN UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN PECAHAN-FRAKTAL

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Kajian ini menyediakan beberapa kaedah baharu untuk menyelesaikan masalah nilai awal (MNA) dan masalah nilai sempadan (MNS) persamaan pembezaan fraktal-pecahan (PPFP) menggunakan matriks operasi(MO) dan rangkaian neural buatan (RNB). Kajian ini tertumpu kepada memperoleh dua kaedah dan menerbitkan dua definisi pengoperasi fraktal-pecahan pembezaan dan kamiran. Bahagian pertama tesis ini membentangkan definisi baharu mengenai pembezaan Caputo teritlak dan pengoperasi kamiran dengan peringkat pecahan dan dimensi fraktal. Menggunakan MO berdasarkan polinomial ortogon (Legendre dan Jacobi), kaedah berangka untuk menyelesaikan pelbagai jenis PPFP diterbitkan. Tesis ini menekankan teori kewujudan dan penyelesaian berangka untuk nilai awal dan sempadan multi-peringkat PPFP. Dalam bab ini, kami mengkaji penumpuan, kewujudan, dan keunikan penyelesaian kepada PPFP, yang bertujuan untuk menentukan kewujudan dan keunikan sekurang-kurangnya satu penyelesaian. Di samping itu, analisis batas ralat dilakukan untuk mengesahkan kesahihan dan penumpuan kaedah. MO memudahkan PPFP ke dalam sistem algebra, menghasilkan masalah yang ringkas serta mudah diselesaikan. Seterusnya, prestasi teknik yang dicadangkan dalam menyelesaikan masalah dunia nyata ditunjukkan. Dalam bahagian kedua tesis, kami membangunkan definisi terbitan fraktal-pecahan Hilfer. Begitu juga, MO den gan kaedah tau terhadap kebolehbezaan pecahan-fraktal Hilfer diitlakkan untuk menyelesaikan PPFP berdasarkan polinomial ortogon. Keputusan berangka menunjukkan bahawa kaedah yang dicadangkan lebih jitu berbanding kaedah lain sedia ada. Polinomial Jacobi, dengan dua parameter, ξ dan ϑ, membawa kepada koleksi polinomial ortogon yang berbeza. Melaraskan parameter ini menghasilkan pelbagai jenis polinomial ortogon, masing-masing dengan ciri unik. Kami juga mengkaji ilustrasi berangka dengan mengubah nilai parameter pecahan dan fraktal serta bilangan sebutan daripada polinomial Legendre teranjak(PLT) yang dipangkas dan mengalihkan polinomial Jacobi teranjak(PJT). Teknik OM kami berdasarkan PLT dan PJT hanya memerlukan beberapa sebutan untuk mendapatkan penyelesaian yang jitu. Di bahagian ketiga, RNB berdasarkan kaedah siri kuasa teritlak dalam terbitan fraktal-pecahan Caputo teritlak (TFPCT) diperoleh kepada penyelesaian anggaran PPFP linear dan tak linear. Akhirnya, RNB yang menggunakan gabungan kaedah siri kuasa dalam TFPCT dibangunkan untuk penye-

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lesaian anggaran PPFP linear peringkat lebih tinggi dengan pekali tetap dan berubah. Pada permulaan, algoritma menggunakan siri terpangkas. Nilai-nilai pekali yang tidak diketahui dalam siri kuasa terpangkas ini kemudiannya ditentukan menggunakan teknik pengoptimuman untuk meminimumkan fungsi kriteria. Penemuan ini menunjukkan penumpuan ke arah pekali model optimum apabila proses pembelajaran berlaku. Berbanding dengan kaedah tradisional lain, pendekatan yang dicadangkan telah terbukti lebih jitu. Definisi dan teknik yang diperoleh melebihi kaedah tradisional dari segi kejituan mewakili kemajuan yang ketara dalam bidang.

Kata Kunci: Matriks operasi, Persamaan pembezaan Fraktal-pecahan, Rangkaian neural buatan, terbitan fraktal-pecahan Umum Caputo, terbitan fraktal-pecahan Hilfer

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Date: 11 July 2024

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LIST OF ABBREVIATIONS

DEs	Differential Equations
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
IVPs	Initial Value Problems
BVPs	Boundary Value Problems
BCs	Boundary Conditions
FFDEs	Fractal-Fractional Differential Equations
FDEs	Fractional Differential Equations
FD	Fractional Derivative
FC	Fractional Calculus
FF	Fractal-Fractional
OM	Operational Matrix
LOM	Legendre Operational Matrix
SLPs	Shifted Legendre Polynomials
JOM	Jacobi Operational Matrix
SJPs	Shifted Jacobi Polynomials
ANNs	Artificial Neural Networks
HFD	Hilfer Fractional Derivative
HFFD	Hilfer Fractal-Fractional Derivative
LWPT	Legendre Wavelet-like Operational Matrix Method
PLSM	Polynomial Least Squares Method
GPs	Generalized Power Series
HOL-FFDEs	Higher Order Linear Fractal-Fractional Differential Equations

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FFrANNs	Fractal-Fractional Artificial Neural Networks
FrNNsM	Fractional Neural Networks Method
IRKHSM	Iterative Reproducing Kernel Hilbert Space Method
FGHAM	Fractional Generalised Homotopy Method
FrDEsANN	Artificial Neural Network for Solving Fractional Order Differential Equations
NNPSM	Neural Network Based on Power Series Method
LeNN	Legendre Artificial Neural Network
FNNsM	Fractional Neural Networks Method
LDGM	Local Discontinuous Galerkin Method
PTSEM	Piecewise Taylor Series Expansion Method
LWS	Lucas Wavelet Scheme
Ps	Power Series
NGCFF	New Generalized Caputo Fractal-Fractional
NGCFFD	Generalized Caputo Fractal-Fractional Derivative
SLOM	Shifted Legendre Operational Matrix
SJOM	Shifted Jacobi Operational Matrix
SL	Shifted Legendre
SJ	Shifted Jacobi
L-MSE	Least Mean Square Error
RMSE	Root Mean Square Error
MAE	Mean Absolute Error
TIC	Theil's Inequality Coefficient
NSE	Nash Sutcliffe Efficiency

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CHAPTER 1

INTRODUCTION

Differential and integral operators are utilised to solve modeling related problems. Differential equations (DEs) are essential tools for modeling various issues in applied science and technology that involves unknown functions and derivatives, including mathematical models of electrical circuits, chemical reactions, mechanical systems, and fluid mechanics. Ordinary Differential Equations (ODEs), Partial Differential Equations (PDEs), Fractional Differential Equations (FDEs), and Fractal-Fractional Differential Equations (FFDEs) are the four forms of DEs. DEs can be classified into linear and nonlinear categories. The significance of nonlinear problems lies in the fact that the majority of phenomena in the world are inherently nonlinear, requiring the use of nonlinear equations for their accurate representation.

An analytical solution is obtained by solving a DE, expressing the dependent variable as an algebraic equation in terms of the independent variable. This solution is presented in a closed form. Conversely, a numerical solution involves approximations for a DE, typically lacking a closed-form representation and relying on computational methods for estimation.

This thesis is divided into two parts. Firstly, we explain on various DEs methods together with their real-world utilizations for solving linear/nonlinear systems of FFDEs. The principal goal of this thesis is to introduce new fractal-fractional differential and integral operators and develop new operational matrix modifications (spectral method) for numerically solving fractal-fractional differential equations (FFDEs). Furthermore, we present an Artificial Neural Networks (ANNs) approach to solve FFDEs. This chapter covers the fundamentals of FFDEs, fractional mathematical models, operational matrices, the tau method, and ANNs. It describes the problem statement, research objectives, and thesis outline.

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1.1 Fractional Calculus

Fractional calculus (FC) is a mathematical discipline concerned with derivatives and integrals of non-integer order. It is widely realized that fractional derivative-based models are much better than integer order models in many situations. Being nonlocal, the fractional derivatives provide excellent tool for understanding various materials and processes' memory and hereditary properties. This is the main advantage of fractional derivatives compared to classical integer order derivatives. As a result of their numerous real-world applications, fractional differential equations (FDEs) are becoming increasingly important. For many years, fractional calculus was considered an abstract mathematical concept.

However, the subject is now used in almost every science branch; numerous applications of the fractional derivative operator are used in many fields including viscoelastic damping (Caputo, 1967), anomalous diffusion processes, signal processing, electrochemistry, fluid flow, chemistry, and others (Oldham and Spanier, 1974; Sun et al., 2018). Around the world, it has been discovered that models based on fractional derivatives outperform integer-order models. For three centuries, fractional calculus became traditional but uncommon amongst science and engineering communities. The Riemann-Liouville, Granwald-Letinkov, and Caputo definitions of the fractional derivative are arguably the most used forms. These existing definitions are similar only in a few cases but are not identical in general (Podlubny, 1998). These properties outline the behaviors of derivatives and integrals of various orders:

1. When a function undergoes zeroth order differentiation or integration, the function remains unchanged.

2. If the order of differentiation or integration is an integer number, the fractional and ordinary operations are the same.

3. Just like the rules for regular derivatives and integrals, fractional operations follow

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linearity. For instance, for any form of fractional differentiation D^{α}:

$$D^{\alpha}(f(x)+g(x)) = D^{\alpha}f(x)+D^{\alpha}g(x).$$

1.2 Fractal-Fractional Derivative

The fractal derivative extends the traditional concept of derivatives to accurately represent the intricate and discontinuous characteristics found in fractal media such as magnetic plasma Chen (2006), heat transfer, wool fibers, groundwater flow (Atangana and Qureshi, 2019), geometric (Akgül, 2021), and porous materials (Fan and He, 2012). The concept of fractal-fractional derivatives introduces a fascinating extension to the traditional idea of derivatives in mathematics. While the standard derivatives we are familiar with describe how a function changes over a small, infinitesimal interval, fractal-fractional derivatives delve into a realm where this change is not confined to integer dimensions. Instead, they explore the idea that change can occur in dimensions that are fractions, or even non-integer values. This notion opens up a rich and intricate understanding of how quantities evolve and interact in systems where the traditional rules of calculus might not fully apply.

The fractal-fractional derivative (FFD) is an amalgamation of two preceeding concepts: the fractal derivative and the fractional derivative. It encompasses two distinct orders, namely fractional-order and fractal-order. Fractal-fractional (FF) differential and integral operators are new concepts that appears superior to existing fractional operators with constant orders. Selecting the fractional order leads to a fractal order system, while opting for a fractal order equal to one results in a fractional order system. The primary motivation for this research lies in the inherent association of fractal-fractional order differential equations with memory-based systems, commonly found in biological systems. Existing FD derivatives are represented by these derivatives, which have both memory and fractal dimension $D^{\alpha,\beta}$. The memory effect and fractal properties included in the FD α and β play an important role in describing real-world phenomena and can be explained using FD and FFD. These novel differential operators include the fractal

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derivative of a continuous function and power law, the exponential decay law, and the extended Mittag-Leffler function. These operators can be converted to classical, fractal, and fractional differential and integral operators in the limit cases, making them upper classes of differential and integral operators. The fractal differential and integral operators are recovered as the fractional order approaches zero. In a nutshell, it is expected that such operators have the capacity to identify self-similarities.

1.3 Operational Matrices

Researchers in applied sciences field often encounter situations where it is not possible to find precise analytical solutions to tackle differential or integral equations-related problems. Thus, there is a strong need to develop effective numerical methods that can provide approximate answers for these types of equations. Popular operational matrices approaches that make use of polynomials and spectral procedures such as Collocation and Tau methods resolve this issue by transforming both differential and integral equations into system of algebraic equations. Polynomials are widely utilized in mathematics due to their immense utility as they can be easily represented and solved using computers, can be used to portray various types of problems, and can be integrated and differentiated effortlessly. Examples of polynomial uses include construction of spline curves and highly accurate estimation of specific functions. The literature agrees that operational matrices methods can be effectively utilized to solve initial as well as boundary value issues for fractional order differential equations.

In the realm of fractional calculus, the derivation of operational matrices for fractional derivatives began with (Saadatmandi and Dehghan, 2010). This process involves considering a set of basis functions and their fractional derivatives. By examining the relationship between the original functions and their fractional derivatives, operational matrices for fractional differentiation can be derived. These matrices enable efficient computation of fractional derivatives in numerical methods, especially for solving fractional

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differential equations.

The goal is to create matrices that streamline the calculation of fractional derivatives, facilitating their implementation and analysis in various applications. These matrices are built using orthogonal polynomials, coupled with methods such as Collocation and Tau, to convert differential and integral equations into systems of algebraic equations. This conversion greatly simplifies their solution using computational software. The choice of polynomials is due to their practicality and widespread utility in mathematics. They are straightforward to define, computationally efficient, and capable of representing diverse functions. Their ease of integration and differentiation makes them valuable tools. Additionally, polynomials allow for the construction of spline curves by assembling them, enabling accurate approximations of various functions.

1.4 Artificial Neural Networks

Artificial neural networks (ANNs) have high learning ability. They have numerous advantages including high adaptability and fast error computation, and is usually utilised to solve ordinary differential equations, partial differential equations Lagaris et al. (1998), fuzzy differential equation (Effati and Buzhabadi, 2012), and fractional differential equations Raja et al. (2010b).

ANNs is highly effective in function approximation because it tackles the matter using differential equations method (in specific as a differential function). Computational intelligence methods are reliable, can improve accuracy and convergence rate, and require less computational time (Sabir et al., 2020; Jafarian et al., 2017).

Artificial intelligence techniques based on neural network models have been extensively used in various applied science and engineering problems such as financial (Coakley and Brown, 2000), medical (Agatonovic and Beresford, 2000), image recog-

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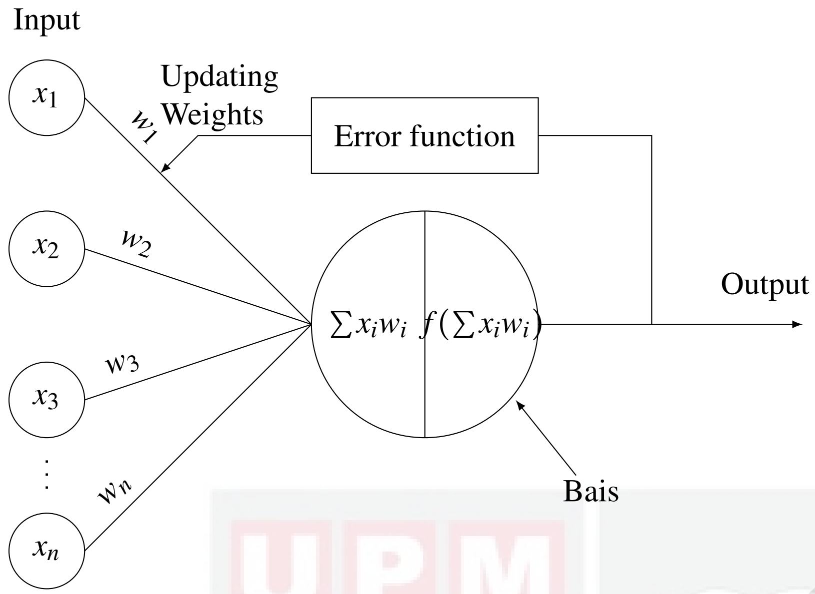

Figure 1.1: Mathematical framework of ANN.

nition (Roy et al., 2015), biology Umar et al. (2021), process optimization and control systems (Chambers and Mount, 2002), Mondal et al. (2023).

The fundamental element of an ANN is known as an artificial neuron (node). This neuron comprises several key components, as depicted in Figure 1.1

1. **Input:** This refers to the signals or data received by the neuron from other neurons or from the input layer of the network. These inputs are weighted based on their importance or significance to the neuron.

2. **Summing Junction:** The neuron sums up the weighted inputs along with a bias term. The bias allows the neuron to adjust the threshold at which it activates.

3. **Activation Function:** This function determines the output of the neuron based on the sum of the weighted inputs and the bias. It introduces non-linearity into the network, signifying that alterations in the first variable do not always lead to a consistent change in the second variable, allowing the network to learn complex patterns and relationships in the data. The criteria for an activation function involve possessing a derivative, which denotes the alteration in the y-axis concerning changes in the x-axis (commonly referred to as slope in Backpropagation), and being a monotonic

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function, implying it is consistently either non-increasing or non-decreasing. There exists a multitude of activation functions documented in the literature; however, the most prevalent ones are outlined as follows:

(a) Linear function:

$$f(x) = ax, \quad a \in \mathbb{R}.$$

Range: $(-\infty, \infty)$.

(b) Sigmoid function:

$$f(x) = \frac{1}{1 + e^{-x}}.$$

Range: $(0, 1)$.

(c) Hyperbolic tangent function:

$$f(x) = \tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}.$$

Range: $(-1, 1)$.

(d) Rectified linear unit (ReLU) function:

$$ReLU(x) = \max\{0, x\}.$$

Range: $[0, \infty)$.

(e) Identity function:

$$f(x) = x.$$

Range: $(-\infty, \infty)$.

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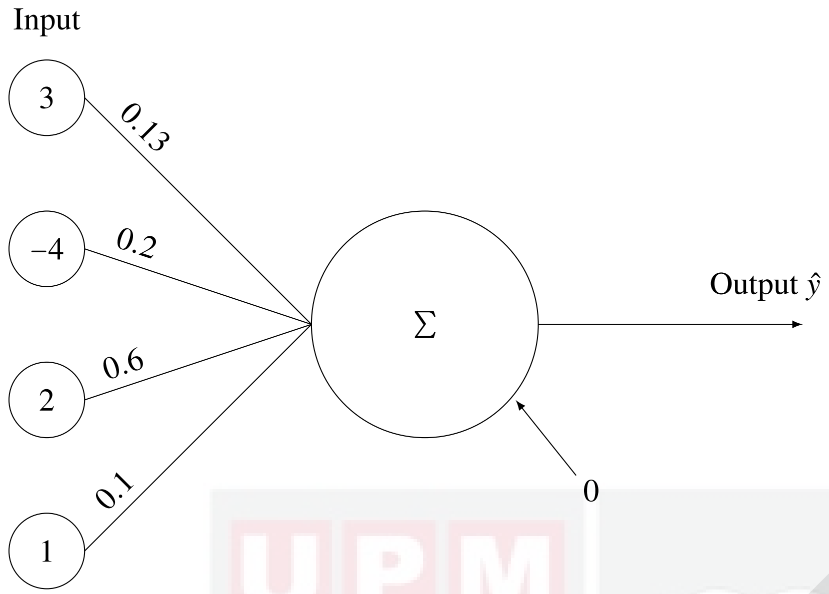

Figure 1.2: ANN with 4 input values.

4. **Bias:** The bias term is a constant value added to the sum of the weighted inputs before passing through the activation function. It helps the neuron to learn and adjust its output.

5. **Output:** After the sum of the weighted inputs and the bias are passed through the activation function, the neuron produces an output. This output is then passed on to other neurons in the network as input.

Consider example of an ANN with four input values: 3, -4, 2, and 1, each with weights of 0.13, 0.2, 0.6, and 0.1 as shown in Figure 1.2. For this particular setup, the bias is set to zero. This example utilizes a common activation function known as sigmoid. The neuron is involved in four processes, as was previously indicated, as follows:

Step 1: The input values are multiplied by their corresponding weights to begin the

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weighting process.

$$x_1 \rightarrow x_1 \times w_1 = 3 \times 0.13 = 0.39,$$

$$x_2 \rightarrow x_2 \times w_2 = -4 \times 0.2 = -0.8,$$

$$x_3 \rightarrow x_3 \times w_3 = 2 \times 0.6 = 1.2,$$

$$x_4 \rightarrow x_4 \times w_4 = 1 \times 0.1 = 0.1.$$

Step 2: The weighted inputs are summed, followed by the addition of the bias term.

$$\begin{aligned} x &\mapsto \left(\sum_{i=1}^{4} x_i \times w_i\right) + b \\ &= 0.39 - 0.8 + 1.2 + 0.1 + 0 = 0.89 \end{aligned}$$

Step 3: Applying the sigmoid function.

$$\begin{aligned} f(x) &= \frac{1}{1 + e^{-x}} \\ &= \frac{1}{1 + e^{-0.89}} \\ &= 0.709. \end{aligned}$$

Step 4: Considering this value as the output of the last layer, the neuron's output is 0.709.

In summary, an artificial neuron in an ANN is composed of inputs, a summing junction where inputs are weighted and summed along with a bias term, an activation function that determines the neuron's output, and finally, the output itself, which is passed on to other neurons in the network. These components work together to process information and learn patterns from the input data. The Error (Loss) function serves as a means to assess the performance of your algorithm in modeling your dataset. When your predictions deviate significantly, the error function yields a higher value. Conversely, when predictions are more accurate, it produces a lower value. While fine-tuning your algorithm to enhance the model, the error function provides feedback on your progress. It

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essentially aids in gauging the disparity between predicted and actual values.

1.5 Orthogonal Polynomials

Approximation theory and computational schemes are the main areas that utilise orthogonal polynomials. Most popular orthogonal polynomials include Legendre polynomials, Jacobi polynomials, Chebyshev polynomials, Laguere polynomials and Hermite polynomials. In specific, this work will zoom in on shifted Legendre and shifted Jacobi polynomials.

1.5.1 Shifted Jacobi Polynomials

Jacobi polynomials (JPs) was introduced by Carl Gustav Jacob Jacobi (1804-1851). to tackle second order homogeneous differential equations of the form

$$(1-x^2)v''(x)+(\vartheta-\xi-(\vartheta+\xi+2)x)v'(x)+n(n+\xi+\vartheta+1)v(x)=0. \tag{1.1}$$

For $\vartheta,\xi>-1$, and $n\in\mathbb{N}$ then a polynomial of order n is solution of Eq.(1.1). It is defined as the JPs (Bojdi et al., 2013; Doha et al., 2012) with two parameters, $P_n^{(\xi,\vartheta)}(x)$, is defined over the interval $[-1,1]$ as,

$$P_n^{(\xi,\vartheta)}(x)=\frac{\Gamma(\xi+n+1)}{n!\,\Gamma(\vartheta+\xi+n+1)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(n+m+\xi+\vartheta+1)}{\Gamma(\xi+n+1)}\left(\frac{2x-1}{2}\right)^m. \tag{1.2}$$

JPs have been verified to be orthogonal on the interval $[-1,1]$ with respect to the weight function $(1-x)^{\xi}(1+x)^{\vartheta}$. Using $x=\frac{2z}{\lambda}-1$ to convert the original interval of $[-1,1]$ into $[0,\lambda]$ will result to the two-parametric shifted JPs as given below,

$$P_{\lambda,i}^{(\xi,\vartheta)}(z)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+\vartheta+1)\Gamma(i+k+\vartheta+\xi+1)}{\Gamma(k+\vartheta+1)\Gamma(i+\xi+\vartheta+1)(i-k)!\,k!\,\lambda^k}z^k,\quad i=0,1,2,3,\ldots. \tag{1.3}$$

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$$P_{\lambda,i}^{(\xi,\vartheta)}(0) = (-1)^i \frac{\Gamma(i+\vartheta+1)}{\Gamma(i+1)\Gamma(\vartheta+1)},$$
$$P_{\lambda,i}^{(\xi,\vartheta)}(\lambda) = \frac{\Gamma(i+\xi+1)}{\Gamma(i+1)\Gamma(\xi+1)},$$

and

$$\max_{z\in[0,\lambda]} |P_{\lambda,i}^{(\xi,\vartheta)}(z)| \leqslant \hat{\Delta}(i,\kappa),$$

where $\hat{\Delta}(i,\kappa) = \frac{\Gamma(i+\kappa+1)}{\Gamma(i+1)\Gamma(\kappa+1)}$ and $\kappa = \max(\xi,\vartheta)$.

The orthogonality condition of JPs on $[0,\lambda]$ is as under

$$\int_0^\lambda P_{\lambda,i}^{(\xi,\vartheta)}(z) P_{\lambda,j}^{(\xi,\vartheta)}(z) W_\lambda^{(\xi,\vartheta)}(z)\, dz = R_{\lambda,j}^{(\xi,\vartheta)} \theta_{(i,j)}, \tag{1.4}$$

where

$$\theta_{(i,j)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.5}$$

the weight function $W_\lambda^{(\xi,\vartheta)}(z)$ has the following form

$$W_\lambda^{(\xi,\vartheta)}(z) = (\lambda - z)^\xi z^\vartheta, \tag{1.6}$$

and

$$R_{\lambda,j}^{(\xi,\vartheta)} = \frac{\lambda^{\xi+\vartheta+1}\Gamma(j+\xi+1)\Gamma(j+\vartheta+1)}{(2j+\xi+\vartheta+1)\Gamma(j+1)\Gamma(j+\xi+\vartheta+1)}. \tag{1.7}$$

Orthogonality of JPs will result to any function $y_1 \in C[0,\lambda]$ may be mentioned as a linear combination of shifted JPs as,

$$y_1 = \sum_{k=0}^{\infty} c_k P_{\lambda,k}^{(\xi,\vartheta)}(z). \tag{1.8}$$

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It is important to obtain the truncated sum of shifted JPs to solve related numerical problems. Thus, Eq. (1.8) can be expressed as,

$$y_1 \simeq \sum_{k=0}^{n} c_k P_{\lambda,k}^{(\xi,\vartheta)}(z). \tag{1.9}$$

Since Eq.(1.9) obtains the best result as $n \longrightarrow \infty$, thus, via Eqs.(1.4), (1.6), and (1.7), c_k can be obtained via,

$$\frac{1}{R_{\lambda,j}^{(\xi,\vartheta)}} \int_0^{\lambda} W_{\lambda}^{(\xi,\vartheta)}(z)\, y_1(z)\, P_{\lambda,j}^{(\xi,\vartheta)}(z)\, dz, \;\; j = 0, 1, \ldots.$$

In vector notation Eq.(1.9) has the following form,

$$y_1 \simeq A_N^T \, \Psi_N(z), \tag{1.10}$$

where

$A_N^T = [c_0,\, c_1, \ldots, c_n]$, $\Psi_N(z) = [P_{\lambda,0}^{(\xi,\vartheta)}(z), P_{\lambda,1}^{(\xi,\vartheta)}(z), \ldots, P_{\lambda,n}^{(\xi,\vartheta)}(z)]$ and $N = n+1$ is the vectors size utilized as a scale to come out with relevant numerical schemes. The coefficient vector is A_N^T and finally $\Psi_N(z)$ represents the function vector. We suggest readers refer to (Doha et al., 2012) for a more detailed study of JPs. Other special orthogonal polynomials linked to shifted Jacobi's polynomials are as follows:

1. $P_{\lambda,i}(z) = P_{\lambda,i}^{(0,0)}(z)$, is the shifted Legendre polynomials by putting $\xi = \vartheta = 0$ in (1.3).

2. $T_{\lambda,i}(z) = \frac{\Gamma(i+1)\Gamma(\frac{1}{2})}{\Gamma(i+\frac{1}{2})} P_{\lambda,i}^{(-\frac{1}{2},-\frac{1}{2})}(z)$, is stated to have shifted Chebyshev polynomials by giving $\xi = \vartheta = -\frac{1}{2}$ in (1.3).

3. $U_{\lambda,i}(z) = \frac{\Gamma(i+2)\Gamma(\frac{1}{2})}{\Gamma(i+\frac{3}{2})} P_{\lambda,i}^{(\frac{1}{2},\frac{1}{2})}(z)$, is stated to have shifted Chebyshev of second kind if $\xi = \vartheta = \frac{1}{2}$ in (1.3).

4. $C_{\lambda,i}^{\xi}(z) = \frac{\Gamma(i+1)\Gamma(\xi+\frac{1}{2})}{\Gamma(i+\xi+\frac{1}{2})} P_{\lambda,i}^{(\xi-\frac{1}{2},\vartheta-\frac{1}{2})}(z)$, is stated to have shifted Gegenbauer (Ultraspherical) polynomials by setting if $\xi = \vartheta$ in (1.3).

5. $V_{\lambda,i}(z) = \frac{\Gamma(2i+1)}{\Gamma(2i-1)} P_{\lambda,i}^{(\frac{1}{2},-\frac{1}{2})}(z)$, is stated to have shifted Chebyshev polynomials of third

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kinds by giving $\xi = \frac{1}{2}$, $\vartheta = -\frac{1}{2}$ in (1.3).

6. $W_{\lambda,i}(z) = \frac{\Gamma(2i+1)}{\Gamma(2i-1)} P_{\lambda,i}^{(-\frac{1}{2},\frac{1}{2})}(z)$, when $\xi = -\frac{1}{2}$, $\vartheta = \frac{1}{2}$ in (1.3) it is said to have shifted Chebyshev polynomials of the fourth order.

1.5.2 Shifted Legendre Polynomials

If both ξ and ϑ parameters are set as zero, the SJ polynomials will be transformed into shifted Legendre (SL) polynomials. SL polynomials have significance because their weight function is one. These kind of polynomials is given as follows,

$$L_i(t) = \sum_{l=0}^{i} \Omega_{i,l}\, t^l, \quad i = 1, 2, 3, \ldots$$

where

$$\Omega_{i,l} = \frac{(-1)^{i+l}\,\Gamma(i+l+1)}{\Gamma(i-l+1)\,\lambda^l\,(l!)^2}.$$

These polynomials are orthogonal on $[0, \lambda]$, the orthogonality relation of these polynomials is given as,

$$\int_0^{\lambda} L_i(t)\,L_j(t)\,dt = \begin{cases} \dfrac{\lambda}{2i+1}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.11}$$

Based on the orthogonality of these polynomials, we can derive a smooth function such as,

$$g(t) = \sum_{i=0}^{\infty} c_i\, L_i(t),$$

where c_i can be obtained by relation

$$c_i = \frac{2i+1}{\lambda} \int_0^{\lambda} g(t)\, L_i(t)\, dt.$$

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1.6 Basic Definitions and Preliminary Concepts

This section introduces the well-known fractal and fractional calculus definitions, such as the Riemann-Liouville (RL), Caputo, generalized Caputo, Hilfer, Caputo fractal-fractional, and Riemann-Liouville fractal-fractional operators. This section will also discuss some special functions, such as the Gamma and the Mittag-Leffler functions, which play important roles in fractal and fractional calculus.

The numerical solutions of FFDEs have been the subject of research in numerical analysis. Various types of numerical methods have been developed for solving FFDEs. In general, multi-order FFDE is described as,

$$D^{\alpha,\beta} y(t) = F\left(t, y(t), D^{\mu_1,\beta 1} y(t), \ldots, D^{\mu_k,\beta_k} y(t)\right), \tag{1.12}$$

or

$$D^{\alpha,\beta} y(t) = \sum_{i=0}^{k} c_i D^{\mu_i,\beta_k} y(t) + f(t), \tag{1.13}$$

with its initial conditions:

$$y^{(i)}(0) = d_i, \ i = 0, 1, \ldots, n,$$

where $n < \alpha < n+1, \ 0 < \mu_1 < \mu_2 < \ldots < \mu_k < \alpha, \ 0 < \beta_1 < \beta_2 < \ldots < \beta_k < \beta$, and $f(t)$ is a known function. Consider a system of multi-order fractal-fractional differential equations:

$$\begin{aligned} D^{\alpha_1,\beta_1} y_1(t) &= G_1\left(t, y_1(t), y_2(t), \ldots, y_m(t)\right) \\ \vdots \qquad & \qquad \vdots \qquad\qquad\qquad\qquad , \\ D^{\alpha_m,\beta_m} y_m(t) &= G_m\left(t, y_1(t), y_2(t), \ldots, y_m(t)\right) \end{aligned} \tag{1.14}$$

subject to initial conditions:

$$y_r^{(i)}(0) = d_{i,r}, \;\; i = 0, 1, \ldots, n, r = 1, .., m. \quad (1.15)$$

1. **Gamma Function**

   The Gamma function, denoted by the Greek capital letter $\Gamma(x)$, is one of the important functions that is thought to be an extension of the factorial function for positive real numbers.

$$\Gamma(n) = (n-1)!, \;\; n \in \mathbb{N}.$$

Definition 1.1

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \;\; Re(x) > 0.$$

These are a few of the most important properties of Gamma function (Owolabi and Atangana, 2019) are given by

$$\Gamma(x+1) = x\Gamma(x), \qquad Re(x) > 0.$$
$$\Gamma(x) = (x-1)!, \qquad x > 0,$$

and

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt = \sqrt{\pi}.$$

2. **Beta Function**

   The Beta function, denoted by $B(u,v)$, is another special function defined by an improper integral (see for e.g. (Owolabi and Atangana, 2019))

$$B(u,v) = \int_0^1 t^{u-1} (1-t)^{v-1}\, dt.$$

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The relationship between the Gamma and Beta functions is given by:

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

3. **Mittag-Leffler Function**

   It is a special function that extends the exponential function and is frequently used in the solutions of fractional differential equations and systems of FFDEs (Owolabi & Atangana, 2019):

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0.$$

   As a special case, if $\alpha = 1$. Then

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Definition 1.2 *(Miller & Ross, 1993).*

The RL fractional derivative of order $\alpha > 0$ and $\alpha, t \in \mathbb{R}$ is defined as:

$${}_a^R\mathcal{D}_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d^n}{dt^n}\right) \int_a^t (t-s)^{n-\alpha-1}\, y(s)ds, \quad t > a, \tag{1.16}$$

where $a \geq 0$, $n-1 < \alpha < n$ and $n \in \mathbb{N}$.

Definition 1.3 *(Podlubny, 1998).*

The Caputo fractional derivative of order $\alpha > 0$ is defined as:

$${}^C\mathcal{D}_{a+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}\, y^{(n)}(s)ds, \quad t > a, \tag{1.17}$$

where $n-1 < \alpha \leqslant n$ and $n \in \mathbb{N}$.

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Definition 1.4 *(Odibat & Baleanu, 2020).*

The generalized fractional integral of a function $y(t)$ of order $\alpha > 0$, $I_{a+}^{\alpha,\rho} y(t)$, is defined by:

$$I_{a+}^{\alpha,\rho} y(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} y(s)\, ds, \tag{1.18}$$

(provided it exists) where $\rho > 0$ and $t > a$.

Definition 1.5 *(Katugampola, 2014).*

The generalized Riemann-type fractional derivative of order $\alpha > 0$ is defined as:

$$^{R}\mathcal{D}_{a+}^{\alpha,\rho} y(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{n-\alpha-1} y(s) ds, \ \ t > a, \tag{1.19}$$

where $\rho > 0$, $a \geq 0$ and $n = \lceil \alpha \rceil$.

Definition 1.6 *(Odibat & Baleanu, 2020).*

The new generalized Caputo-type fractional derivative of order $\alpha > 0$ is defined as:

$$^{C}\mathcal{D}_{a+}^{\alpha,\rho} y(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{n-\alpha-1} \left(s^{1-\rho}\frac{d}{ds}\right)^n y(s)\, ds, \ \ t > a, \tag{1.20}$$

where $\rho > 0$, $a \geq 0$, $n-1 < \alpha < n$, $n = \lceil \alpha \rceil$, and $y(t) \in C^n[a,b]$.

In addition, for the new generalized Caputo fractional derivative (Odibat and Baleanu, 2020; Jarad et al., 2017), we have:

$$^{C}\mathcal{D}^{\alpha,\rho} C = 0, \ \ C \ \text{is a constant}. \tag{1.21}$$

Moreover, if $n-1 < \alpha < n$, $k > n-1$ and $k \notin \mathbb{N}$,

$$^{C}\mathcal{D}_{a+}^{\alpha,\rho} (t^\rho - a^\rho)^k = \begin{cases} \frac{\rho^\alpha\, \Gamma(k+1)}{\Gamma(k-\alpha+1)} (t^\rho - a^\rho)^{k-\alpha} & , \ k \in \mathbb{N}_0 \text{ and } k \geqslant \lceil \alpha \rceil \text{ or } k \in \mathbb{N} \text{ and } k > \lfloor \alpha \rfloor, \\ 0 & , \ k \in \mathbb{N}_0 \text{ and } k < \lceil \alpha \rceil. \end{cases} \tag{1.22}$$

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Definition 1.7 *(Atangana, 2017).*

Let $y(t)$ be differentiable on (a,b). If y is the fractal differentiable of order β on (a,b), then the fractal-fractional derivative of y of order α and β in Caputo sense with power law kernel is defined as:

$$^{C}\mathcal{D}_{a+}^{\alpha,\beta}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}\,\frac{dy(s)}{ds^{\beta}}\,ds, \tag{1.23}$$

where

$$\frac{dy}{dt^{\beta}} = \lim_{t\to s}\frac{y(t)-y(s)}{t^{\beta}-s^{\beta}},$$

and $0 < n-1 < \beta, \alpha \leqslant n.$

Definition 1.8 *(Atangana, 2017).*

Let $y(t)$ be differentiable on (a,b); if y is the fractal differentiable of order β on (a,b), then the fractal-fractional derivative of y of order α and β in Riemann-Liouville sense with power law kernel is defined as:

$$^{RL}\mathcal{D}_{a+}^{\alpha,\beta}y(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{ds^{\beta}}\int_{a}^{t}(t-s)^{n-\alpha-1}\,y(s)\,ds, \tag{1.24}$$

where

$$\frac{dy}{dt^{\beta}} = \lim_{t\to s}\frac{y(t)-y(s)}{t^{\beta}-s^{\beta}},$$

and $0 < n-1 < \beta, \alpha \leq n.$

Definition 1.9 *(Atangana, 2017).*

Let y be continuous on an open interval **I***, the fractal-Laplace transform of order α is defined by:*

$$^{F}\mathcal{L}_{p}^{\alpha}(y(t)) = \int_{0}^{\infty} t^{\alpha-1}y(t)\exp(-pt)\,dt, \;\; \alpha > 0. \tag{1.25}$$

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1.7 Problem Statement

In recent years, FFDEs have garnered considerable interest in science and engineering due to their capacity to model numerous complicated phenomena (Atangana, 2017). Nonetheless, the numerical solutions of these equations remain difficult due to the non-local and non-integer order of the associated derivatives. Thus, the existing numerical approaches for solving FFDEs encounter several concerns, including low precision numerically, and large processing costs. These negative aspects are considerable barrier to the accurate and efficient resolution of FFDE-based real-world situations. Therefore, this research aimed to produce effective numerical approaches for solving FFDEs. The proposed methods were based on generalising numerical approaches, including OM and ANN techniques.

The field of differential equations has profoundly impacted numerous scientific disciplines, offering a mathematical framework for understanding changes across various systems. Among these, fractal-fractional differential equations stand out for their potential to model phenomena in fractal materials-structures that exhibit complex patterns at every scale. Despite their significance, the investigation into solving fractal-fractional differential equations remains surprisingly sparse. This gap in research is partly due to the prevailing perception that fractional derivatives, which are crucial to this area of study, are not adequately suited for fractal materials. This inadequacy stems from the current limitations in the definitions of fractal-fractional derivatives. These definitions are not only scarce but also insufficiently versatile, rendering them incapable of addressing a wide range of problems inherent to fractal materials.

Consequently, there exists a critical need for comprehensive research aimed at developing new or improved definitions of fractal-fractional derivatives. Such advancements would not only enable a more effective solution of fractal-fractional differential equations but also significantly enhance our understanding and modeling capabilities of phenomena in fractal materials. This thesis aims to bridge this gap by proposing novel approaches to

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define and solve fractal-fractional differential equations, thereby offering new perspectives and tools for researchers dealing with fractal problems.

1.8 Objectives of the Study

The following are the main objectives of this thesis:

1. To define the definitions of generalized Caputo fractal-fractional differential, integral, and the Hilfer fractal-fractional differential operator.

2. To derive operational matrices of derivatives with fractional order and fractal dimension based on the definitions operators defined in objective 1 to solve various types of multi-order linear/non-linear and systems of FFDEs.

3. To apply artificial neural networks with fractional order and fractal dimension that utilized the generalized Caputo fractal-fractional derivative sense for solving linear and nonlinear multi-order FFDEs with an order of range $0 < \alpha \leq 1$.

4. To apply artificial neural networks in the fractal domain to find solutions for higher-linear multi-order FFDEs with an order of $\alpha > 1$ with variable and constant coefficients.

1.9 Scope of the Study

The research will focus on presenting two definitions for generalized Caputo and Hilfer fractal-fractional differential and integral operators. The primary objective is to develop operational matrix and artificial neural network approaches utilizing these new fractal-fractional derivatives. These methods will be employed to solve system and multi-order fractal-fractional differential equations, with the utilization of operational matrices based on orthogonal polynomials and collocation points to simplify the problem into a system of algebraic equations.

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1.10 Motivation

Fractal-fractional differential equations (FFDEs) are widely applicable, yet their solutions have been thoroughly investigated. As computer technology advances, the need for appropriate numerical methods to solve FFDEs becomes increasingly important. Although the OM method and ANNs have proven effective for solving initial and boundary value problems in fractional differential equations, their potential for FFDEs remains largely unexplored. This thesis is motivated by the desire to fill this gap in research. Firstly, by introducing two new definitions of generalized Caputo and Hilfer fractal-fractional differential and integral operators. Secondly, by extending the OM method and ANNs to effectively solve FFDEs. These methods are utilized to handle a range of issues, including various types of multi-order FFDEs. The proposed approaches are engineered to be pragmatic, guaranteeing the precision of the outcomes obtained.

1.11 Outline of the Study

The study is divided into seven chapters as described in the following. The present chapter introduces readers to the thesis, or in specific on fundamentals of fractional calculus, FFDEs, operational matrices, orthogonal polynomials and ANNs that will be used in the later chapters of the thesis. Chapter 1 discusses on explaining the fundamentals of fractional calculus. Also, it describes the problem statement, research objectives, and thesis outline. Chapter 2 provides literature review including brief history of fractional calculus and the use of generalized Caputo and Hilfer fractional derivatives to solve fractional differential equations, fractal-fractional calculus, operational matrices, and artificial neural networks.

In Chapter 3, we start our investigation by introducing new fractal-fractional operators namely new generalized Caputo fractal-fractional differential and integral operators. Then in section 3.2, we develop a computationally efficient method for solving different

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types of FFDEs using Legendre polynomials combined with the operational matrix. In section 3.4, the method is combined with Jacobi polynomials to find approximate solutions for various FFDEs. In section 3.6, examples of problems are provided to determine the efficiency and accuracy of different methods including multi-order linear, nonlinear, and systems of FFDEs with IVPs and PVPs. Results obtained in this section will be compared with results from other studies.

The new fractal-fractional of Hilfer derivative for operational matrices is covered in Chapter 4. In addition, we propose a new method for deducing an operational matrix of derivatives using new FF definitions, which is the Hilfer FFD, for solving three classes of different types of multi-order IVP and BVP FFDEs using both Jacobi and Legendre polynomials. In Chapter 5, we solve FFD problems corresponding to multi-order FFDEs using combined truncated generalized power series and ANNs. Chapter 6 uses artificial intelligence techniques to estimate a solution for FFDEs of high-order linear with variable and constant coefficients based on a mix of power series method and neural network (NNs) approach. In each chapter, we discuss applications of the proposed concepts. Finally, Chapter 7 provides the study's conclusion and recommends future work.

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