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POSITIVE LINEAR OPERATORS AND APPROXIMATION PROPERTIES

By

MOHAMMAD AYMAN MURSALEEN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To the loving memory of my mother

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosphy

POSITIVE LINEAR OPERATORS AND APPROXIMATION PROPERTIES

By

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February 2024

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A fundamental result in approximation theory is the Weierstrass approximation theorem, stating that every continuous function on $[0,1]$ (or continuous 2π periodic function on $[0,2\pi]$) can be approximated by an algebraic (or trigonometric) polynomial. Bernstein provided a concise and elegant proof of this theorem by introducing Bernstein polynomials. Szász and Mirakjan independently studied the Szász-Mirakjan operators to handle functions in $C[0,\infty)$. Various extensions of Bernstein operators have been devised to approximate functions in $L^p[0,1](1 \leq p < \infty)$, including Bernstein-Kantorovich and Bernstein-Durrmeyer operators. This thesis focuses on classical calculus, introducing Stancu-type generalizations of Baskakov-Durrmeyer operators and α-Stancu-Schurer-Kantorovich operators by exploring new operators using q-calculus, starting with Stancu-type modifications of generalized Baskakov-Szász operators and Phillips operators. Then, shifting knots of q-Bernstein-Kantorovich operators, a family of summation-integral type hybrid operators with shape parameter α, and the q-Baskakov-Kantorovich operators via wavelets are

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introduced. The computation of moments and central moments, determining the order of approxi-mation, discussion of basic results and approximation properties and also study of estimates and rate of convergence for these operators by employing various useful tools like Voronovskaja type result, Peetre's K-functional and asymtotic error constant.

Keywords: Bernstein polynomials; Korovkin theorem; Peetre's K-functional; q calculus; Szász operators.

SDG: GOAL 4: Quality Education; GOAL 9: Industry, Innovation, and Infrastructure; GOAL 11: Sustainable Cities and Communities.

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Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGOPERASI LINEAR POSITIF DAN SIFAT PENGHAMPIRAN

Oleh

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Satu hasil asas dalam teori penghampiran ialah teorem penghampiran Weierstrass, menyatakan bahawa setiap fungsi selanjar pada $[0,1]$ (atau fungsi selanjar dengan tempoh 2π pada $[0,2\pi]$) boleh dihampiri oleh polinomial algebra (atau trigonometri). Bernstein telah menyediakan bukti yang ringkas dan menarik bagi teorem ini dengan memperkenalkan polinomial Bernstein. Szász dan Mirakjan secara bebas mengkaji pengoperasi Szász-Mirakjan untuk mengendalikan fungsi dalam $C[0,\infty)$. Pelbagai perluasan pengoperasi Bernstein telah direka untuk menghampiri fungsi dalam $L^p[0,1]$ $(1 \leq p < \infty)$, termasuk pengoperasi Bernstein-Kantorovich dan Bernstein-Durrmeyer. Tesis ini, memberi tumpuan kepada kalkulus klasik, memperkenalkan pengitlakan

jenis Stancu bagi pengoperasi Baskakov-Durrmeyer dan pengoperasi α-StancuSchurer-Kantorovich, dengan meneroka pengoperasi baharu menggunakan q-kalkulus, bermula dengan pengubahsuaian jenis Stancu bagi pengoperasi Baskakov-Szász yang diitlakkan dan pengoperasi Phillips. Kemudian, titik anjakan bagi pengoperasi q-Bernstein-Kantorovich, satu keluarga pengoperasi hibrid jenis jumlahan-kamiran dengan parameter bentuk α, dan pengoperasi q-Baskakov-Kantorovich melalui gelombang kecil diperkenalkan. Pengiraan momen dan momen pusat, menentukan susunan penghampiran, perbincangan hasil asas dan sifat penghampiran dan juga kajian anggaran dan kadar penumpuan untuk pengendali ini dengan menggunakan pelbagai alat berguna seperti hasil jenis Voronovskaja, Peetre's K-fungsional dan pemalar ralat asimtotik.

Kata Kunci: Bernstein polinomial; Korovkin teorem; Peetre's K-fungsional; q kalkulus; Szász pengendali.

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LIST OF ABBREVIATIONS

$\mathbb{N}$	Set of all natural numbers
$\mathbb{R}$	Set of all natural numbers
$\mathbb{R}_0^+$	Set of all non-negative real numbers
$\mathbb{C}$	Set of all complex numbers
$C[a,b]$	Space of all continuous functions on $[a,b]$
$C^s[a,b]$	$C[a,b]$ with continuous derivatives of order s
$L^p[a,b]$	Space of all Lebesgue integrable functions on $[a,b], p \geq 1$
$B_\rho(\mathbb{R}_0^+)$	$\{h: |h(u) \leq K_h \rho(u)|, K_h$ is a constant depending upon $h\}$
$C_\rho(\mathbb{R}_0^+)$	$B_\rho(\mathbb{R}_0^+) \cap C(\mathbb{R}_0^+)$
$C_\rho^2(\mathbb{R}_0^+)$	$\left\{h \in C_\rho(\mathbb{R}_0^+): \lim_{u\to\infty} \frac{|h(u)|}{\rho(u)}\right.$ exists $\}$
$\binom{r}{j}$	Binomial coefficients
$\begin{bmatrix} r \\ j \end{bmatrix}_q$	q −Binomial coefficients
$w(h,\theta)$	Modulus of continity
$\mathrm{Lip}_M\ \eta$	Lipschitz class of functions
$[n]_q$	q −integer
$D_q h$	q −derivative of h
$\delta(A)$	Natural density of $A \subseteq \mathbb{N}$
$\delta_q(A)$	q-density of $A \subseteq \mathbb{N}$
$St - \lim$	Statistical limit
$St_q - \lim$	q-statistical limit

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

A challenging and complex function can be solved by using a straightforward and easily calculable function in the area of mathematics known as the Approximation Theory. Weierstrass, in 1885, originally came to the conclusion that the set of algebraic polynomials is dense in the set of continuous real-valued functions defined on closed interval. That is, for any $h \in C[a,b]$ there exists an algebraic polynomial $Q(u)$ having real coefficients such that for $\epsilon > 0, |h(u) - Q(u)| < \epsilon$ for every $u \in [a,b]$.

In approximation theory, we have two primary problems. The first one is qualitativewhich investigates what limitations are faced by a sequence of operators in approximating identity operator. While, the second problem deals with quantitative-which investigates how fast a sequence of operators approaches the identity operator.

The positive approximation process by positive linear operators possesses a crucial role in the approximation theory when dealing with the approximation of continuous functions, which is accomplished in an unforced and very natural fashion. Bernstein operators were presented by S.N. Bernstein in the year 1912 and were the first recognized and acclaimed positive linear operators. Bernstein polynomials thus provide one way to prove the Weierstrass approximation theorem that every real-valued continuous function on a real interval $[a,b]$ can be uniformly approximated by polynomial functions over $\mathbb{R}$. Since then these operators have been generalized,

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modified and extended to deal with the approximation of the functions of different natures.

During the 50s of the previous century, a significant breakthrough was achieved in the approximation theory. For example, it was studied by Popoviciu (1940), Bohman (1952) and Korovkin (1953). Moreover, in 1953, Korovkin discovered a straightforward, clearcut standard for determining when a sequence of positive linear operators will uniformly converge to identity operator. It states that if a sequence of positive linear operators on the space $C[0,1]$ approximates the monomials $t^j, j = 0,1,2$, known as the test functions, then it approximates any continuous function on $[0,1]$.

If the definition domain of function h is unbounded, i.e., $\mathbb{R}_0^+ = [0,\infty)$, then this theorem holds true specifically for continuous functions having finite limit at infinity. Apart from that, in this particular instance, however, the test functions $t^j, j = 0,1,2$ have been replaced by other test functions (e.g. e^{-jt} or $\left(\frac{t}{1+t}\right)^j, j = 0,1,2,$).

Test functions have a significant role in Korovkin's type approximation, which is why they have been given their own unique moniker- Korovkin's test functions.

The following quantities are necessary tools for performing research, including the process of approximating functions using positive linear operators T. This includes the calculation of test function $T(e_j;u)$ (also known as moments) and the central moments $T\left((e_1 - e_0;u)^j;u\right)$ of order j, where $e_j(t) = t^j, j \in \mathbb{N} \cup \{0\}$. It is possible to demon-strate the convergence of sequence of positive linear operators to the identity

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operator by implementing the famous theorem of Korovkin. This initially requires determining of the first three moments.

For quantitative approximation, the estimation of remainder terms $T(h) - h$ is performed by utilizing modulus of smoothness of distinct types and Peetre's K-functionals. Shisha and Bond (1968) provided the estimation of the remainder, utilizing the classical modulus of continuity.

Our aim of this research is to fill some gaps in the existing literature by constructing new sequences of positive linear operators which may provide more flexibility and better approximation as well as a faster rate of convergence to the functions of different classes.

1.2 Preliminaries

Positive Linear Operators

In this section, several fundamental properties and definitions of positive linear operators will be presented. For the definitions, notations and examples used in this chapter and onward, we refer to Altomare and Campiti (1994) and Aral et al. (2013).

For a function $h \geq 0$, we mean $h(u) \geq 0$ for all u in the domain of h.

Definition 1.2.1 *Let X and Y be any two normed linear spaces. The mapping : $X \to Y$ is known as a linear operator if*

$$T(\eta u + \beta v) = \eta T(u) + \beta T(v), \text{ for all } u, v \in X, \eta, \beta \in \mathbb{R}$$

and is positive, if

$$T(u) \geq 0, \text{ for all } u \in X \text{ having the property } u \geq 0.$$

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Remark 1.2.2 *(i) The set* $\mathcal{T}(X, Y) = \{T: X \to Y \mid T$ *denotes a linear operator* $\}$ *is a real vector space.*

(ii) To stress the argument of the function $T(h) \in Y$, *we employ the symbols* $T(h; u)$.

Properties of Positive Linear Operators

Proposition 1.2.3 *Let* $T: X \to Y$ *represent a positive linear operator. Then, the following holds:*

(i) If $h, g \in X$ *with* $h \leq g$, *then* $T(h) \leq T(g)$. *(Monotonicity)*

(ii) For every $h \in X$, *yields* $|T(h)| \leq T(|h|)$.

Proposition 1.2.4 *(Hölder inequality for positive linear operators): Provided that* X *is both lattice subspace and subalgebra as well as* $T: X \to Y$ *is a positive linear operator, then for any real number* $p, l > 1$ *satisfying* $1/p + 1/l = 1$, *we have*

$$T(|h \cdot g|; u) \leq \left(T(|h|^p; u)\right)^{\frac{1}{p}} \cdot \left(T(|g|^l; u)\right)^{\frac{1}{l}}, \text{ for every } h, g \in X.$$

Remark 1.2.5 *Fix* $p = l = 2$ *in the above inequality, we obtain the following important inequality celebrated as Cauchy-Schwarz inequality for positive linear operators:*

$$|T(h \cdot g; u)| \leq \sqrt{T(h^2; u)} \cdot \sqrt{T(g^2; u)}$$

Definition 1.2.6 *Suppose* $T: X \to Y$, *where* X *and* Y *are the two normed linear spaces of real functions. The the norm of the operator* T *is defined by*

$$\| T \| = \sup_{\substack{h \in X \\ \|h\| = 1}} \| T(h) \| = \sup_{\substack{h \in X \\ 0 < \|h\| \leq 1}} \| T(h) \| \qquad (1.2.1)$$

which is known as the operator norm.

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1.3 Approximation of Functions by Positive Linear Operators

The smoothness characteristics of the function determine the degree of approximation by positive linear operators, and pertinent methods for determining the smoothness of functions are defined by distinct types of moduli of continuity.

Different types of moduli of smoothness

In Jackson (1911), the authors provided the definition of the first modulus of continuity (smoothness) in the Doctorate thesis, defined by

$$\omega(h;\theta) := \sup\{|h(u+t)-h(u)|: u, u+t \in [a,b], 0 \le t \le \theta\}$$

where $h \in C[a,b]$ and $\theta \ge 0$.

It served as the foundation for the quantitative approximation theory that is practised today. A natural modulus of smoothness, established in 1987 by Ditzian and Totik, is viewed as a superior tool for dealing with embedding theorems, inverse theorems, as well as the rate of best approximation. (given in Ditzian and Totik (1987), p. 1-4).

The Ditzian-Totik modulus of smoothness is provided by

$$\omega^m(h,\theta)_p = \sup_{0<t\le\theta} \|\Delta_t^m h\|_p \tag{1.3.2}$$

Here, $\Delta_t^m h$ represents the m-th order forward difference having step length t.

Remark 1.3.1 *The moduli of smoothness of first and second order are the primary metrics to evaluate the degree of convergence of positive linear operators towards the identity operator. For* $h \in C[a,b]$ *and* $\theta \ge 0$, *the modulii of continuity are defined by*

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$$\omega_1(h,\theta) := \sup\{|h(u+t)-h(u)| : u, u+t \in [a,b], 0 \le t \le \theta\}$$
$$\omega_2(h,\theta) := \sup\{|h(u+t)-2h(u)+h(u-t)| : u, u \pm t \in [a,b], 0 \le t \le \theta\}$$

The majority of the error estimates in this study is given by the two moduli of smoothness. For example, Ditzian-Totik second order modulus is donated by $\omega^2(h,\cdot)$. In the following property, we denote $\omega_1 = \omega$.

Proposition 1.3.2 *Let* $h \in C[a,b]$ and $\theta \ge 0$.

(i) For any $u \ne v \in [a,b], |h(v)-h(u)| \le \omega(h,|v-u|)$.

(ii) h *is uniformly continuous if and only if* $lim_{\theta\to 0}\omega(h,\theta) = 0$.

(iii) If $0 < \theta_1 < \theta_2$, *then* $\omega(h,\theta_1) \le \omega(h,\theta_2)$.

(iv) For any $k > 0, \omega(h,k\theta) \le (1+k)\omega(h,\theta)$.

(v) If h' *exists and bounded on* $[a,b]$, *then* $\omega(h,\theta) \le M\theta$, *for some constant* M.

(vi) For every $\theta > 0$, *we have*

$$|h(v)-h(u)| \le \left(1+\frac{|v-u|}{\theta}\right)\omega(h,\theta) \tag{1.3.3}$$

We occasionally provide estimations when higher-order moduli are involved. According to Schumaker (1981) definition, the modulus of smoothness of higher order is given by:

Definition 1.3.3 *For* $m \ge 1, \theta \ge 0$ and $h \in C[a,b]$, the modulus of smoothness of order m is written as

$$\omega_m(h,\theta) = \sup\{|\Delta_t^m h(u)| : 0 < t \le \theta, u, u+mt \in [a,b]\} \tag{1.3.4}$$

where $\Delta_t^m h(u)$ denotes the m^{th} order forward difference of $h(u)$ having step length t, given by

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$$\Delta_t^m h(u) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} h(u + jt),\ m \in \mathbb{N}$$

Lipschitz class of functions

A function h defined on an interval $[a, b]$ satisfies the Lipschitz condition of exponent $\eta > 0$ if

$$|h(v) - h(u)| \leq M|v - u|^\eta,\ M > 0 \tag{1.3.5}$$

In this case, we say that $h \in \mathrm{Lip}_M\ \eta$.

Remark 1.3.4 (i) A function complying with a Lipschitz condition is uniformly continuous.

(ii) $h \in Lip_M\ \eta, \eta > 0$ *iff inequality* $\omega(h, \theta) \leq M\theta^\eta$ *holds in the interval* $[a, b]$.

(iii) If $h \in Lip_M\ \eta$ *and* $\eta > 1$, *then* h *is a constant function.*

K-functional and its relationship with moduli

Half a century ago, Peetre (1968) established a function, known as Peetre's Kfunctional, to investigate the interpolation between two Banach spaces. Here, it is an essential metric in measuring the smoothness of a function depending on how effectively it may be approximated by functions that are smoother. The K-functional can be defined in a setting as general as that described in Ditzian and Totik (1987). This can be applied specifically for the optimal approximation polynomials.

The classical definition of the K-functional is given as follows

Definition 1.3.5 For any $h \in C[a, b], \theta \geq 0$ and $s \geq 1$, the Peetre's K-functional of *order* s *is defined by*

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$$K_s(h,\theta)_{[a,b]} = \inf\{\| h - g \|_\infty + \theta \|g^{(s)}\|_\infty : g \in C^s[a,b]\} \qquad (1.3.6)$$

Provided that there is no ambiguity regarding the interval of h, we may write $K_s(h,\theta)_{[a,b]}$ as $K_s(h,\theta)$.

It is evident that the quantity given in (1.3.6) reflects several approximation properties of h. Here, the inequality given by $K_s(h,\theta) < \varepsilon, \theta > 0$ implies that h may be approximated with error $\| h - g \|_\infty < \varepsilon$ in $C[a,b]$ by an element $g \in C^s[a,b]$, provided that the norm is not too large and $\|g^{(s)}\|_\infty \leq \frac{\varepsilon}{\theta}$.

The lemma given below gathers several properties of $K_s(h,\cdot)$. They were proven by Butzer and Berens (1967), which may also be discovered in more current research on approximation theory, for example, Daubechies (1992); Gonska (1998); Schumaker (1981).

Lemma 1.3.6 (see Proposition 3.2.3 in Butzer and Berens (1967)) Let $K_s(h,\cdot)$ be defined as in (1.3.6). Then

(i) The mapping $K_s(h,\theta): \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous at* $\theta = 0$, *in which,*

$$\lim_{\theta \to 0^+} K_s(h,\theta) = 0 = K_s(h,0)$$

(ii) For each $h \in C[a,b]$, *the mapping* $K_s(h,\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ *is monotonically increasing as well as a concave function.*

(iii) For arbitrary $k, \theta \geq 0$, *and fixed* $h \in C[a,b]$, *the following inequality holds:*

$$K_s(h,k\theta) \leq max\{1,k\} \cdot K_s(h,\theta).$$

(iv) For arbitrary $h_1, h_2 \in C[a,b]$, *we have* $K_s(h_1+h_2,\theta)K_s(h_1,\theta)K_s(h_2,\theta), \theta \geq 0.$

(v) For each $\theta \geq 0$ *fixed,* $K_s(\cdot,\theta)$ *is a seminorm on* $C[a,b]$, *provided that*

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$$K_s(h,\theta) \leq \| h \|_\infty$$

for all $h \in C[a,b]$.

The theorem given below forms the relationship between the moduli of smoothness as well as K-functional. Here, K_s and ω_s are related via the equivalence relation given by Johnen (1972)

Theorem 1.3.1 There exist constants C_1 and C_2, which depends only on s as well as $[a,b]$ provided that

$$C_1\omega_s(h,\theta) \leq K_s(h,\theta^s) \leq C_2\omega_s(h,\theta) \tag{1.3.7}$$

for all $h \in C[a,b]$ and $\theta > 0$.

Generally, there are no known sharp constants occurring in the inequality given above. Nevertheless, there exist two exceptional cases for $s = 1,2$ (see Păltănea (2010)).

For the applications, it is sufficient to consider the condition in which $s = 2$ holds. For further information, one can refer to Stănilă (2014).

Big-O and small-o notations

Landau's symbol is used to explain the asymptotic behavior of functions in mathematics, computer science, and complexity theory. It was named after Edmund Landau, a German number theorist. These symbols mostly tell us how quickly a function either increases or decreases in value. The growth rate of a function is also known as its order, which is the reason the letter ' O ' is employed for the symbol that Landau constructed.

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Definition 1.3.7 (Big-O): Let $h(u)$ and $g(u)$ be two functions defined on several subsets of real numbers. We say $h(u) = O(g(u))$ if and only if there exist constants N and K in which

$$|h(u)| \leq K|g(u)|, \text{ for all } u > N$$

In a less formal sense, the expression $h(u) = O(g(u))$ indicates that the function h does not increase quicker than the function g. In addition to the Big-O notation, another one of Landau's symbols, the Small-o sign, is employed in mathematical notation. The expression $h(u) = o(g(u))$ indicates that the function h grows far more slowly than the function g, and in comparison, the comparison is useless.

Definition 1.3.8 (Small-o): $h(u) = o(g(u))$ known as Small-o if and only if for every $K > 0$, there is a positive real number N provided that

$$|h(u)| < K|g(u)|, \text{ for all } u > N$$

If $g(u) \neq 0$, then this is equivalent to $\lim_{u\to\infty} \frac{h(u)}{g(u)} = 0$.

1.4 Problem Statement

In Approximation Theory, there exists a vast array of operators constructed by researchers worldwide, aiming to generalize or modify Bernstein operators for the approximation of functions across different classes, settings, and spaces. Despite the considerable body of work in this field, a notable research gap persists concerning the optimal design and application of these operators to achieve superior approximation quality and rates of convergence. This gap arises from the inherent complexity and diversity of functions encountered in real-world applications, which demand tailored

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approximation techniques like introduction of various parameters for flexibilty, effective representation and analysis.

Specifically, existing literature often lacks comprehensive exploration and understanding of how these operators perform in various function spaces and when applied to functions with specific characteristics such as discontinuities, singularities, or oscillatory behavior. Furthermore, while numerous operators have been proposed, there remains a need to identify the most efficient and effective ones for different scenarios, considering factors such as computational complexity, approximation accuracy, and stability.

In this study, we aim to bridge this research gap by introducing and investigating various sequences of operators designed to address the aforementioned challenges.

These operators include Stancu-type generalized Baskakov-Durrmeyer operators, Stancu-Schurer-Kantorovich operators with shape parameter α, Stancu variants of generalized Baskakov-Szász operators, Stancu type Phillips operators, q-analog of Bernstein-Stancu-Kantorovich operators, Baskakov-Gamma operators, and q-analog of Baskakov operators by Wavelets.

Our problem is twofold: first, to develop a deeper theoretical understanding of the approximation properties of these operators, including their convergence behavior, stability, and ability to accurately represent various function spaces. Secondly, we seek to provide practical insights into the selection and application of these operators in

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real-world scenarios, considering their computational efficiency and adaptability to different function types like wavelets and spaces like L_p-space.

By addressing this research gap, our study aims to contribute to the advancement of approximation theory, providing researchers and practitioners with valuable tools and methodologies for effectively approximating complex functions encountered in diverse scientific and engineering fields.

1.5 Objectives

The objectives of the study are to:

(i) Analyze the statistical convergence and approximation properties of the Stancu-type generalization of Baskakov-Durrmeyer operators using inverse Pólya-Eggenberger distribution.

(ii) Examine the Voronovskaja type result and convergence analysis.

(iii) Analyze the Stancu type modification of generalized Baskakov-Szász operators and study q-statistical approximation techniques.

(iv) Examine Phillips operators with real parameters and determine the order of convergence in terms of the maximal Lipschitz function and Peetre's K-functional.

(v) Study convergence features and obtain quantitative estimates of the Voronovskaja-type.

(vi) Study the local behavior of approximating operators through the wavelets and to obtain a general formula for the moments of these newly defined operators.

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These objectives are designed to address the specific goals and topics outlined in each chapter while utilizing relevant mathematical tools and techniques such as Korovkin type theorems, various modulus of continuity, and q-calculus.

1.6 Scope of the Study

The present study will provide:

(i) The general procedure to construct the Stancu variants of positive linear operators,

(ii) To improve the rate of convergence and better approximation,

(iii) How to construct wavelets aided operators to study local behavior of approximating operators,

(iv) How to use statistical and q-statistical convergence techniques in improving the convergence results for the sequence of positive linear operators.

1.7 Outline of the Study

The present thesis consists of nine chapters:

In the first chapter, we present some ancillary results, definitions and notations which form the background of the subsequent chapters, such as the concept of positive linear operators and provide instances of their features. The modulus of smoothness of the first and second order, the Ditzian-Totik modulus of smoothness, and the Peetre's K-functional are a few of the significant tools that quantify the quantitative approximation of a function by positive linear operators.

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Second chapter is Literature Review in which we provide a quick summary of Korovkin's type theorem as well as an introduction to q-calculus and the notion of statistical convergence.

In Chapter 3, we introduce Stancu type generalization of Baskakov-Durrmeyer operators by using inverse Pólya-Eggenberger distribution and discuss some basic results and approximation properties. In addition to that, we investigate the statistical convergence for each of these operators.

The objective of Chapter 4 is to present a sequence of α-Stancu-Schurer-Kantorovich operators. With the aid of modulus of continuity, we compute moments and central moments and determine the order of approximation. Also we obtain the Voronovskaja type result. Next, numerical and graphical representations of error analysis and convergence of the operators for specific functions are provided. In addition, twodimensional α-Stancu-Schurer-Kantorovich operators are built, together with their rate of convergence, graphical depiction, and numerical error estimates.

Chapter 5 deals with the Stancu type modification of generalized Baskakov-Szász operators. In order to calculate moments for these new operators, we obtain a recurrence relation first. We investigate a variety of approximation properties as well as q-statistical approximation.

Chapter 6 is to incorporate non-negative real parameters to enable approximation results for the Stancu variant of Phillips operators. We focus on the uniform modulus of smoothness in a straightforward manner before going on to the approximation in weighted Korovkin's space. Our study's objectives and consequences are to

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exhaustively develop the Phillips operators' universally approximated findings. We determine the order of convergence in terms of the maximal Lipschitz function and Peetre's K-functional. In addition, a theorem of the Voronovskaja type is also demonstrated.

In Chapter 7, we examine the convergence and related features of q-BernsteinKantorovich operators, particularly the shifting knots of real positive numbers. We design the q-calculus-generated shifting knots of Bernstein-Kantorovich operators. Specifically, we investigate the convergence features of our novel operators in the space of continuous functions and the space of Lebesgue integrable functions. Using modulus of continuity and integral modulus of continuity, we determine the degree of convergence. In addition, we create quantitative estimates of the Voronovskaja-type.

In Chapter 8, a distinctive family of summation-integral type hybrid operators with shape parameter α is constructed. Using Korovkin's theorem and the modulus of smoothness, basic estimates, rate of convergence, and order of approximation are also addressed. Using Peetre's K-functional, the Lipschitz class, and the second-order modulus of smoothness, we examine the local approximation results for these operators.

In the ninth chapter, we use wavelets to define q-analog of the Kantorovich variant of Baskakov type operator, and we study the L_p-approximation, weighted q-statistical approximation and rate of convergence. In order to construct the operators of the q-

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Baskakov type, we make use of Daubechies' compactly supported wavelets which allow us to study the local behaviour of approximating operators.

The tenth and final chapter contains conclusion of our whole work followed by the key ideas for the the future works.

At the end of thesis, we give a list references of papers and books which have been a helping source of the present research work followed by the list of publications where all our results are published.

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