EXPONENTIATED-BASED BURR TYPE X DISTRIBUTIONS WITH CENSORED DATA AND COVARIATE

By

OH YIT LENG

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Chairman : Lim Fong Peng, PhD
Faculty : Faculty of Science

In an attempt to create distributions with greater flexibility to accommodate survival data with various hazard function forms, a number of extended Burr type X distributions have been developed recently. For instance, the Weibull Burr type X, beta Burr type X, and gamma Burr type X distributions. Previous studies have demonstrated that the hazard functions of these distributions can take various forms, such as increasing, decreasing, and bathtub, but not unimodal, which is frequently observed in survival analysis. In order to solve this issue, the aim of this study is to propose three distributions with greater flexibility in fitting hazard functions in various forms, particularly the unimodal: exponentiated Weibull Burr type X, exponentiated beta Burr type X, and exponentiated gamma Burr type X distributions. We begin by deriving the probability density function and cumulative distribution function of the three proposed distributions as well as their important statistical characteristics, including the quantile function, moment, moment generating function, order statistics, and Renyi entropy. To explore the performance of the three proposed distributions, simulation studies with various sample sizes and censoring rates for data with and without censored data and covariate are conducted after it. This study considers two types of censoring: random censoring and type-I censoring. Besides, for the simulation studies, we consider cases with single covariates. For these simulation studies, the inverse transform approach is used to simulate the event time. Meanwhile, we estimate the parameters of each of the three proposed distributions using the maximum likelihood estimation approach. Lastly, we use three real data sets: two complete data sets and one with censored data and covariates, to demonstrate the effectiveness and adaptability of the three suggested distributions. The two complete data sets are Data Set 1, which represents the failure time of 84 aircraft windshields, and Data Set 2, which represents the remission time of 128 patients with bladder cancer. Data Set 1 has an increasing hazard function, whereas Data Set 2 has a unimodal hazard function. Data Set 3 contains the recurrence time of 86 bladder cancer patients with censored data and three covariates. The findings of this study have demonstrated that the hazard functions of the three proposed distributions can take the forms of increasing, decreasing, bathtub, and unimodal. Additionally, even with censored data and covariate are present, the parameters of the three proposed distributions can be estimated using the maximum likelihood estimation approach. Finally, the three proposed distributions fit the two complete data sets better than various extended Burr type X distributions and their sub-models, and they are formidable rivals to all other competing distributions, including the non-nested distributions used in this study, as demonstrated in the real data applications. Nevertheless, the three proposed distributions can be used to fit survival data with unimodal hazard function as demonstrated in applications of Data Set 3.

Keywords: Beta-G, Burr Type X, Exponentiated, Gamma-G, Weibull-G

SDG: GOAL 3: Good Health and Well-Being

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

TABURAN BURR JENIS X BERASASKAN EXPONEN DENGAN DATA TERSENSOR DAN KOVARIAT

Oleh

OH YIT LENG

Januari 2024

Pengerusi : Lim Fong Peng, PhD
Fakulti : Fakulti Sains

Baru-baru ini, banyak taburan lanjutan taburan Burr jenis X telah dijanakan bagi tujuan menciptakan taburan dengan fleksibiliti yang lebih tinggi untuk menyeleras data mandirian dengan pelbagai bentuk fungsi bahaya. Sebagai contoh, taburan Burr jenis X Weibull, taburan Burr jenis X beta, dan taburan Burr jenis X gamma. Kajian sebelumnya telah menunjukkan bahawa fungsi bahaya taburan ini ada berbagai bentuk, seperti meningkat, menurun, dan tab mandi, tetapi tidak unimod, yang sering muncul dalam analisis mandirian. Untuk menyelesaikan masalah ini, tujuan kajian ini adalah untuk menperkenalkan tiga taburan dengan fleksibiliti yang lebih tinggi dalam menyeleras fungsi bahaya dalam pelbagai bentuk, terutamanya unimod: taburan Burr jenis X Weibull eksponen, taburan Burr jenis X beta eksponen, dan taburan Burr jenis X gamma eksponen. Kami bermula dengan membentangkan fungsi ketumpatan kebarangkalian dan ungkapan fungsi taburan kumulatif bagi ketiga-tiga taburan yang dicadangkan serta ciri-cri penting statistik mereka, termasuk fungsi kuantil, momen, fungsi penjanaan momen, statistik urutan, dan entropi Renyi. Untuk meneroka prestasi ketiga-tiga taburan yang dicadangkan, kajian simulasi dengan pelbagai saiz sampel

dan kadar pengecualian untuk data dengan dan tanpa data tersensor dan kovariat dijalankan kemudianya. Kajian ini mempertimbangkan dua jenis sensor: sensor rawak dan sensor tipe I. Di samping itu, untuk kajian simulasi, kami mempertimbangkan kes dengan satu kovariat sahaja. Untuk kajian simulasi ini, transformasi songsang digunakan untuk mensimulasikan masa hayat. Sebagai tambahan, kami menganggarkan parameter bagi ketiga-tiga taburan yang dicadangkan dengan menggunakan pendekatan penganggaran kebolehjadian maksimum. Akhirnya, kami menggunakan tiga set data sebenar: dua set data lengkap dan satu dengan data tersensor dan kovariate, untuk menunjukkan fleksibiliti dan penyesuaian ketiga-tiga taburan yang dicadangkan. Dua set data lengkap ialah Data Set 1, yang mewakili masa kegagalan 84 tingkap kapal terbang, dan Data Set 2, yang merupakan masa remisi 128 pesakit dengan kanser pundi kencing. Data Set 1 mempunyai fungsi bahaya yang meningkat, manakala Data Set 2 mempunyai fungsi bahaya unimod. Data Set 3 mengandungi masa pengulangan 86 pesakit kanser pundi kencing dengan data tersensor dan tiga kovariat. Hasil kajian ini telah menunjukkan bahawa fungsi bahaya ketiga-tiga taburan yang dicadangkan boleh mengambil bentuk meningkat, menurun, tab mandi, dan unimod. Sebagai tambahan, walaupun dengan data tersensor dan kovariate, parameter ketiga-tiga taburan yang dicadangkan boleh dianggarkan menggunakan pendekatan penganggaran kebolehjadian maksimum. Akhirnya, ketiga-tiga taburan yang dicadangkan sesuai dengan dua set data lengkap dengan lebih baik berbanding dengan pelbagai taburan lanjutan Burr jenis X dan sub-model mereka, dan mereka adalah saingan yang mengerunkan kepada semua taburan bersaing lain, termasuk taburan tak bersarang yang digunakan dalam kajian ini, seperti yang ditunjukkan dalam aplikasi data sebenar. Sebagai tambahan, ketiga-tiga taburan yang dicadangkan boleh digunakan untuk mencocokkan data mandirian dengan fungsi bahaya unimod seperti yang ditunjukkan dalam aplikasi Data Set 3.

Kata Kunci: Beta-G, Burr Jenis X, Exponen, Gamma-G, Weibull-G

SDG: MATLAMAT 3: Kesihatan dan Kesejahteraan Baik

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Lim Fong Peng, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Chen Chuei Yee, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

Wendy Ling Shin Yie, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

Loh Yue Fang, PhD
Assistant Professor and Dean
Faculty of Business and Management
UCSI University
(Member)

ZALILAH MOHD SHARIFF, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 12 September 2024

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LIST OF ABBREVIATIONS

pdf	Probability Density Function
cdf	Cumulative Distribution Function
mgf	Moment Generating Function
MLE	Maximum Likelihood Estimation Approach
AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
CAIC	Corrected Akaike Information Criterion
HQ	Hanna-Quinn Information Criterion
BX	Burr Type X
GBX	Gamma Burr Type X
BBX	Beta Burr Type X
EGEBX	Exponentiated Generalized Burr Type X
WBX	Weibull Burr Type X
EWBXII	Exponentiated Weibull Burr Type XII
EBXIIP	Exponentiated Burr Type XII Poisson
GMOBXII	Marshall-Olkin Extended Burr-XII
BBXII	Beta Burr Type XII
EWBX	Exponentiated Weibull Burr Type X
EBBX	Exponentiated Beta Burr Type X
EGBX	Exponentiated Gamma Burr Type X
AVE	Average of Maximum Likelihood Estimates
RMSE	Root Mean Square
SE	Standard Error
MLEs	Maximum Likelihood Estimates

$-l$	Negative Log-likelihood

CHAPTER 1

INTRODUCTION

1.1 Background of Study

Survival analysis is an analysis of the duration of time until the occurrence of an event known as death (Collet, 2003). It is also well-known as reliability, duration, and event history analyses. Survival analysis is introduced to determine the probability of an individual surviving longer than a certain time and the rate of death at a particular time. These can be determined from the survival and hazard functions, which are the two crucial functions in survival analysis.

In survival analysis, there are three types of modelling: parametric, semi-parametric, and nonparametric. For parametric modelling, the survival time is assumed to follow a well-known distribution such as Weibull, exponential, Burr type X, and gamma distributions. These distributions have one crucial feature, that is they take only positive values. In contrast, nonparametric modelling, known as distribution-free modelling, does not require any assumptions to be made about the underlying distribution of the survival times. The Kaplan-Meier method is one of the most popular nonparametric methods used in survival analysis. Lastly, semi-parametric modelling combines parametric and nonparametric modelling, which consists of both parametric and nonparametric components.

It is vital to fit the survival data with a suitable distribution in parametric modelling. Although several distributions can be used to model survival data with various characteristics, there are cases where the survival data could not be well fitted with the existing distribution, especially survival data with bathtub and unimodal hazard functions. Since the bathtub and unimodal hazard functions are very common in survival analysis (Silva et al., 2010), developed new distribution with higher flexibility in modelling survival data with hazard functions in different shapes has become necessary.

Several attempts have been made to explore new distributions with higher flexibility that can fit hazard functions in different shapes (Singla et al., 2012; Pescim et al., 2010). These new distributions developed through several generalized families of distributions by adding more parameters into the baseline distribution. For example, Paranaiba et al. (2011) proposed a five-parameter distribution, namely beta Burr type XII distribution, by using beta generalized family of distributions (Eugene et al., 2002) along with Burr type XII as the baseline distribution. In the meantime, De Pascoa et al. (2011) used the generalized gamma distribution (Stacy, 1962) and the Kumaraswamy generalized family distributions (Cordeiro & de Castro, 2011) to develop the Kumaraswamy generalized gamma distribution, a five-parameter distribution. These new distributions then can be further reduced to some well-known distributions. For instance, the beta generalized Weibull distribution (Singla et al., 2012) can be reduced to the beta generalized exponential, beta Weibull, exponentiated Weibull, generalized Rayleigh, beta exponential, generalized exponential, Weibull, Rayleigh, and exponential distributions. Due to this, the new distributions are very flexible and can be used as an alternative distribution in modelling different data types (De Pascoa et al., 2011). It has been proven that the new distributions provide a better model fit and higher flexibility than its sub-models (Carrasco et al., 2008; Singla et al., 2012; Pescim et al., 2010).

On the other hand, some new distributions are obtained by adding more parameters to the extended distributions. Abouelmagd et al. (2017) extended the Weibull Burr type X distribution (Ibrahim et al., 2017) and developed the exponentiated Weibull Burr type X distribution. The proposed distribution is greatly flexible and able to model survival data with different shapes of hazard rate functions, including constant, decreasing, increasing, J-shape, unimodal and bathtub. Ibrahim and Khaleel (2018) introduced an extended version of exponentiated Weibull known as exponentiated Kumaraswamy exponentiated Weibull distribution. It has been proven that this extended distribution has more sub-models than exponentiated Weibull distribution, and its flexibility has improved. These have shown that developing new distributions with higher flexibility in modelling various shapes of hazard functions is always of interest to researchers. This study aims to propose three new distributions extending the Burr type X distribution, namely exponentiated gamma Burr type X, exponentiated beta Burr type X, and exponentiated Weibull Burr type X. These proposed distributions are expected to be more flexible to cover more shapes of hazard functions, especially unimodal hazard function and can be applied in different areas.

1.2 Problem Statement

Burr (1942) used the differential equation method to develop twelve cumulative frequency functions, namely Burr type I to Burr type XII. Among these distributions, Burr type X and Burr type XII have received the greatest attention and are often used in survival analysis (Mead, 2014; Raqab & Kundu, 2006).

Burr type X distribution introduced by Burr (1942) is well-known as one-parameter Burr type X. This distribution plays a vital role in many areas such as biological, health, agricultural, engineering, and survival analysis. In literature, several studies on

Burr type X distribution have been established, such as Jaheen (1995), Ahmad et al. (1997), Jaheen and Al-Matrafi (2002), Aludaat et al. (2008), Shayib and Haghighi (2011), Khaleel et al. (2017), and Ahmed et al. (2021). Then Surles and Padgett (2001) introduced a scaled Burr type X distribution (BX) by adding a scale parameter to the one-parameter Burr type X distribution. The BX distribution is suitable for modelling strength and survival data. It is appropriate to model survival data with increasing and bathtub hazard functions. Besides, its hazard function can be non-monotone, which can be useful in many areas (Raqab & Kundu, 2006). In addition, its probability density and cumulative distribution functions are closed form. Hence, it can be applied conveniently for data with or without censored observations (Raqab & Kundu, 2006). In the past, several studies have explored Burr type X distribution, such as Raqab and Kundu (2005, 2006).

Several generalizations of Burr type X distribution with various numbers of parameters have been formed to improve the flexibility of BX distribution such as the gamma Burr type X (GBX) (Khaleel et al., 2016), beta Burr type X (BBX) (Merovci et al., 2016), exponentiated generalized Burr type X (EGEBX) (Khaleel et al., 2018), Weibull Burr type X (WBX) (Ibrahim et al., 2017), beta Kumaraswamy Burr type X (Madaki et al., 2018), beta Burr type X (one parameter) (Khaleel et al., 2017), and exponentiated Burr type X (Ahmed et al., 2021) distributions.

Khaleel et al. (2016) proposed a three-parameter extended Burr type X distributions whereas Merovci et al. (2016), and Ibrahim et al. (2017) proposed two four-parameter extended Burr type X distributions. These extended Burr type X distributions are very flexible. Their hazard function has a wide variety of shapes, including increasing, decreasing, and bathtub shapes. However, these distributions do not provide a reasonable fit in fitting survival data with unimodal hazard function. In this context, the unimodal hazard function refers to a function that reaches a pick after some time and then decreases. It often happens in survival analysis (Paranaiba et al., 2011; Oluyede et al., 2018). A distribution must therefore be able to exhibit a unimodal hazard function. Additionally, various distributions have been created by adding additional parameters to the baseline distribution with the aim of increasing the flexibility of the current distribution, such as new beta power very flexible Weibull (Khan et al., 2023), modified exponential-Weibull (Al-Essa et al., 2023), and type I half-logistic modified Weibull (Elbatal et al., 2019). As a result, this study proposes three extended distributions: exponentiated Weibull Burr type X, exponentiated beta Burr type X and exponentiated gamma Burr type X distribution by adding an additional shape parameter to GBX, BBX, and WBX distributions. The additional shape parameter is added to the distributions via the exponentiated type of distribution (Gupta et al.,1998). We anticipate that the three proposed distributions will be more flexible in modelling hazard functions of various shapes, particularly the unimodal hazard function. Perhaps the flexibility and versatility of the proposed distributions make them attractive as a tentative model for cases where the underlying distribution is unknown and can be considered as an alternative model for some well-known distributions.

1.3 Objective of the Study

This study aims to modify the existing extended Burr type X distributions to form three extended distributions with high level of flexibility in modelling survival data with different hazard functions, especially bathtub and unimodal. Several objectives have been identified to achieve the aim of this study and the objectives are listed below.

1. To extend the existing extended Burr type X distributions by using the exponentiated type of distributions to form the exponentiated Weibull Burr type X, exponentiated beta Burr type X, and exponentiated gamma Burr type X distributions and incorporate covariates into these distributions.
2. To derive the maximum likelihood estimates and statistical properties of the three extended distributions.
3. To investigate the performance of the three extended distribution using simulations studies with different sample sizes and parameter values in the present of censored observations and covariates.
4. To illustrate the performance of the three extended distributions through three real data sets: failure time of 84 aircraft windshields, remission time of 128 bladder cancer patients, and recurrence time of 86 blander cancer patients.

1.4 Organization of Study

This study is constructed and outlined as follows. We first explore several frequently utilized distributions in survival analysis in Chapter 2, and then we look at the survival function and hazard function. Subsequently, several important statistical properties and useful functions used in this study are also covered in Chapter 2. Also in Chapter 2, we review the maximum likelihood method, likelihood function for censored data, Cox proportional mode, censored data, and covariates. Apart from that, the extended Burr type X distributions, such as the BBX, GBX, and WBX distributions, are discussed in Chapter 2 along with numerous generalized families of distributions. In chapter 2, we also explore the non-nested distributions of the Burr type X distribution. The exponentiated Weibull Burr type XII (EWBXII), exponentiated Burr type XII Poisson (EBXIIP), generalized Marshall-Olkin extended Burr-XII (GMOBXII), and beta Burr type XII (BBXII) distributions are among the non-nested distributions. Following that, in Chapters 3, 4, and 5, we discuss the three extended distributions: exponentiated Weibull Burr type X (EWBX), exponentiated beta Burr type X (EBBX), and exponentiated gamma Burr type X (EGBX) distributions. In these chapters, we derive their pdf, cdf, statistical properties, and likelihood function. Additionally, we conduct simulation studies in these chapters to evaluate how well the three proposed distributions perform for cases with and without censored data and covariate. In Chapter 6, we adapt three real data sets to demonstrate the performance of the three proposed distributions. Finally, Chapter 7 presents the findings and conclusions of this study. This chapter also discusses the limitations of this study and provides suggestions for future research.

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