DIFFERENTIAL GAMES OF MANY PLAYERS FOR INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS

By

KAZIMIROVA RUZAKHON YUNUS KIZI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To my parents

Yunusjon Kazimirov and Paridakhon Kazimirova and also to my friend

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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October 2024

Chairman : Risman bin Mat Hasim, PhD
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This thesis focuses on studying evasion and pursuit differential games involving multiple pursuers and a single evader for infinite systems of differential equations in the Hilbert space l_2. Three problems are examined in this thesis. In the first problem, an evasion differential game with multiple pursuers and one evader is considered, described by an infinite system of binary differential equations in two cases. Solutions are presented for the evasion problem with negative coefficients in the first case and nonnegative coefficients in the second case, under the geometric constraints imposed on the players' controls. The goal of the pursuers is to bring the state of at least one controlled system to the origin of l_2, while the evader aims to prevent this. A sufficient condition for evasion is obtained for any initial state of the players, and an evasion strategy for the evader is constructed. The second problem involves a multiple-pursuer and single-evader evasion differential game with integral constraints for an infinite system of two-block differential equations. For this problem, it is shown that if the evader's control resource is greater than or equal to the combined resources of the pursuers, evasion is possible from any initial position in the infinite system. Both the first and second problems are

approached by reducing the game in Hilbert space l_2 to an equivalent differential game in finite-dimensional Euclidean space. An explicit evasion strategy is proposed, guaranteeing successful evasion. The third problem addresses a pursuit differential game involving multiple pursuers and one evader, described by an infinite system of second-order differential equations, where the players' control functions are subject to integral constraints. To solve this problem, the case of one pursuer and one evader is first examined, and this result is subsequently applied to obtain the main result of the problem. Additionally, a condition in terms of the players' energies is derived, providing a sufficient condition for successful pursuit. Strategies for the pursuers are constructed to ensure the capture of the evader.

Keywords: Differential Game, Control, Strategy, Infinite System of Differential Equations, Integral Constraint

SDG: GOAL 4: Quality Education

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PERMAINAN PEMBEZAAN RAMAI PEMAIN UNTUK SISTEM PERSAMAAN PEMBEZAAN TIDAK TERHINGGA

Oleh

KAZIMIROVA RUZAKHON YUNUS KIZI

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Tesis ini memberi tumpuan kepada kajian permainan pembezaan pengelakan dan pengejaran yang melibatkan ramai pengejar dan satu pengelak untuk sistem persamaan pembezaan tidak terhingga dalam ruang Hilbert l_2. Tiga masalah dikaji dalam tesis ini. Dalam masalah pertama, permainan pembezaan pengelakan dengan ramai pengejar dan satu pengelak dipertimbangkan, diterangkan oleh sistem persamaan pembezaan binari tidak terhingga dalam dua kes. Penyelesaian dibentangkan untuk masalah pengelakan dengan pekali negatif dalam kes pertama dan pekali bukan negatif dalam kes kedua, di bawah kekangan Geometri yang dikenakan pada kawalan pemain. Matlamat pengejar adalah untuk membawa keadaan sekurang-kurangnya satu sistem terkawal ke asalan l_2, sementara matlamat pengelak untuk mencegahnya. Syarat mencukupi untuk mengelak diperolehi untuk mana-mana keadaan awal pemain, dan strategi pengelak dibina. Masalah kedua melibatkan permainan pembezaan pengelakan ramai pengejar dan satu pengelak dengan kekangan kamiran untuk sistem persamaan pembezaan dua blok tidak terhingga. Untuk masalah ini, ditunjukkan bahawa jika sumber kawalan pengelak lebih besar daripada atau sama dengan sumber gabungan pengejar,

pengelakan adalah tidak mustahil dari mana-mana kedudukan awal dalam sistem tak terhingga. Kedua-dua masalah pertama dan kedua didekati dengan menurunkan permainan ke dalam ruang Hilbert l_2 kepada permainan pembezaan setara dalam ruang Euclidean berdimensi terhingga. Strategi pengelakan yang jelas dicadangkan, menjamin pengelakan yang berjaya. Masalah ketiga menangani permainan pembezaan pengejaran yang melibatkan ramai pengejar dan satu pengelak, yang diterangkan oleh sistem persamaan pembezaan peringkat kedua tidak terhingga, di mana fungsi kawalan pemain tertakluk kepada kekangan kamiran. Bagi menyelesaikan masalah ini, kes satu pengejar dan satu pengelak diperiksa terlebih dahulu, dan hasil ini kemudiannya diterapkan untuk mendapatkan hasil utama masalah tersebut. Di samping itu, syarat dari segi tenaga pemain diperoleh, dengan anggapan syarat yang mencukupi untuk pengejar berjaya. Strategi untuk pengejar dibina untuk memastikan penangkapan pengelak.

Kata Kunci: Permainan Pembezaan, Kawalan, Strategi , Sistem tak Terhingga Persamaan Pembezaan, Kekangan Integral

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LIST OF ABBREVIATIONS

u_i	Control parametr of the i-th pursuer
v_i	Control parametr of the i-th evader
P	Pursuer
E	Evader
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	The n - dimensional Euclidean space
l_2	Hilbert space
L_2	Space of square-integrable functions
$C(0,T;l_2)$	Space of continuous functions on $[0;T]$ with value in l_2
Ω	Domain in $\mathbb{R}^n$
$\partial\Omega$	Boundary of the domain Ω
$H(x_0,r)$	Ball of radius r centered at x_0
U_i	Strategy of the i-th pursuer
V_j	Strategy of the j -th evader
A^*	The transpose of matrix A
x_i	State of the i-th pursuer
$y_i(t)$	State of the i -th evader
e	Unit vector
$\dot{x}$	First derivative of x
$\dot{y}$	First derivative of y
$\langle\, \rangle$	Inner product
$\|\| \ \|\|$	Norm for Hilbert space

CHAPTER 1

INTRODUCTION

The background information on differential game issues is presented in this chapter. Additionally, it includes important definitions, motivation for the study , and often used standard formulas. It is also described how the thesis is organized, as well as the objectives of the study, scope and limitations of the thesis.

1.1 Introduction

Differential games are a group of mathematical problems related to the mathematical modeling and analsis of conflicts in the framework of dynamical systems, which are governed by differential equations. More specifically, a state variable or variables evolve over time according to a differential equation. At the center of differential games lies the fundamental conflict between two parties known as ”Pursuers and Evaders”. The goal of a pursuer is to capture evading agents, while the goal of an evader is to avoid being captured by a pursuer. This is a zero-sum game where the cost or payoff is the time to capture. Basic questions arise: ”What route should an evader or pursuer take to achieve their goal of avoiding or ensuring capture? And, under optimal play, by either pursuer or evader, is capture at all possible?”

A differential game can involve one pursuer and one evader, many pursuers and one evader, one pursuer and many evaders, and many pursuers and many evaders. For example, the one pursuer, one evader differential game is the Homicidal Chauffeur Game proposed by Isaacs (1965) and clearly expressed that a differential game is described by the differential equation

$$\dot{x}(t) = f[x(t), u(t), v(t), t],$$

where t is time, x is a state vector, u and v are control vectors manipulated, within constraints, by the players. In this game, a pedestrian who can move slowly but can change direction very easily is competing against someone driving a car that is faster but not as good at turning corners, called Dubin's car. In this situation , the driver attempts to run over the pedestrian. The question to be solved is:

1. Under what circumstances, and with what strategy, can the driver of the car guarantee that he can always catch the pedestrian or conversely, the pedestrian guarantee that he can indefinitely elude the car?
2. If the pedestrians demise is guaranteed, what is the chauffeurs optimal strategy that will minimize the time-to-capture of the pedestrian, and what is the latters strategy to maximize his time?

This problem has been investigated thoroughly in works such Merz (1971), Marchal (1974), Falcone (2006), Patsko and Turova (2011) .

The notion of a differential game is connected to the optimal control. In the optimal control problem, a single control and a single set of criteria are analyzed to be maximized. On the other hand, the differential game involves two controls, $u(t)$ and $v(t)$, which represent the pursuer and evader controls, respectively.

Differential games of many players for infinite systems of differential equations refer to mathematical models in which multiple decision-makers, known as players, interact within a continuous-time framework governed by an infinite system of differential equations. Each player seeks to optimize their own objective function while considering the dynamic evolution of the system, influenced by the decisions of all players. The game of pursuit and evasion examines conflict problems in systems that are driven by a model. The model that describes the behavior of players which is determined by the players input through their respective control functions included in the model. The model is usually a system of differential equations, and each player aims to control the state of the system as much as possible in order to achieve their goal. The system of differential equations is usually the space where pursuit and evasion differential game problems are played.

The movement of players in a differential game is described by differential equations and is subjected to at least one of the constraints which are state constraints, integral constraint, and geometric constraint. The state or phase constraint describes the limitation of player positions, where movement of players can only occur within an area in a given space at all times. The integral constraint is described in integral form, and usually refers to the resources of players, such as fuel, food, energy, and finance that could be exhausted by consumption. Geometric constraint for the control parameter of the pursuer, $u(t)$, in subset X of $\mathbb{R}^n$ is written as $u(t) \in X \subset \mathbb{R}^n$.

The concept of Game of Kind is fundamental in differential games and, given the initial conditions and problem parameters, its solution provides the answer to the question of which team wins the game.

The significance of differential games in game theory lies in their ability to model and analyze complex, dynamic interactions among strategic decision-makers in various fields such as economics, engineering, biology, and military strategy. They allow us to study scenarios where the timing and duration of actions are crucial and where players have continuous control over their decisions. For instance, in economics, differential games are used to model strategic interactions in markets where firms compete over time, taking into account factors like investment, production, and pricing decisions. In engineering, they are applied to study optimal control problems in dynamic systems, such as those involving resource allocation or trajectory planning for autonomous vehicles. In biology, differential games help understand the dynamics of predator-prey relationships and evolutionary strategies. Moreover, in military strategy, they are utilized to analyze conflicts involving the continuous deployment and maneuvering of forces over time.

1.2 Preliminaries

Below are several significant definitions pertaining to differential game theory

1.2.1 Hilbert space

Definition 1.1 *(Inner product space)(Debnath and Mikusinski, 2005)*
Let E be a complex vector space. A mapping $\langle\cdot,\cdot\rangle : E\times E \to C$ is called inner product in E if for any $x,y,z\in E$ and $\alpha,\beta\in C$ the following conditions are satisfied:

i. $\langle x,y\rangle = \overline{\langle y,x\rangle}$;
ii. $\langle \alpha x+\beta y,z\rangle = \alpha\langle x,z\rangle+\beta\langle y,z\rangle$;
iii. $\langle x,x\rangle \geq 0$;
iv. $\langle x,x\rangle = 0$ *implies* $x=0$.

A vector space with an inner product is called an inner product space.

Example 1.1 $\mathbb{R}^n$ *is a Hilbert space with an inner -product defined by*

$$\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$.

Example 1.2 *(Debnath and Mikusinski, 2005) (n-dimensional inner product space)* *The space C^n of ordered n-tuples $x = (x_1, ..., x_n)$ of complex numbers is a Hilbert space , with inner product defined by:*

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k},\ x = (x_1, ..., x_n),\ y = (y_1, ..., y_n)$$

Example 1.3 *The vector space of sequences of real numbers*

$$l_2 = \left\{ \xi = (\xi_1, \xi_2, ...) \mid \sum_{n=1}^{\infty} \xi_n^2 < \infty \right\}$$

is a Hilbert space where the inner product is defined by

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n.$$

Definition 1.2 *(Young, 1988)*
Let E be a real complex vector space. A norm on E is a mapping $|| \cdot || : E \to R$ which satisfies the following conditions:

i. $||x|| > 0$ *if* $x \neq 0$,
ii. $||\lambda x|| = |\lambda| ||x||$ *for all scalars* λ *and vectors* $x \in E$,
iii. $||x + y|| \leq ||x|| + ||y||$ *for all* $x, y \in E$,

A normed space is a pair $(E, || \cdot ||)$ where E is a real or complex vector space and $|| \cdot ||$ is a norm on E.

Definition 1.3 *The norm of vector x in an inner product space is defined by*

$$||x|| = \sqrt{\langle x, x \rangle}.$$

Theorem 1.1 *(Cauchy -Schwartz inequality).*

Let $(E, \langle\cdot,\cdot\rangle)$ be an inner-product space space. For any two elements x and y of inner product space , we have

$$|\langle x,y\rangle| \leq ||x|| \cdot ||y||.$$

The equality $|\langle x,y\rangle| = ||x|| \cdot ||y||$ holds true if and only if x and y are linearly dependent .

For this proof see Debnath and Mikusinski (2005)

Definition 1.4 *(Bierens, 2009) A sequence of elements x_n of a metric space with metric $d(\cdot,\cdot)$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exists an $n_0(\varepsilon) \in N$ such that for all $k,m \geq n_0(\varepsilon)$, $d(x_k,x_m) < \varepsilon$.*

Definition 1.5 *(Debnath and Mikusinski, 2005) A sequence of vectors (x_n) in a normed space is called Cauchy sequence if for every $\varepsilon > 0$ there exists a number M such that $||x_m - x_n|| < \varepsilon$ for all $m,n > M$.*

Definition 1.6 *An inner product space E is said to be complete if every Cauchy (convergent) sequence from E converges in norm to a limit in E. A complete inner product space is called a Hilbert space.*

Hilbert spaces provide a mathematical framework for dealing with infinite-dimensional spaces, which is particularly important in the context of systems of differential equations where the solutions are functions defined on an infinite domain.

1.2.2 Measurable Functions

Definition 1.7 *A real-valued function $f(x)$ is called piecewise continuous, if its domain can be represented as union of finite number of intervals on each the function $f(x)$ is continuous.*

Definition 1.8 *Let Φ be a set and A is a set of subsets of Φ. Then A is called a σ-algebra on Φ if*

i. $\varnothing \in A$

ii. *if $B \in A$, then $\Phi \setminus B \in A$*

iii. *$B_1, B_2, \ldots$ is a sequence of elements A, then $\bigcup\limits_{k=1}^{\infty} B_k \in A$.*

Example 1.4 *Let Φ be a set. Then clearly $\{\varnothing, \Phi\}$ is a σ-algebra on Φ.*

Definition 1.9 *(Axler, 2020) A measurable space is an ordered pair (Φ, A), where Φ is a set and A is σ - algebra on Φ.*

Definition 1.10 *The smallest σ -algebra on R containing all open subsets of R is called the collection of Borel subsets of R. An element of this σ -algebra is called a Borel set.*

Definition 1.11 *Let (Φ, A) be a measurable space. A function $f : \Phi \to R$ is called A -measurable if*

$$f^{-1}(B) \in A$$

for every Borel set $B \subseteq R$, where $f^{-1}(B) = \{x \in \mathbb{R} \mid f(x) \in B\}$.

Example 1.5 *If $A = \{\varnothing, \Phi\}$, then the only A-measurable functions from Φ to R are the constant functions.*

Example 1.6 *If A is the set of all subsets of Φ, then every function from Φ to R is A-measurable.*

Definition 1.12 *Assume $\Phi \subseteq R$. If $f^{-1}(B)$ is a Borel set for each Borel set $B \subseteq R$, a function $f : X \to R$ is said to be Borel measurable.*

1.3 Motivation

Why infinite system of Differential Equations?

We study the controlled, distributed system described by the parabolic equation :

$$y_t - Ay = -u + v, \;\; y|_{t=0} = y_0(x), \;\; y(x,t) = 0, \;\; x \in \partial\Omega \tag{1.3.1}$$

where $y = y(x,t)$ is the unknown function, $x = (x_1, x_2, ..., x_n) \in \Omega \subset \mathbb{R}^n$, $n \geq 1$, Ω is bounded domain, and the boundary $\partial\Omega$ of the domain Ω is assumed to be piecewise smooth, $t \in [0,T]$, and T is a given positive number, $u = u(x,t)$ and $v = v(x,t)$ are the control functions of the first player (the pursuer) and the second player (the evader), respectively, $u(x,t), \;\; v(x,t) \in L_2(H_T)$,

$$H_T = \{(x,t) | x \in \Omega, 0 < t < T\}$$

is an open cylinder in $\mathbb{R}^{n+1}$, $y_0(x) \in L_2(\Omega)$, $G_T = \{(x,t) | x \in \partial\Omega, \; 0 < t < T\}$ is the lateral surface of the cylinder H_T

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right), \;\; a_{ij}(x) = a_{ji}(x),$$

$a_{ij}(x)$ are a bounded measurable functions. Also, there is a positive number k such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geq k\sum_{i=1}^{n}\eta_i^2, \text{ for all } (\eta_1,\eta_2,...,\eta_n) \in R^n \text{ and } x \in \Omega.$$

Further, recall that $W_2^1(\Omega)$ is the Hilbert space of elements of $L_2(\Omega)$ whose first order generalized derivatives are square integrable on Ω, $\mathring{W}_2^{1,0}(\Omega)$ is the subspace of $W_2^1(\Omega)$ in which smooth compactly supported functions form a dense subset, $W_2^{1,0}(H_T)$ is the Hilbert space of elements of $L_2(H_T)$ whose generalized derivatives $\frac{\partial y}{\partial x_i}, i = 1,2,...,n,$ are square integrable on H_T , and $\mathring{W}_2^{1,0}(H_T)$ is the subspace of $W_2^{1,0}(H_T)$ in which smooth functions vanishing near H_T form a dense set. The inner product in $L_2(\Omega)$ and $W_2^1(\Omega)$ are defined by the formulas

$$\langle u,v\rangle_{L_2} = \int_{\Omega} u(x)v(x)\,dx\,, \;\; \langle u,v\rangle_{W_2^1} = \int_{\Omega}(u(x)v(x)+u_x(x)v_x(x))\,dx\,.$$

respectively, and norms in these spaces are defined by the formulas

$$||u||_{L_2} = \sqrt{(u,u)_{L_2}}, \;\; ||u||_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

respectively.

If the above-mentioned conditions are satisfied (Ladyzhenskaya, 1973), then for any $u(x,t)$, $v(x,t) \in L_2(H_T)$ and $y_0(x) \in L_2(\Omega)$, problem (1.3.1) has a unique generalized solution $y = y(x,t)$ in the class $\mathring{W}_2^{1,0}(H_T)$.

Moreover, the solution has the form (Ladyzhenskaya, 1973),

$$y_0(x) = \sum_{i=1}^{\infty} y_{i0}\varphi(x)\,, \qquad (1.3.2)$$

where the functions $y_i(t), 0 \le t \le T,\ i = 1,2,...,$ form a solution of the Cauchy problem for the infinite system of differential equations and initial conditions

$$\dot{y}_i = \lambda_i y_i - u_i(t) + v_i(t),\ \ y_i(0) = y_{i0},\ \ i = 1,2,..., \tag{1.3.3}$$

$\lambda_1, \lambda_2, ..., \lambda_i, ...$ are the generalized eigenvalues of the operator A (Chernousko, 1992), all these eigenvalues are negative, and $\lambda_i \to -\infty$ as $i = 1,2,..$, the functions $\varphi_1(x), \varphi_2(x), ..., \varphi_i(x), ...$ form orthonormal complete system of generelized eigenfunctions of A in $L_2(\Omega)$, and $u_i(t)$ $v_i(t)$ and y_{i0} are the Fourier coefficients of $u(x,t)$, $v(x,t)$ and $y_0(x)$, respectively, in the system $\{\varphi_i(x)\}$, that is

$$u(x,t) = \sum_{i=1}^{\infty} u_i(t)\varphi_i(x),\ \ y_0(x,t) = \sum_{i=1}^{\infty} y_{i0}(t)\varphi_i(x).$$

In addition, series (1.3.2) converges uniformly in $L_2(H_T)$, and its sum $y(x,t)$ belongs to the space $\mathring{W}_2^{\,1}(\Omega)$ for each $t \in [0,T]$ and is a continuous function of the variable t in the norm of $\mathring{W}_2^{\,1}(\Omega)$ (Ladyzhenskaya, 1973). Thus, there is a close relationship between the control that are problems described by partial differential equations (1.3.1) and infinite system of differential equations (1.3.3). The papers such as Satimov and Tukhtasinov (2005), Tukhtasinov (1995), Ibragimov (2002) suggest studying differential game problems described by an infinite system of differential equations. It is assumed that variables $\lambda_k,\ k = 1,2,...,$ in (1.3.3) are any real numbers. For an infinite system of binary differential equations in the Hilbert space l_2, pursuit differential game of one pursuer and one evader was studied by Tukhtasinov et al. (2021) and Ibragimov et al. (2022), when the pursuers control set contains the evader's control set. As a result, all of this acts as a motivating aspect for us to research differential games problems of many players described by an infinite system of ordinary differential equations

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1.4 Objectives of the Thesis

The objectives of the thesis are as follows:

- to obtain the sufficient conditions for which evasion is possible in the differential game described by the following infinite systems of two-block differential equations in the Hilbert space l_2:

$$\begin{aligned} \dot{x}_{ij} &= -\alpha_j x_{ij} - \beta_j y_{ij} + u^1_{ij} - v^1_j, \quad x_{ij}(0) = x^0_{ij}, \\ \dot{y}_{ij} &= \ \ \beta_j x_{ij} - \alpha_j y_{ij} + u^2_{ij} - v^2_j, \quad y_{ij}(0) = y^0_{ij}, \end{aligned} \tag{1.4.1}$$

  where x_{ij}, $y_{ij} \in \mathbb{R}$, $i = 1,2,...,m$, $j = 1,2,...$, are the state variables, α_j, β_j are real numbers and $\alpha_j \geq 0$, u^1_{ij}, u^2_{ij}, $j = 1,2,...$, are the control parameters of the i-th pursuer, $i = 1,2,...,m$, and v^1_j, v^2_j, $j = 1,2,...$, are the control parameters of the evader.

- to construct a strategy for the evader subject to geometric constraints, in the differential game (1.4.1).

- to construct a strategy for the evader subject to integral constraints, in the differential game (1.4.1).

- to obtain a sufficient condition of completion of pursuit for the differential game of many pursuers and one evader in a Hilbert space l_2 described by the following infinite system of second order differential equations

$$\ddot{x}_{ik} = -\mu_{ik} x_{ik} - u_{ik} + v_k, \ \ x_{ik}(t_0) = x^0_{ik}, \ \ \dot{x}_{ik}(t_0) = x^1_{ik}, \tag{1.4.2}$$

  where x_{ik}, x^0_{ik}, u_{ik}, $v_k \in \mathbb{R}^1$, $i = 1,2,...,m$, $k = 1,2,...$, are the state variables, $u_i = (u_{i1}, u_{i2}, ...)$ and $v = (v_1, v_2, ...)$ are the control parameters of the i-th pursuer and the evader.

- to construct strategies for the pursuers in the multiple pursuer differential game with integral constraints described by equation (1.4.2).

1.5 Scope and Limitation

In this thesis, we focus our attention on differensial games of many players for infinite system of differential equations in Hilbert space and limited only to systems of infinite ordinary differential equations (1.4.1) and (1.4.2).

1.6 Methodology

In evasion differential games, there are three steps to prove a statement that evasion is possible.

Step 1 : Construction the strategy of evader, whereas behavior of the pursuer is assumed to be any meaning that the pursuer applies any control $u(t)$.

Step 2 : Admissibility of constructed strategy.

Step 3 : Proof that evasion is possible.

In pursuit games, proof a statement about completion of pursuit consists of three steps.

Step 1 : Construction the strategies for the pursuers, whereas behavior of the evader is assumed to be any meaning that the evader applies any control $v(t)$.

Step 2 : Admissibility of constructed strategy.

Step 3 : Proof that pursuit can be completed.

1.7 Organization of the Thesis

The thesis is divided into six chapters.

Chapter 1 presents an overview of the fundamental concepts of differential game theory. It covers important definitions, provides motivation, methodology for the study, and introduces frequently used standard formulas.

Chapter 2 presents a comprehensive and concise examination of the historical development of differential games. Additionally, it includes a thorough analysis of significant research carried out in this particular area of differential games, both in finite dimensional spaces and in Hilbert space.

Chapter 3 begins with an introductory section. We investigate an evasion differential game that involves multiple pursuers and one evader for an infinite system of binary differential equations in Hilbert space. This investigation is conducted under two conditions. In the first case, we present the problem statement for the system with negative coefficients and for the second system with nonnegative coefficients, and we also provide definitions and important inequalities.In the case of negative coefficients, the solution of the infinite system of differential equations exists on any finite interval $[0,T]$. In the case of positive coefficients, they must be bounded from above in order the infinite system of differential equations to have a solution. Afterwards, we proceed to investigate the reduction of the differential game in Hilbert space to a game in Euclidean space. By constructing admissible strategies, it is proven that evasion is possible in the game described by the infinite system of equations, regardless of the initial states. The chapter concludes with a comprehensive summary.

Chapter 4 focuses on a differential game of evasion from many pursuers with integral constraints on the controls of players in Hilbert space. The problem statement is given , and the main result is shown by constructing a strategy for the evader to guarantee that evasion is possible in the game within the given time period. The chapter ends with a conclusion.

Chapter 5 studies a pursuit differential game of m pursuers and one evader described by the infinite system of second order differential equations. We begin by introducing the chapter and then move on to solving the case of the one evader and one pursuer. The following part looks at a game where there are multiple pursuers and one evader. The chapter ends by obtaining a condition in terms of the energies of players that is sufficient for the completion of pursuit in the game.

Chapter 6 provides a comprehensive summary of the main findings and contributions presented in this thesis. Furthermore, it presents recommendations for future research works.

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