APPROXIMATE SOLUTION FOR TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

By

ALSIDRANI FAHAD ABDULAZIZ A

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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This work is dedicated to all of my beloved
&
To my mother Norah
&
To my wife Nada and son Abdulaziz

You have enhanced my abilities and made me stronger and more fulfilled than I could have ever imagined.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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This thesis investigates one-dimensional time-dependent partial differential equations, focusing on two types of fractional derivative definitions and their properties. The primary goal is to derive semianalytical approximate series solutions for the spatial variable ν within a bounded interval $[\mathrm{a}, \mathrm{b}]$, where a and b are real numbers. Three powerful numerical methods are employed to obtain approximate analytical solutions for fractional order partial differential equations: the variational iteration method (VIM), the Adomian decomposition method (ADM), and the homotopy analysis method (HAM). These techniques balance the simplicity of analytical solutions with the accuracy of numerical approaches. The study includes a comprehensive convergence analysis of the approximate series solutions obtained from VIM, ADM, and HAM. The differential equation under investigation is derived from the traditional Fornberg-Whitham equation and the Helmholtz equation by replacing the integer order time derivative with noninteger derivatives of order μ in the range $n-1 < \mu \leq n$, for $n \in \mathbb{N}$, incorporating variable coefficients. Novel approaches are developed to compute the Laplace transform in the Atangana-Baleanu fractional derivative operator, enhancing the performance and accuracy of the semianalytical methods. The research extends to validate the effectiveness of fractional order methods. To demonstrate the applicability of these techniques, computational analyses of various test problems are provided, featuring two fractional derivatives and variable coefficients. Comparisons reveal that the absolute differences between the approximate solutions derived from VIM, ADM, and HAM decrease with the parameter μ approaches to the integer order. The findings indicate that the differences between ADM and HAM are consistently smaller than those involving VIM, signifying that while all methods yield similar results, ADM and HAM show closer alignment and potential excellence in specific scenarios. According to the results and graphical representation, it can be seen that the proposed methods are efficient in obtaining an analytical solution for time-fractional differential equations.

Keywords: Adomian Decomposition Method, Fractional Derivatives, Homotopy Analysis Method, Partial Differential Equations, Variational Iteration Method.

SDG: GOAL 4: Quality Education.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENYELESAIAN PENGHAMPIRAN UNTUK PERSAMAAN PEMBEZAAN SEPARA PECAHAN MASA DENGAN PEKALI PEMBOLEHUBAH

Oleh

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Tesis ini mengkaji persamaan pembezaan separa satu dimensi yang bergantung kepada masa, dengan fokus pada dua jenis definisi terbitan pecahan dan sifat-sifatnya. Matlamat utama adalah untuk memperoleh penyelesaian hampiran siri separa-analitik untuk pemboleh ubah ruang ν dalam selang terhad $[a, b]$, di mana a dan b adalah nombor nyata. Tiga kaedah berangka yang digunakan untuk mendapatkan penyelesaian hampiran analitik bagi persamaan pembezaan separa peringkat pecahan: kaedah lelaran bervariasi (KLB), kaedah penghuraian Adomian (KPA), dan kaedah analisis homotopi (KAH). Teknik-teknik ini mengimbangi kesederhanaan penyelesaian analitik dengan kejituan pendekatan berangka. Kajian ini merangkumi analisis penumpuan yang komprehensif terhadap penyelesaian hampiran siri yang diperoleh daripada KLB, KPA dan KAH. Persamaan pembezaan yang dikaji diperoleh daripada persamaan tradisional Fornberg-Whitham dan persamaan Helmholtz dengan menggantikan terbitan masa peringkat integer dengan terbitan bukan integer peringkat μ dalam julat $n - 1 < \mu \leq n$, untuk $n \in \mathbb{N}$, yang menggabungkan pekali berubah. Pendekatan baharu dibangunkan untuk mengira transformasi Laplace dalam operator terbitan pecahan Atangana-Baleanu, meningkatkan prestasi dan kejituan kaedah separa-analitik. Kajian ini dilanjutkan untuk mengesahkan keberkesanan kaedah peringkat pecahan. Untuk menunjukkan kebolehgunaan teknik-teknik ini, analisis pengiraan pelbagai masalah ujian disediakan, yang menampilkan dua terbitan pecahan dan pekali berubah. Perbandingan menunjukkan

bahawa perbezaan mutlak antara penyelesaian yang diperoleh daripada KLB, KPA dan KAH berkurangan dengan parameter μ. Penemuan menunjukkan bahawa perbezaan antara KPA dan KAH adalah lebih kecil secara konsisten berbanding dengan yang melibatkan KLB, mencadangkan bahawa walaupun semua kaedah menghasilkan hasil yang serupa, KPA dan KAH menunjukkan keselarasan yang lebih dekat dan potensi kecemerlangan dalam senario tertentu. Menurut hasil dan perwakilan grafik, dapat dilihat bahawa kaedah yang dicadangkan adalah berkesan dalam memperoleh penyelesaian analitik untuk persamaan pembezaan pecahan-masa.

Kata Kunci: Kaedah Analisis Homotopi, Kaedah Lelaran Bervariasi, Kaedah Penghuraian Adomian, Persamaan Pembezaan Separa, Terbitan Pecahan.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

TFPDE	Time Fractional Partial Differential Equation
MTFAD	Multi Term Fractional Advection-Diffusion
LADM	Laplace Adomian Decomposition Method
FVIM	Fractional Variational Iteration Method
OHAM	Optimal Homotopy Asymptotic Method
VITM	Variational Iteration Transform Method
FPDE	Fractional Partial Differential Equation
ABRL	Atangana Baleanu Riemann Liouville
LVIM	Laplace Variational Iteration Method
LHAM	Laplace Homotopy Analysis Method
ADM	Adomian Decomposition Method
HPT	Homotopy Perturbation Method
ODE	Ordinary Differential Equation
SDM	Shehu Decomposition Method
FDE	Fractional Diffusion Equation
VIM	Variational Iteration Method
PDE	Partial Differential Equation
GLT	Generalized Locally Toeplitz
HAM	Homotopy Analysis Method
DPSM	Direct Power Series Method
ABC	Atangana Baleanu Caputo
HWM	Hermite Wavelet Method
RBF	Radial Basis Function
FD	Fractional Derivative
WSC	Wide Sense Criterion
SSC	Strict Sense Criterion
FC	Fractional Calculus
AB	Atangana Baleanu
CF	Caputo Fabrizio

CHAPTER 1

INTRODUCTION

1.1 Fundamentals of Fractional Calculus

In the heart of all various applied sciences, everything shows itself in the mathematical relation that most of these problems and phenomena are modeled by ordinary differential equations (ODEs) or partial differential equations (PDEs). The field of fractional calculus (FC) is almost as old as conventional calculus itself. It attracts attention to modeling problems concerning nonlocality and memory effect concepts that are not well described by the classical calculus. It represents a conception of classical differentiation and integration of nonnegative integer order to an arbitrary order (constant or variable). In recent years, several books on fractional calculus have been published. Researchers have developed new numerical methods to solve fractional differential equations more efficiently. Applications of fractional calculus can be found in various fields, including physics, engineering, and biology. The growing interest in fractional calculus highlights its potential to address complex real-world problems (Podlubny, 1998; Kilbas et al., 2006; Sabatier et al., 2007; Chen et al., 2009; Baleanu et al., 2011; Sabatier et al., 2015; Guo et al., 2015; Daftardar-Gejji, 2019; Gorenflo et al., 2020; Mainardi, 2022).

1.1.1 Origin of the Fractional Derivative

The historical development of fractional calculus dates back to several centuries and involves contributions from multiple mathematicians. The concept of fractional derivatives (FD) has more than 325 years of history, yet it is still an exciting research topic, and interested researchers are actively working on problems of fractional order derivatives. The roots of this concept can be traced back to a letter from ĹHôpital to Leibnitz in 1695, in which the meaning of the derivative of a function of order 1/2. Later investigations and further developments were by other mathematicians, such as Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1823 – 1826, Liouville in 1832 – 1873, Riemann in 1847, Holmgren in 1865 – 1867, and Gruňwald 1867 – 1872. These early mathematicians made significant contributions to the field of fractional calculus in its early stages. They explored the possibility of extending differentiation and integration to noninteger orders (Podlubny, 1998; Kilbas et al., 2006; Guo et al., 2015; Daftardar-Gejji, 2019). However, the formalization of fractional calculus took several more years to develop. Since then, numerous scholars attempted to offer a clear definition of a fractional derivative. The principal approach employed by the majority of these researchers involved the utilization of an integral formulation for the representation of the fractional derivative. We briefly introduce the commonly used fractional derivative formulation as frequently discovered in scholarly literature. In 1819, the first mention of fractional derivative in a published paper by Lacroix (Ross, 2006). Starting with $\psi = \varsigma^{m}$, where m, $n \in \mathbb{N}$ and $m \geq n$, Lacroix found the nth derivative of ψ and he further obtained the generalized form with $\Gamma(\cdot)$ the well-known Gamma function by

$$\frac{d^{n}}{d\varsigma^{n}}\psi = \frac{m!}{(m-n)!}\varsigma^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\varsigma^{m-n}. \tag{1.1}$$

In particular, he computed the following derivative of fractional order, when $m = 1$ and $n = 1/2$, he obtained

$$\frac{d^{\frac{1}{2}}}{d\varsigma^{\frac{1}{2}}}\psi = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}\varsigma^{\frac{1}{2}} = \frac{2\sqrt{\varsigma}}{\sqrt{\pi}}. \tag{1.2}$$

It can be seen Equation (1.1) indicates that the fractional derivative of a constant denoted by ς^{0} deviates from the conventional expectation of yielding zero. A specific illustration of this notice arises when the parameters are assigned values such that $m = 0$ and $n = 1/2$, we obtain

$$\frac{d^{\frac{1}{2}}}{d\varsigma^{\frac{1}{2}}}\varsigma^{0} = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})}\varsigma^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi\varsigma}}. \tag{1.3}$$

In this particular formatting, the fractional derivative assumes a non-zero value. The next stage was taken from Fourier in 1822. He defined the fractional derivative through the so-called Fourier transform (FT), which generalized the integer-order derivative into the fractional order derivatives by

$$\mathcal{D}^{\mu}\psi(\varsigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(\xi)\nu^{\mu}\cos\left[\nu(\varsigma-\xi)+\frac{\mu\pi}{2}\right]d\nu d\xi. \tag{1.4}$$

Abel in 1823, discussed the fractional derivative as well.

The comprehensive development and formalization of fractional calculus occurred in the 19th and 20th centuries. Riemann and Liouville established the foundation of modern fractional calculus in the mid-19th century. Riemann introduced the concept of fractional integration, and Liouville further developed the theory by introducing the fractional derivative. Liouville made progress work in fractional derivatives and properly involved it in potential theory. If a function, denoted as $\psi(\varsigma)$, admits expansion into an infinite series, then its fractional derivative can be derived through the following

$$\mathcal{D}^{\mu}\psi(\varsigma) = \sum_{r=0}^{\infty} \mathcal{C}_r \mathcal{A}_r^{\mu} e^{\mathcal{A}_r \varsigma}, \tag{1.5}$$

and if the function $\psi(\varsigma)$ cannot be expanded into an infinite series, then its fractional derivative is obtained by using the Gamma function as follows:

$$\begin{aligned} \mathcal{D}^{\mu}\varsigma^{-\alpha} &= \frac{(-1)^{\mu}}{\Gamma(\alpha)} \int_0^{\infty} \xi^{\alpha+\mu-1} e^{-\varsigma\xi} d\xi, \\ &= \frac{(-1)^{\mu}\Gamma(\alpha+\mu)\varsigma^{-\alpha-\mu}}{\Gamma(\alpha)}, \quad \alpha > 0. \end{aligned} \tag{1.6}$$

In 1870, the Riemann-Liouville (RL) derivative of order μ for a given power function kernel was defined by

$$^{RL}\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma) = \frac{1}{\Gamma(1-\mu)} \frac{d}{d\varsigma} \int_a^{\varsigma} \psi(\xi)(\varsigma-\xi)^{-\mu} d\xi. \tag{1.7}$$

The singularity of the Riemann-Liouville fractional derivative occurs when the parameter ς is equal to ξ. In order to avoid this singularity, alternative definitions have been introduced. One such alternative is the Caputo fractional derivative, which was first introduced by (Caputo, 1967). This approach is often preferred in practical applications because of its enhanced capability to address initial conditions. The μ-th Caputo fractional derivative $^{C}\mathcal{D}_{\varsigma}^{\mu}$ of the function $\psi(\varsigma)$ is formally defined as follows:

$$^{C}\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma) = \frac{1}{\Gamma(n-\mu)} \int_a^{\varsigma} \psi^{(n)}(\xi)(\varsigma-\xi)^{n-\mu-1} d\xi. \tag{1.8}$$

Further, Hadamard in 1892 proposed a nonlocal fractional derivative with singular logarithmic function kernel with memory of order μ defined by (Jarad et al., 2012)

$$^{H}\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma) = \left(\varsigma\frac{d}{d\varsigma}\right)^{\alpha} \frac{1}{\Gamma(\alpha-\mu)} \int_a^{\varsigma} \psi(\xi) \log\left(\frac{\varsigma}{\xi}\right)^{\alpha-\mu-1} \xi^{-1} d\xi. \tag{1.9}$$

The proposition of a fractional derivative employing a logarithmic function kernel of order $n-1<\mu<n$, where $n\in\mathbb{N}$, has been reintroduced in the work by (Beghin et al., 2015). The formulation is defined as follows:

$$\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma)=\frac{1}{\Gamma(\alpha-\mu)}\int_{\frac{1-a}{b}}^{\varsigma}\psi(\xi)\log\Big(\frac{a+b\varsigma}{a+b\xi}\Big)^{\alpha-\mu-1}\Big(\Big(\frac{a}{b}+\xi\Big)\frac{d}{d\xi}\Big)^{\alpha}\frac{b}{a+b\xi}d\xi. \quad (1.10)$$

Remark 1.1 : If we set $a=0$ and $b=1$ in Equation (1.10), it is also known as the Hadamard fractional derivative.

Furthermore, (Caputo and Fabrizio, 2015) proposed a fractional derivative with an exponential function kernel. By changing the kernel $(\varsigma-\xi)^{-\mu}$, where $0<\mu<1$ with the function $\exp(-\mu(\varsigma-\xi)/(1-\mu))$ and $1/\Gamma(1-\mu)$ with $\mathcal{P}(\mu)/(1-\mu)$, we obtain the following new definition of fractional time derivative defined by

$${}^{CF}\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma)=\frac{\mathcal{P}(\mu)}{1-\mu}\int_{a}^{\varsigma}\psi'(\xi)\exp\Big(\frac{\mu(\varsigma-\xi)}{\mu-1}\Big)d\xi, \quad (1.11)$$

where $\mathcal{P}(\mu)$ denotes a normalization function such that $\mathcal{P}(0)=\mathcal{P}(1)=1$.

The main recent advancement within the domain of fractional calculus occurred in the year 2016 when Atangana and Baleanu introduced a novel definition for the fractional derivative characterized by a nonlocal and nonsingular kernel by the Mittag-Leffler (ML) function. This formulation, referred to as the Atangana-Baleanu derivative, was accompanied by the explication of relevant properties. Besides, the Atangana-Baleanu derivative was effectively employed in the solution of a fractional heat transfer model. The Atangana-Baleanu fractional derivative operator in both the Caputo and Riemann-Liouville senses are defined for the orders within the range of $0<\mu\leq 1$, where $\psi(\varsigma)\in L^1(a,b)$ with $a<b$. The different formulations in the Caputo and Riemann-Liouville senses are denoted as the Atangana-Baleanu fractional derivative operator in the Caputo sense and the Atangana-Baleanu fractional derivative operator in the Riemann-Liouville sense, respectively, by (Atangana and Baleanu, 2016) as follows:

$${}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma)=\frac{\mathcal{P}(\mu)}{1-\mu}\int_{a}^{\varsigma}\psi'(\xi)\,E_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi \quad (1.12)$$

and

$$^{ABR}\mathcal{D}^{\mu}_{\varsigma}\psi(\varsigma) = \frac{\mathcal{P}(\mu)}{1-\mu}\frac{d}{d\varsigma}\int_{a}^{\varsigma}\psi(\xi)\,E_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi, \tag{1.13}$$

where $E_{\mu}(\varsigma)$ represents the one-parameter Mittag-Leffler function.

1.1.2 The Fractional Taylor's Formula

The Taylor expansion for fractional derivatives can be expressed through the framework of fractional calculus. This approach extends the classical Taylor series by incorporating the fractional order derivatives, allowing for the representation of functions with noninteger order differentiation. This generalization not only broadens the applicability of the Taylor series but also provides a powerful tool for modeling complex dynamical systems where traditional integer order calculus may fall short. The resulting fractional Taylor series captures the complex behavior of such systems, offering deeper insights and enhanced analytical capabilities in fields like physics, engineering, and applied mathematics. The general form of the Taylor expansion of a function $\psi(\xi)$ with its center at a specific point a is formally established as follows:

$$\psi(\xi) = \psi(a) + \frac{\psi'(a)(\xi-a)}{1!} + \frac{\psi''(a)(\xi-a)^2}{2!} + \frac{\psi'''(a)(\xi-a)^3}{3!} + \cdots. \tag{1.14}$$

The fractional Taylor series has been established within the framework of the Riemann-Liouville derivative as elaborated by (Trujillo et al., 1999) under specific conditions on the function $\psi(\xi)$. This development has led to the formulation known as the generalized Taylor's formula. Trujillo et al. provided a comprehensive treatment, which includes the derivation of a generalized mean value theorem (GMVT) and exploration of various applications arising from the generalized Taylor's formula. Similarly, an analogous investigation has been undertaken for the Caputo fractional derivative as defined by (Odibat and Shawagfeh, 2007). For a positive real number μ, $a \in \Theta$, and $E \subset \Theta$ is an interval such that for all $\xi \in E$ and $a \leq \xi$, then

$${}_{a}I_{\mu}(E) = \{\psi \in \Psi(\Theta) : {}_{a}I^{\mu}\psi(\xi) \quad \text{exists and it is finite}\}, \tag{1.15}$$

where $\Psi(\Theta)$ stands for the set of real functions of a single real variable with domain in Θ. Assume that μ and q such that $0 < \mu \leq 1$ and $q \in \mathbb{N}$. Let ψ denotes a continuous function defined in the interval $(a, b]$, subject to the following conditions

1. ${}_a\mathcal{D}^{p\mu}\psi \in C((a, b])$ and ${}_a\mathcal{D}^{p\mu}\psi \in {}_aI_\mu([a, b])$, $\forall\ p = 1, \cdots, q$.

2. ${}_a\mathcal{D}^{(q+1)\mu}\psi$ is continuous on $[a, b]$.

3. If $\mu < 1/2$ then, $\forall\ p \in \mathbb{N}$, $1 \le p \le q$, such that $(p+1)\mu < 1$, ${}_a\mathcal{D}^{(p+1)\mu}\psi(\xi)$ is α-continuous in $\xi = a$ for some α, $1-(p+1)\mu \le \alpha \le 1$, or a-singular of order μ.

Then $\forall\ \xi \in (a, b]$, the generalized Riemann-Liouville Taylor's formula is defined by

$$\begin{aligned} \psi(\xi) &= \sum_{p=0}^{q} \frac{\mathcal{C}_p(\xi-a)^{(p+1)\mu-1}}{\Gamma((p+1)\mu)} + \frac{{}_a\mathcal{D}^{(q+1)\mu}\psi(\nu)}{\Gamma((q+1)\mu+1)}(\xi-a)^{(q+1)\mu}, \quad a \le \nu \le \xi \\ \mathcal{C}_p &= \Gamma(\mu)\Big[(\xi-a)^{1-\mu}\ {}_a\mathcal{D}^{p\mu}\psi(\xi)\Big](a^+), \quad \forall\ p = 0, 1, \cdots, q. \end{aligned} \tag{1.16}$$

The representation of the successive fractional derivative is indicated by

$${}_a\mathcal{D}^{q\mu} = \underbrace{{}_a\mathcal{D}^{\mu}{}_a\mathcal{D}^{\mu}\cdots{}_a\mathcal{D}^{\mu}}_{q\ \text{times}} \tag{1.17}$$

In case the parameter $\mu = 1$, both the generalized Riemann-Liouville Taylor's formula and the Caputo Taylor's formula reduce to the classical Taylor's formula.

1.1.3 The Leibniz Rule for Fractional Derivatives

The Leibniz rule, also known as the product rule for fractional derivatives, is one of the numerous attractive properties of fractional calculus. The Leibniz rule provides a manner to compute the fractional derivative of a product of two functions, extending the ordinary product rule from the classical calculus.

Theorem 1.1 : *For* $n \in \mathbb{N}$ *time differentiable functions* $\psi(\xi)$ *and* $\phi(\xi)$ *on* $[a, b]$ *then the derivative of their product is given by*

$$(\psi(\xi)\phi(\xi))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \psi^{(k)}(\xi)\phi^{(n-k)}(\xi). \tag{1.18}$$

The binomial coefficient can indeed be expressed in terms of the Gamma function

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}. \tag{1.19}$$

For $n = 1$, we obtain $(\psi(\xi)\phi(\xi))' = \psi(\xi)\phi'(\xi) + \psi'(\xi)\phi(\xi)$.

Theorem 1.2 : *(Min, 2016) Given an analytic function* $\psi(\xi)$ *defined on the interval* $[a, b]$*, it is observed that the Riemann-Liouville derivative operator satisfies the following*

$${}_{a}^{RL}\mathcal{D}_{\xi}^{\mu}\psi(\xi) = \sum_{m=0}^{\infty}\binom{\mu}{m}\frac{\psi(\xi)^{(m)}}{\Gamma(m-\mu+1)}(\xi-a)^{m-\mu}, \quad \mu \in \mathbb{R}. \tag{1.20}$$

Theorem 1.3 : *(Williams, 2007) Assume that* $\psi(\xi)$ *and* $\phi(\xi)$ *are analytic functions on* $[a, b] \subset \mathbb{R}$ *with all their derivatives are continuous. Then the Leibniz rule holds for* $\mu \in \mathbb{R}$.

$${}_{a}^{RL}\mathcal{D}_{\xi}^{\mu}(\psi(\xi)\phi(\xi)) = \sum_{k=0}^{\infty}\binom{\mu}{k}\Big[{}^{RL}\mathcal{D}^{k}\psi(\xi)\Big]\Big[{}^{RL}\mathcal{D}^{\mu-k}\phi(\xi)\Big]. \tag{1.21}$$

Proof:

Since the functions $\psi(\xi)$ and $\phi(\xi)$ analytic on $[a, b]$, then $\psi\phi$ is also analytic on $[a, b]$.

$$\begin{aligned}
{}_{a}^{RL}\mathcal{D}_{\xi}^{\mu}(\psi(\xi)\phi(\xi)) &= \sum_{m=0}^{\infty}\binom{\mu}{m}\frac{(\psi(\xi)\phi(\xi))^{(m)}}{\Gamma(m-\mu+1)}(\xi-a)^{m-\mu} \\
&= \sum_{m=0}^{\infty}\binom{\mu}{m}\frac{(\xi-a)^{m-\mu}}{\Gamma(m-\mu+1)}\sum_{k=0}^{m}\binom{m}{k}\psi^{(k)}(\xi)\phi^{(m-k)}(\xi) \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{\mu}{m}\binom{m}{k}\frac{\psi^{(k)}(\xi)\phi^{(m-k)}(\xi)(\xi-a)^{m-\mu}}{\Gamma(m-\mu+1)} \\
&= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\binom{\mu}{k}\binom{\mu-k}{m-k}\frac{\psi^{(k)}(\xi)\phi^{(m-k)}(\xi)(\xi-a)^{m-\mu}}{\Gamma(m-\mu+1)} \\
&= \Big[\sum_{k=0}^{\infty}\binom{\mu}{k}\psi^{(k)}(\xi)\Big]\Big[\sum_{m=k}^{\infty}\binom{\mu-k}{m-k}\frac{\phi^{(m-k)}(\xi)(\xi-a)^{(m-k)-(\mu-k)}}{\Gamma((m-k)-(\mu-k)+1)}\Big] \\
&= \sum_{k=0}^{\infty}\binom{\mu}{k}\Big[{}^{RL}\mathcal{D}^{k}\psi(\xi)\Big]\Big[{}^{RL}\mathcal{D}^{\mu-k}\phi(\xi)\Big].
\end{aligned}$$

The fractional formula of the Leibniz rule for differentiation Caputo type derivative was given by (Baleanu and Trujillo, 2010) as follows:

$${}_{a}^{C}\mathcal{D}_{\xi}^{\mu}(\psi(\xi)\phi(\xi)) = \sum_{k=0}^{\infty}\binom{\mu}{k}\Big[{}^{RL}\mathcal{D}^{k}\psi(\xi)\Big]\Big[{}^{RL}\mathcal{D}^{\mu-k}\phi(\xi)\Big] - \frac{\psi(a)\phi(a)(\xi-a)^{\mu}}{\Gamma(1-\mu)}. \tag{1.22}$$

1.2 Special Functions of Fractional Calculus

The specialized mathematical functions in the field of mathematical physics have occurred to address the demands of applied sciences, offering solutions to integer-order differential equations derived from mathematical physics models. This section provides fundamental theories related to special functions, serving as a foundational framework for subsequent chapters. Specifically, it offers essential insights into the primary functions integral to the theory of arbitrary-order differentiation and the theory of fractional differential equations. In particular, it contains comprehensive information on critical functions such as the Gamma and Beta functions, the Mittag-Leffler functions, and the Wright functions, which readers may refer to (Podlubny, 1998; Gorenflo et al., 2007; Mainardi et al., 2010; Guo et al., 2015; Gorenflo et al., 2020; Mainardi, 2022).

1.2.1 Gamma Function

The Gamma function $\Gamma(\cdot)$ plays a crucial role in various areas of mathematics, including complex analysis, number theory, and probability theory. Let u be a complex number with $\mathrm{Re}(u) > 0$, then the integral

$$\Gamma(u) = \int_0^\infty \eta^{u-1} e^{-\eta} d\eta, \tag{1.23}$$

is known as the Gamma function or the Euler integral of the second kind and converges absolutely. It is defined for all complex numbers except for the non-positive integers. The presented function contains a generalization of the factorial, which can be expressed in the form $\Gamma(u) = (u-1)!$. It is obvious that through the application of integration by parts, the fundamental property of the Gamma function can be derived as follows:

$$\Gamma(u+1) = \int_0^\infty \eta^{u+1-1} e^{-\eta} d\eta = (-e^{-\eta}\eta^u)\Big|_{\eta=0}^{\eta=\infty} + u\int_0^\infty \eta^{u-1} e^{-\eta} d\eta = u\Gamma(u). \tag{1.24}$$

Proposition 1.1 : *Some important properties of the Gamma function include*

1. $\Gamma(u+1) = u\Gamma(u)$.

2. $\Gamma(u+1) = u!, \quad \forall u \in \mathbb{N}$.

3. $\Gamma(1/2) = \sqrt{\pi}$.

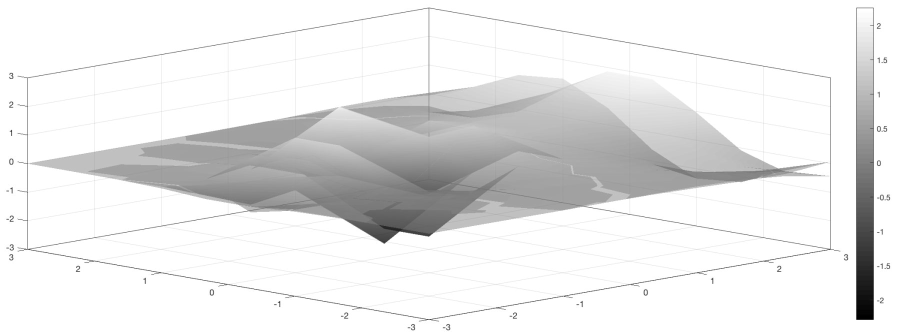

Figure 1.1: Complex plot of Gamma function

1.2.2 Beta Function

The Beta function is a special function defined as follows:

$$\beta(u,v) = \int_0^1 \eta^{u-1}(1-\eta)^{v-1} d\eta, \tag{1.25}$$

where Re(u) > 0 and Re(v) > 0 which is also called the Euler integral of the first kind.

Corollary 1.1 : *The Beta function can be written in the form of Gamma function as follows:*

$$\beta(u+v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{1.26}$$

1.2.3 Mittag-Leffler Function

The Mittag-Leffler (ML) function relates to the Gamma function, which plays a crucial role in fractional calculus and the behavior of special functions. The Mittag-Leffler function with two-parameters is a special function allowing a wider range of applications, which is defined as follows:

$$\mathrm{E}_{\mathrm{u,v}}(\eta) = \sum_{\mathrm{m}=1}^{\infty} \frac{\eta^{\mathrm{m}}}{\Gamma(\mathrm{mu}+\mathrm{v})}, \quad \mathrm{Re}(\mathrm{u}) > 0. \tag{1.27}$$

For $\mathrm{v} = 1$, we obtain the so-called one-parameter Mittag-Leffler function

$$\mathrm{E}_{\mathrm{u},1}(\eta) = \sum_{\mathrm{m}=0}^{\infty} \frac{\eta^{\mathrm{m}}}{\Gamma(\mathrm{mu}+1)} \equiv \mathrm{E}_{\mathrm{u}}(\eta). \tag{1.28}$$

Corollary 1.2 : *The Mittag-Leffler function arises as a generalization of the exponential function in fractional calculus. According to (Petrás, 2011), the most well-known relationships for the Mittag-Leffler function can be summarized as follows:*

1. $\mathrm{E}_{1,1}(\eta) = \mathrm{e}^{\eta}$.
2. $\mathrm{E}_{1,2}(\eta) = \dfrac{\mathrm{e}^{\eta}-1}{\eta}$.
3. $\mathrm{E}_{2,1}(\eta) = \cosh(\sqrt{\eta})$.
4. $\mathrm{E}_{2,1}(-\eta^2) = \cos(\eta)$.

1.2.4 Wright Function

The entire function of η denoted as $\mathrm{W}_{\mathrm{u,v}}(\eta)$ is a special function that generalizes various elementary functions which are defined as follows:

$$\mathrm{W}_{\mathrm{u,v}}(\eta) = \sum_{\mathrm{m}=0}^{\infty} \frac{\eta^{\mathrm{m}}}{\Gamma(\mathrm{m}+1)\Gamma(\mathrm{mu}+\mathrm{v})}, \quad \mathrm{u} > -1 \quad \text{and} \quad \mathrm{v} \in \mathbb{C}. \tag{1.29}$$

The function denoted as $\mathrm{W}_{\mathrm{u,v}}(\eta)$ exhibits convergence across the whole η-complex plane, and it is identified as the Wright function, named in honor of the British mathematician E. M. Wright, who introduced it in the 1940s (Gorenflo et al., 2007; Mainardi et al., 2010). This function has gained standing in recent scholarly works focusing on partial differential equations of fractional order. The integral representation of the Wright function is expressed as follows:

$$\mathrm{W}_{\mathrm{u,v}}(\eta) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Ha}} \xi^{\mathrm{v}} \mathrm{e}^{\xi+\eta\xi^{-\mathrm{u}}} \mathrm{d}\xi, \quad \mathrm{u} > -1 \quad \text{and} \quad \mathrm{v} \in \mathbb{C}, \tag{1.30}$$

where Ha denotes the Hankel path. The Wright function has applications in diverse areas of mathematics, including differential equations, mathematical physics, and probability theory. It often arises in problems involving fractional calculus, delay differential equations (DDE), and solutions to certain types of differential equations. It can be used as a powerful tool for expressing solutions to differential equations that cannot be easily expressed in terms of elementary functions.

Remark 1.2 : In the case $u = 0$, the Wright function Equation (1.29) is reduced to the exponential function with constant factor $1/\Gamma(v)$

$$\begin{aligned} W_{0,v}(\eta) &= \sum_{m=0}^{\infty} \frac{\eta^m}{\Gamma(m+1)\Gamma(v)} \\ &= \frac{e^\eta}{\Gamma(v)}. \end{aligned} \tag{1.31}$$

If $u = 0$ and $v = 1$, we have

$$\begin{aligned} W_{0,1}(\eta) &= E_{1,1}(\eta) \\ &= e^\eta. \end{aligned} \tag{1.32}$$

1.3 Integral Transforms

Integral transforms have proven to be powerful tools in addressing fractional partial differential equations. In this exploration, we focus on employing a range of transforms, such as the Laplace transform, the Mellin transform, the Sumudu transform, the Elzaki transform, or the Shehu transform, to convert partial differential equations into ordinary differential equations. The integral transformation simplifies the solution process, making it possible to apply well-established techniques. The modified numerical methods depend on combining a numerical approach with an appropriate transformation operator, making it easier to achieve analytical or numerical solutions for fractional partial differential equations that involve fractional order derivatives. For more details regarding the integrals transform, encompassing their properties for specific instances, we direct the interested reader to (Spiegel, 1965; Watugala, 1993; Sheng et al., 2011; Elzaki, 2011; Luchko and Kiryakova, 2013; Maitama and Zhao, 2019; Magar et al., 2022). In the following, we provide the most employed integral transforms in fractional calculus.

Defined a set of function

$$\mathbb{A} = \left\{ \psi(\varsigma) : \exists\, \mathfrak{m}, \mathfrak{y}_1, \mathfrak{y}_2 > 0,\ |\psi(\varsigma)| \leq \mathfrak{m} e^{\frac{|\varsigma|}{\mathfrak{y}_k}}, \quad \text{if} \quad \varsigma \in (-1)^k \times [0, \infty) \right\}. \tag{1.33}$$

In what follows, we assume that $\psi \in L^1(a, b)$ with $b > a$, $\varsigma > 0$, and $\psi(\varsigma) \in \mathbb{A}$.

1. The Laplace integral transform of a function $\psi(\varsigma)$ is given as follows:

$$\mathbb{L}_\varsigma[\psi(\varsigma); \mathfrak{y}] = \mathcal{F}(\mathfrak{y}) = \int_{-\infty}^{\infty} \psi(\varsigma) e^{-\mathfrak{y}\varsigma} d\varsigma, \tag{1.34}$$

   and the inverse Laplace transform of $\mathcal{F}(\mathfrak{y})$ is defined by the following complex integral

$$\psi(\varsigma) = \mathbb{L}_\varsigma^{-1}[\mathcal{F}(\mathfrak{y}); \varsigma] = \frac{1}{2\pi i} \lim_{\epsilon \to \infty} \int_{\gamma - i\epsilon}^{\gamma + i\epsilon} \mathcal{F}(\mathfrak{y}) e^{\mathfrak{y}\varsigma} d\mathfrak{y}, \tag{1.35}$$

   where $\text{Re}(\mathfrak{y}) = \gamma$ is the vertical line in the complex plane such that γ is greater than the real part of all singularities of $\mathcal{F}(\mathfrak{y})$.

2. If $\mathfrak{y} > 0$ is any complex variate and $\psi(\varsigma)$ is a function of a real variable ς, such that

$$\mathbb{M}_\varsigma[\psi(\varsigma); \mathfrak{y}] = \mathcal{M}(\mathfrak{y}) = \int_0^{\infty} \psi(\varsigma) \varsigma^{\mathfrak{y}-1} d\varsigma, \tag{1.36}$$

   exists, then the function $\mathcal{M}(\mathfrak{y})$ is called the Mellin integral transform of $\psi(\varsigma)$. Under certain conditions and for $\text{Re}(\mathfrak{y}) = \gamma$, the inverse of Mellin integral transform is defined as follows:

$$\psi(\varsigma) = \mathbb{M}_\varsigma^{-1}[\mathcal{M}(\mathfrak{y}); \varsigma] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{M}(\mathfrak{y}) \varsigma^{-\mathfrak{y}} d\mathfrak{y}. \tag{1.37}$$

3. The Sumudu integral transform of a function $\psi(\varsigma)$ is defined as follows:

$$\mathbb{S}_\varsigma[\psi(\varsigma); \mathfrak{y}] = \mathcal{S}(\mathfrak{y}) = \frac{1}{\mathfrak{y}} \int_0^{\infty} \psi(\varsigma) e^{-\frac{\varsigma}{\mathfrak{y}}} d\varsigma, \tag{1.38}$$

   and for $\text{Re}(\mathfrak{y}) = \gamma$, the inverse of the Sumudu integral transform is defined as follows:

$$\psi(\varsigma) = \mathbb{S}_\varsigma^{-1}[\mathcal{S}(\mathfrak{y}); \varsigma] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{S}\left(\frac{1}{\mathfrak{y}}\right) e^{\mathfrak{y}\varsigma} \frac{d\mathfrak{y}}{\mathfrak{y}}. \tag{1.39}$$

4. The Elzaki integral transform of a function $\psi(\varsigma)$ is closely related to the Laplace transform and Sumudu transform, which is defined by

$$\mathbb{E}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{V}(\mathfrak{y}) = \mathfrak{y}\int_0^{\infty}\psi(\varsigma)e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma. \tag{1.40}$$

5. The Shehu integral transform of a function $\psi(\varsigma)$ of exponential order is defined as follows:

$$\mathbb{SH}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{U}(w,\mathfrak{y}) = \int_0^{\infty}\psi(\varsigma)e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma, \quad w,\mathfrak{y} > 0. \tag{1.41}$$

Lemma 1.1 : If a function $\psi(\varsigma)$ is piecewise continuous on every finite interval in $[0,\infty)$ and $|\psi(\varsigma)| \leq Ke^{a\varsigma}$ for all $\varsigma \in [0,\infty)$, then $\mathbb{L}_{\varsigma}[\psi(\varsigma);\mathfrak{y}]$ exists for all $\mathfrak{y} > a$.

1.3.1 Some Properties of Integral Transforms

The integral transforms have many important properties. In this section, we turn our attention to proving some of them, and here are some key properties of integral transforms.

1.3.1.1 Linear Property

Definition 1.1 : *An operator $\mathbb{O}$ defined on a vector space $\mathbb{S}$ over a field $\mathbb{F}$ is said to be a linear if for any $\psi(\varsigma)$ and $\phi(\varsigma)$ in $\mathbb{F}$ and for any scalar α and γ in $\mathbb{F}$, the following holds*

$$\mathbb{O}(\alpha\psi + \gamma\phi)(\varsigma) = \alpha\mathbb{O}\psi(\varsigma) + \gamma\mathbb{O}\phi(\varsigma). \tag{1.42}$$

1.3.1.2 Convolution Property

Definition 1.2 : *The convolution of two functions* $\psi(\varsigma)$ *and* $\phi(\varsigma)$ *is defined as follows:*

$$(\psi * \phi)(\varsigma) = \int_0^{\varsigma} \psi(\xi)\phi(\varsigma - \xi)\mathrm{d}\xi = \int_0^{\varsigma} \psi(\varsigma - \xi)\phi(\xi)\mathrm{d}\xi. \tag{1.43}$$

1.3.1.3 Derivative Property

The transform of a derivative of a function is related to the transform of the original function. More additional details will be presented in the next section 1.4.

Proposition 1.2 : *Let* $\psi(\varsigma)$ *and* $\phi(\varsigma)$ *be two functions of* $\varsigma \geq 0$*, where* $\mathcal{F}(\mathfrak{y})$ *denotes the Laplace transform of* $\psi(\varsigma)$ *and* $\mathcal{G}(\mathfrak{y})$ *denotes the Laplace transform of* $\phi(\varsigma)$*, then*

1. *for* $\alpha, \gamma \in \mathbb{R}$*, the Laplace transform with respect to* ς *of sum functions* $\psi(\varsigma)$ *and* $\phi(\varsigma)$ *is given as follows:*

$$\begin{aligned} \mathbb{L}_{\varsigma}[(\alpha\psi(\varsigma) + \gamma\phi(\varsigma)); \mathfrak{y}] &= \alpha\mathbb{L}_{\varsigma}[\psi(\varsigma); \mathfrak{y}] + \gamma\mathbb{L}_{\varsigma}[\phi(\varsigma); \mathfrak{y}] \\ &= \alpha\mathcal{F}(\mathfrak{y}) + \gamma\mathcal{G}(\mathfrak{y}). \end{aligned} \tag{1.44}$$

2. *The convolution property of the Laplace transform is given as follows:*

$$\begin{aligned} \mathbb{L}_{\varsigma}[(\psi(\varsigma) * \phi(\varsigma)); \mathfrak{y}] &= \mathbb{L}_{\varsigma}\Big[\int_0^{\varsigma} \psi(\xi)\phi(\varsigma - \xi)\mathrm{d}\xi\Big] \\ &= \mathbb{L}_{\varsigma}[\psi(\varsigma); \mathfrak{y}]\mathbb{L}_{\varsigma}[\phi(\varsigma); \mathfrak{y}] \\ &= \mathcal{F}(\mathfrak{y})\mathcal{G}(\mathfrak{y}). \end{aligned} \tag{1.45}$$

Proof:

From the convolution integral and Equation (1.34), it follows that

$$
\begin{aligned}
\mathbb{L}_\varsigma[(\psi*\phi)(\varsigma);\mathfrak{y}] &= \mathbb{L}_\varsigma\Big[\int_0^\infty \psi(\xi)\phi(\varsigma-\xi)\mathrm{d}\xi\Big] \\
&= \int_0^\infty \Big[\int_0^\infty \psi(\xi)\phi(\varsigma-\xi)\mathrm{d}\xi\Big]\mathrm{e}^{-\mathfrak{y}\varsigma}\mathrm{d}\varsigma \\
&= \int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(\varsigma-\xi)\mathrm{e}^{-\mathfrak{y}\varsigma}\mathrm{d}\varsigma\Big]\mathrm{d}\xi.
\end{aligned}
$$

By setting $\mathrm{u}=\varsigma-\xi \implies \mathrm{du}=\mathrm{d}\varsigma$, it implies that

$$
\begin{aligned}
\mathbb{L}_\varsigma[(\psi*\phi)(\varsigma);\mathfrak{y}] &= \int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(\mathrm{u})\mathrm{e}^{-\mathfrak{y}(\mathrm{u}+\xi)}\mathrm{du}\Big]\mathrm{d}\xi \\
&= \int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(\mathrm{u})\mathrm{e}^{-\mathfrak{y}\mathrm{u}}\mathrm{e}^{-\mathfrak{y}\xi}\mathrm{du}\Big]\mathrm{d}\xi \\
&= \int_0^\infty \psi(\xi)\mathrm{e}^{-\mathfrak{y}\xi}\Big[\int_0^\infty \phi(\mathrm{u})\mathrm{e}^{-\mathfrak{y}\mathrm{u}}\mathrm{du}\Big]\mathrm{d}\xi \\
&= \int_0^\infty \psi(\xi)\mathrm{e}^{-\mathfrak{y}\xi}\mathrm{d}\xi\mathcal{G}(\mathfrak{y}) \\
&= \mathcal{F}(\mathfrak{y})\mathcal{G}(\mathfrak{y}).
\end{aligned}
$$

Proposition 1.3 : *The Elzaki transform meets the following properties.*

1. *Linearity property. For* $\alpha,\gamma\in\mathbb{R}$, *then*

$$
\mathbb{E}_\varsigma[(\alpha\psi(\varsigma)+\gamma\phi(\varsigma));\mathfrak{y}] = \alpha\mathcal{V}(\mathfrak{y})+\gamma\mathcal{K}(\mathfrak{y}). \tag{1.46}
$$

2. *Convolution property.*

$$
\begin{aligned}
\mathbb{E}_\varsigma[(\psi*\phi)(\varsigma));\mathfrak{y}] &= \mathbb{E}_\varsigma\Big[\int_0^\varsigma \psi(\xi)\phi(\varsigma-\xi)\mathrm{d}\xi\Big] \\
&= \frac{\mathcal{V}(\mathfrak{y})\mathcal{K}(\mathfrak{y})}{\mathfrak{y}}.
\end{aligned} \tag{1.47}
$$

Proof:

From the convolution integral and Equation (1.40), it follows that

$$
\begin{aligned}
\mathbb{E}_\varsigma[(\psi*\phi)(\varsigma);\mathfrak{y}] &= \mathbb{E}_\varsigma\Big[\int_0^\infty \psi(\xi)\phi(\varsigma-\xi)\mathrm{d}\xi\Big] \\
&= \mathfrak{y}\int_0^\infty \Big[\int_0^\infty \psi(\xi)\phi(\varsigma-\xi)\mathrm{d}\xi\Big]\mathrm{e}^{-\frac{\varsigma}{\mathfrak{y}}}\mathrm{d}\varsigma \\
&= \mathfrak{y}\int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(\varsigma-\xi)\mathrm{e}^{-\frac{\varsigma}{\mathfrak{y}}}\mathrm{d}\varsigma\Big]\mathrm{d}\xi.
\end{aligned}
$$

By setting $\mathrm{u}=\varsigma-\xi \implies \mathrm{du}=\mathrm{d}\varsigma$, it implies that

$$\begin{aligned}\mathbb{E}_\varsigma[(\psi*\phi)(\varsigma);\mathfrak{y}] &= \mathfrak{y}\int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(u)e^{-\frac{(u+\xi)}{\mathfrak{y}}}du\Big]d\xi\\ &= \mathfrak{y}\int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(u)e^{-\frac{u}{\mathfrak{y}}}e^{-\frac{\xi}{\mathfrak{y}}}du\Big]d\xi\\ &= \mathfrak{y}\int_0^\infty \psi(\xi)e^{-\frac{\xi}{\mathfrak{y}}}\Big[\int_0^\infty \phi(u)e^{-\frac{u}{\mathfrak{y}}}du\Big]d\xi\\ &= \mathfrak{y}\int_0^\infty \psi(\xi)e^{-\frac{\xi}{\mathfrak{y}}}d\xi\frac{\mathcal{K}(\mathfrak{y})}{\mathfrak{y}}\\ &= \frac{\mathcal{V}(\mathfrak{y})\mathcal{K}(\mathfrak{y})}{\mathfrak{y}},\end{aligned}$$

where $\mathcal{V}(\mathfrak{y}) = \mathbb{E}_\varsigma[\psi(\varsigma);\mathfrak{y}]$ and $\mathcal{K}(\mathfrak{y}) = \mathbb{E}_\varsigma[\phi(\varsigma);\mathfrak{y}]$ are the Elzaki transform of the functions $\psi(\varsigma)$ and $\phi(\varsigma)$, respectively.

Proposition 1.4 : *Let $\gamma\psi(\varsigma)$ and $\alpha\phi(\varsigma)$ be in a set $\mathbb{A}$, then $\gamma\psi(\varsigma)+\alpha\phi(\varsigma)\in\mathbb{A}$, where $\gamma,\alpha\in\mathbb{R}\setminus\{0\}$, then we have the following*

1. *The Shehu transform is a linear operator.*

$$\mathbb{SH}_\varsigma[(\gamma\psi(\varsigma)+\alpha\phi(\varsigma));\mathfrak{y}] = \gamma\mathbb{SH}_\varsigma[\psi(\varsigma);\mathfrak{y}] + \alpha\mathbb{SH}_\varsigma[\phi(\varsigma);\mathfrak{y}], \tag{1.48}$$

where $\mathbb{SH}_\varsigma[\cdot]$ is the Shehu transformation.

Proof:

From Equation (1.41), it follows that

$$\begin{aligned}\mathbb{SH}_\varsigma[(\gamma\psi(\varsigma)+\alpha\phi(\varsigma));\mathfrak{y}] &= \int_0^\infty (\gamma\psi(\varsigma)+\alpha\phi(\varsigma))e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma\\ &= \int_0^\infty e^{-\frac{\mathfrak{y}\varsigma}{w}}\gamma\psi(\varsigma)d\varsigma + \int_0^\infty \alpha\phi(\varsigma)e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma\\ &= \gamma\int_0^\infty \psi(\varsigma)e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma + \alpha\int_0^\infty \phi(\varsigma)e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma\\ &= \gamma\mathbb{SH}_\varsigma[\psi(\varsigma);\mathfrak{y}] + \alpha\mathbb{SH}_\varsigma[\phi(\varsigma);\mathfrak{y}].\end{aligned}$$

2. *The Shehu transform meets the following convolution equality.*

$$\mathbb{SH}_\varsigma[(\psi(\varsigma)*\phi(\varsigma));\mathfrak{y}] = \mathcal{U}(w,\mathfrak{y})\mathcal{V}(w,\mathfrak{y}). \tag{1.49}$$

Proof:

From the convolution integral and Equation (1.41), it follows that

$$\begin{aligned}
\mathbb{SH}_\varsigma[(\psi * \phi)(\varsigma);\mathfrak{y}] &= \mathbb{SH}_\varsigma\Big[\int_0^\infty \psi(\xi)\phi(\varsigma-\xi)d\xi\Big] \\
&= \int_0^\infty \Big[\int_0^\infty \psi(\xi)\phi(\varsigma-\xi)d\xi\Big]e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma \\
&= \int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(\varsigma-\xi)e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma\Big]d\xi.
\end{aligned}$$

By setting $u = \varsigma - \xi \implies du = d\varsigma$, it implies that

$$\begin{aligned}
\mathbb{SH}_\varsigma[(\psi * \phi)(\varsigma);\mathfrak{y}] &= \int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(u)e^{-\frac{\mathfrak{y}(u+\xi)}{w}}du\Big]d\xi \\
&= \int_0^\infty \psi(\xi)\Big[\int_0^\infty \phi(u)e^{-\frac{\mathfrak{y}u}{w}}e^{-\frac{\mathfrak{y}\xi}{w}}du\Big]d\xi \\
&= \int_0^\infty \psi(\xi)e^{-\frac{\mathfrak{y}\xi}{w}}\Big[\int_0^\infty \phi(u)e^{-\frac{\mathfrak{y}u}{w}}du\Big]d\xi \\
&= \int_0^\infty \psi(\xi)e^{-\frac{\mathfrak{y}\xi}{w}}d\xi\mathcal{V}(w,\mathfrak{y}) \\
&= \mathcal{U}(w,\mathfrak{y})\mathcal{V}(w,\mathfrak{y}),
\end{aligned}$$

where $\mathcal{U}(w,\mathfrak{y})$, $\mathcal{V}(w,\mathfrak{y})$ are Shehu transforms of $\psi(\varsigma)$ and $\phi(\varsigma)$, respectively.

3. *The Sumudu transform is a linear operator.*

$$\begin{aligned}
\mathbb{S}_\varsigma[(\gamma\psi(\varsigma) + \alpha\phi(\varsigma));\mathfrak{y}] &= \gamma\mathbb{S}_\varsigma[\psi(\varsigma);\mathfrak{y}] + \alpha\mathbb{S}_\varsigma[\phi(\varsigma);\mathfrak{y}] \\
&= \gamma\mathcal{U}(\mathfrak{y}) + \alpha\mathcal{V}(\mathfrak{y}).
\end{aligned} \tag{1.50}$$

4. *The Sumudu transform of convolution is given by*

$$\begin{aligned}
\mathbb{S}_\varsigma[(\psi(\varsigma) * \phi(\varsigma));\mathfrak{y}] &= \mathfrak{y}\mathbb{S}_\varsigma[\psi(\varsigma);\mathfrak{y}]\mathbb{S}_\varsigma[\phi(\varsigma);\mathfrak{y}] \\
&= \mathfrak{y}\mathcal{U}(\mathfrak{y})\mathcal{V}(\mathfrak{y}).
\end{aligned} \tag{1.51}$$

Proof:

From the convolution integral and Equation (1.38), it follows that

$$\begin{aligned}
\mathbb{S}_{\varsigma}[(\psi * \phi)(\varsigma);\mathfrak{y}] &= \mathbb{S}_{\varsigma}\Big[\int_0^{\infty} \psi(\xi)\phi(\varsigma-\xi)d\xi\Big] \\
&= \frac{1}{\mathfrak{y}}\int_0^{\infty}\Big[\int_0^{\infty} \psi(\xi)\phi(\varsigma-\xi)d\xi\Big]e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma \\
&= \frac{1}{\mathfrak{y}}\int_0^{\infty} \psi(\xi)\Big[\int_0^{\infty} \phi(\varsigma-\xi)\psi(\xi)e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma\Big]d\xi.
\end{aligned}$$

By setting $u = \varsigma - \xi \implies du = d\varsigma$, it implies that

$$\begin{aligned}
\mathbb{S}_{\varsigma}[(\psi * \phi)(\varsigma);\mathfrak{y}] &= \frac{1}{\mathfrak{y}}\int_0^{\infty} \psi(\xi)\left[\int_0^{\infty} \phi(u)e^{-\frac{(u+\xi)}{\mathfrak{y}}}du\right]d\xi \\
&= \frac{1}{\mathfrak{y}}\int_0^{\infty} \psi(\xi)\Big[\int_0^{\infty} \phi(u)e^{-\frac{u}{\mathfrak{y}}}e^{-\frac{\xi}{\mathfrak{y}}}du\Big]d\xi \\
&= \frac{1}{\mathfrak{y}}\int_0^{\infty} \psi(\xi)e^{-\frac{\xi}{\mathfrak{y}}}\Big[\int_0^{\infty} \phi(u)e^{-\frac{u}{\mathfrak{y}}}du\Big]d\xi \\
&= \frac{1}{\mathfrak{y}}\int_0^{\infty} \psi(\xi)e^{-\frac{\xi}{\mathfrak{y}}}d\xi\mathfrak{y}\mathcal{V}(\mathfrak{y}) \\
&= \mathfrak{y}\mathcal{U}(\mathfrak{y})\mathcal{V}(\mathfrak{y}),
\end{aligned}$$

where $\mathcal{U}(\mathfrak{y})$, $\mathcal{V}(\mathfrak{y})$ are the Sumudu transforms of $\psi(\varsigma)$ and $\phi(\varsigma)$, respectively.

Lemma 1.2 : *(Belgacem et al., 2019; Meddahi et al., 2021) For* $\mathrm{Re}(u)$, $\mathrm{Re}(v) > 0$ *and* $\lambda \in \mathbb{R}$, *then the integral transforms of the two-parameters Mittag-Leffler function* $E_{u,v}(\lambda\varsigma^{u})$ *are given as follows:*

1. *Laplace transform of* $\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u})$ *is*

$$\mathbb{L}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u});\mathfrak{y}] = \frac{\mathfrak{y}^{u-v}}{\mathfrak{y}^{u}-\lambda}. \tag{1.52}$$

2. *Sumudu transform of* $\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u})$ *is*

$$\mathbb{S}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u});\mathfrak{y}] = \frac{\mathfrak{y}^{v-1}}{1-\lambda\mathfrak{y}^{u}}. \tag{1.53}$$

3. *Shehu transform of* $\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u})$ *is*

$$\mathbb{SH}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u});\mathfrak{y}] = \frac{(\frac{w}{\mathfrak{y}})^{v}}{1-\lambda(\frac{w}{\mathfrak{y}})^{u}}. \tag{1.54}$$

4. *Elzaki transform of* $\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u})$ *is*

$$\mathbb{E}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u});\mathfrak{y}] = \frac{\mathfrak{y}^{v+1}}{1-\lambda\mathfrak{y}^{u}}. \tag{1.55}$$

Table 1.1: A brief table of Laplace transforms

$\psi(\omega) = \mathbb{L}_{\omega}^{-1}\{\mathcal{F}(\eta)\}$	$\mathcal{F}(\eta) = \mathbb{L}_{\omega}\{\psi(\omega)\}$
1	$\frac{1}{\eta}$
ω	$\frac{1}{\eta^2}$
ω^2	$\frac{2}{\eta^3}$
$\omega^n, \quad n = 1, 2, 3 \ldots$	$\frac{n!}{\eta^{n+1}}$
$\omega^{\alpha}, \quad \mathrm{Re}(\alpha) > -1$	$\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}}$
ω^{n-1}	$\frac{(n-1)!}{\eta^n}, \quad n = 1, 2, 3, \ldots$
$\omega^{2\alpha}, \quad \mathrm{Re}(\alpha) > -\frac{1}{2}$	$\frac{\Gamma(2\alpha+1)}{\eta^{2\alpha+1}}$
$\omega^{2\alpha-2}, \quad \mathrm{Re}(\alpha) > \frac{1}{2}$	$\frac{\Gamma(2\alpha-1)}{\eta^{2\alpha-1}}$
$\sin a\omega$	$\frac{a}{\eta^2+a^2}$
$\cos a\omega$	$\frac{\eta}{\eta^2+a^2}$
$e^{a\omega}$	$\frac{1}{\eta-a}$
$\omega e^{a\omega}$	$\frac{1}{(\eta-a)^2}$

Note: The Laplace transform of a power function $\omega^{\alpha-\gamma}$ is given by

$$\mathbb{L}_{\varsigma}[\omega^{\alpha-\gamma}; \eta] = \frac{(\alpha-\gamma)!}{\eta^{\alpha-\gamma+1}}, \quad \mathrm{Re}(\alpha-\gamma) > 0. \tag{1.56}$$

Proof:

1. From Equations (1.34) and (1.27), it follows that

$$\begin{aligned}
\mathbb{L}_\varsigma[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u);\mathfrak{y}] &= \int_0^\varsigma \varsigma^{v-1}\sum_{k=0}^{\infty}\frac{(\lambda\varsigma^u)^k}{\Gamma(ku+v)}e^{-\mathfrak{y}\varsigma}d\varsigma \\
&= \sum_{k=0}^{\infty}\int_0^\varsigma \frac{\varsigma^{v-1}\lambda^k\varsigma^{ku}}{\Gamma(ku+v)}e^{-\mathfrak{y}\varsigma}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(ku+v)}\int_0^\varsigma \varsigma^{v+ku-1}e^{-\mathfrak{y}\varsigma}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(ku+v)}\mathbb{L}_\varsigma[\varsigma^{v+ku-1};\mathfrak{y}] \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(ku+v)}\frac{\Gamma(ku+v)}{\mathfrak{y}^{v+ku}} \\
&= \frac{1}{\mathfrak{y}^v}\sum_{k=0}^{\infty}\frac{\lambda^k}{\mathfrak{y}^{ku}} \\
&= \frac{1}{\mathfrak{y}^v}\left[1+\frac{\lambda}{\mathfrak{y}^u}+\frac{\lambda^2}{\mathfrak{y}^{2u}}+\frac{\lambda^3}{\mathfrak{y}^{3u}}+\frac{\lambda^4}{\mathfrak{y}^{4u}}+\cdots\right] \\
&= \frac{1}{\mathfrak{y}^v}\left[1+\left(\frac{\lambda}{\mathfrak{y}^u}\right)+\left(\frac{\lambda}{\mathfrak{y}^u}\right)^2+\left(\frac{\lambda}{\mathfrak{y}^u}\right)^3+\left(\frac{\lambda}{\mathfrak{y}^u}\right)^4+\cdots\right] \\
&= \frac{1}{\mathfrak{y}^v}\frac{1}{1-\frac{\lambda}{\mathfrak{y}^u}} \\
&= \frac{\mathfrak{y}^{u-v}}{\mathfrak{y}^u-\lambda}.
\end{aligned}$$

Proof:

2. From Equations (1.38) and (1.27), it follows that

$$\begin{aligned}
\mathbb{S}_\varsigma[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^u);\mathfrak{y}] &= \frac{1}{\mathfrak{y}}\int_0^\varsigma \varsigma^{v-1}\sum_{k=0}^{\infty}\frac{(\lambda\varsigma^u)^k}{\Gamma(ku+v)}e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(ku+v)}\frac{1}{\mathfrak{y}}\int_0^\varsigma \varsigma^{v+ku-1}e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(ku+v)}\mathbb{E}_\varsigma[\varsigma^{v+ku-1};\mathfrak{y}] \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(ku+v)}\Gamma(ku+v)\mathfrak{y}^{v+ku-1} \\
&= \mathfrak{y}^{v-1}\sum_{k=0}^{\infty}\lambda^k\mathfrak{y}^{ku} \\
&= \mathfrak{y}^{v-1}[1+\lambda\mathfrak{y}^u+\lambda^2\mathfrak{y}^{2u}+\lambda^3\mathfrak{y}^{3u}+\lambda^4\mathfrak{y}^{4u}+\cdots] \\
&= \mathfrak{y}^{v-1}[1+(\lambda\mathfrak{y}^u)+(\lambda\mathfrak{y}^u)^2+(\lambda\mathfrak{y}^u)^3+(\lambda\mathfrak{y}^u)^4+\cdots] \\
&= \frac{\mathfrak{y}^{v-1}}{1-\lambda\mathfrak{y}^u}.
\end{aligned}$$

Proof:

3. From Equations (1.41) and (1.27), it follows that

$$\begin{aligned}
\mathbb{SH}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u});\mathfrak{y}] &= \int_0^{\varsigma}\varsigma^{v-1}\sum_{k=0}^{\infty}\frac{(\lambda\varsigma^{u})^{k}}{\Gamma(ku+v)}e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(ku+v)}\int_0^{\varsigma}\varsigma^{v+ku-1}e^{-\frac{\mathfrak{y}\varsigma}{w}}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(ku+v)}\mathbb{SH}_{\varsigma}[\varsigma^{v+ku-1};\mathfrak{y}] \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(ku+v)}\Gamma(ku+v)\left(\frac{w}{\mathfrak{y}}\right)^{v+ku} \\
&= \left(\frac{w}{\mathfrak{y}}\right)^{v}\sum_{k=0}^{\infty}\lambda^{k}\left(\frac{w}{\mathfrak{y}}\right)^{ku} \\
&= \left(\frac{w}{\mathfrak{y}}\right)^{v}\left[1+\lambda\left(\frac{w}{\mathfrak{y}}\right)^{u}+\lambda^{2}\left(\frac{w}{\mathfrak{y}}\right)^{2u}+\cdots\right] \\
&= \left(\frac{w}{\mathfrak{y}}\right)^{v}\left[1+\left(\lambda\left(\frac{w}{\mathfrak{y}}\right)^{u}\right)+\left(\lambda\left(\frac{w}{\mathfrak{y}}\right)^{u}\right)^{2}+\cdots\right] \\
&= \frac{\left(\frac{w}{\mathfrak{y}}\right)^{v}}{1-\lambda\left(\frac{w}{\mathfrak{y}}\right)^{u}}.
\end{aligned}$$

Proof:

4. From Equations (1.40) and (1.27), it follows that

$$\begin{aligned}
\mathbb{E}_{\varsigma}[\varsigma^{v-1}E_{u,v}(\lambda\varsigma^{u});\mathfrak{y}] &= \mathfrak{y}\int_0^{\varsigma}\varsigma^{v-1}\sum_{k=0}^{\infty}\frac{(\lambda\varsigma^{u})^{k}}{\Gamma(ku+v)}e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(ku+v)}\mathfrak{y}\int_0^{\varsigma}\varsigma^{v+ku-1}e^{-\frac{\varsigma}{\mathfrak{y}}}d\varsigma \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(ku+v)}\mathbb{E}[\varsigma^{v+ku-1};\mathfrak{y}] \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(ku+v)}\Gamma(ku+v)\mathfrak{y}^{v+ku+1} \\
&= \mathfrak{y}^{v+1}\sum_{k=0}^{\infty}\lambda^{k}\mathfrak{y}^{ku} \\
&= \mathfrak{y}^{v+1}[1+\lambda\mathfrak{y}^{u}+\lambda^{2}\mathfrak{y}^{2u}+\lambda^{3}\mathfrak{y}^{3u}+\lambda^{4}\mathfrak{y}^{4u}+\cdots] \\
&= \mathfrak{y}^{v+1}[1+(\lambda\mathfrak{y}^{u})+(\lambda\mathfrak{y}^{u})^{2}+(\lambda\mathfrak{y}^{u})^{3}+(\lambda\mathfrak{y}^{u})^{4}+\cdots] \\
&= \frac{\mathfrak{y}^{v+1}}{1-\lambda\mathfrak{y}^{u}}.
\end{aligned}$$

1.4 Fractional Integrals and Fractional Derivatives

Numerous almost analogous formulations for fractional integrals and derivatives find application across various functional domains. This segment outlines the three most commonly employed definitions within the domain of fractional calculus, namely the Riemann-Liouville, Caputo, and Gruñwald-Letnikov definitions. Interested readers are encouraged to refer to this for further details (De Oliveira and Tenreiro Machado, 2014; Jiang et al., 2018; Valério et al., 2022). The following are some of the notations used in this section.

1. $\Omega = [a, b]$ be a finite interval on the real axis $\mathbb{R}$.
2. $\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma) = \psi_{\varsigma}^{(\mu)}(\nu,\varsigma) = \dfrac{\partial^{\mu}\psi(\nu,\varsigma)}{\partial\varsigma^{\mu}}$.

Definition 1.3 : *(Elbeleze et al., 2014) The real function $\psi(\varsigma), \varsigma > 0$ is said to be in the space $\mathbb{C}_{\mu}, \mu \in \mathbb{R}$ if there exists a real number $\eta > \mu$, such that $\psi(\varsigma) = \varsigma^{\eta}\psi_1(\varsigma)$ where $\psi_1(\varsigma) \in \mathbb{C}(0,\infty)$. The function $\psi(\varsigma)$ is said to be in the space $\mathbb{C}_{\mu}^{n}$ if $\psi^{n} \in \mathbb{R}_{\mu}$ for $n \in \mathbb{N}$.*

1.4.1 Riemann-Liouville Fractional Integral

Definition 1.4 : *Let $\mu \in (0,1)$ and $\psi(\nu,\varsigma) \in L^1(\Omega)$. The left and right partial Riemann-Liouville integrals of order μ of $\psi(\nu,\varsigma)$ with respect to $\varsigma > 0$ are defined, respectively, by the expression*

$$^{RL}\mathcal{I}_{a,\varsigma}^{\mu}\psi(\nu,\varsigma) = \frac{1}{\Gamma(\mu)}\int_{a}^{\varsigma}\psi(\nu,\xi)(\varsigma-\xi)^{\mu-1}d\xi, \tag{1.57}$$

and

$$^{RL}\mathcal{I}_{\varsigma,b}^{\mu}\psi(\nu,\varsigma) = \frac{1}{\Gamma(\mu)}\int_{\varsigma}^{b}\psi(\xi,\nu)(\xi-\varsigma)^{\mu-1}d\xi, \tag{1.58}$$

for almost all $(\nu,\varsigma) \in \Omega$ and $\Gamma(\cdot)$ is the well-known Gamma function.

Proposition 1.5 : *(Hamed and Mohamed, 2016) Some of the properties of the operator* ${}^{RL}\mathcal{I}_{a}^{\mu}$. *For* $\psi(\nu,\varsigma) \in L^1(\Omega)$ *and* $\mu,\ \alpha \geq 0$

1. ${}^{RL}\mathcal{I}_{a}^{\alpha}[{}^{RL}\mathcal{I}_{a}^{\mu}\psi(\nu,\varsigma)] = {}^{RL}\mathcal{I}_{a}^{\mu}[{}^{RL}\mathcal{I}_{a}^{\alpha}\psi(\nu,\varsigma)]$.
2. ${}^{RL}\mathcal{I}_{a}^{\alpha}[{}^{RL}\mathcal{I}_{a}^{\mu}\psi(\nu,\varsigma)] = {}^{RL}\mathcal{I}_{a}^{\alpha+\mu}\psi(\nu,\varsigma)$.
3. ${}^{RL}\mathcal{I}_{a}^{\mu}(\xi - a)^{\gamma} = \dfrac{\Gamma(\gamma+1)(\xi-a)^{\mu+\gamma}}{\Gamma(\mu+\gamma+1)}, \quad \gamma \geq -1.$

Remark 1.3 : If the function $\psi(\varsigma)$ is continuously differentiable or $\psi(\varsigma)$ is only continuous for $\varsigma \geq 0$, then it follows that

$$\lim_{\mu\to 0} {}^{RL}\mathcal{I}^{\mu}\psi(\varsigma) = \psi(\varsigma). \tag{1.59}$$

Example 1.1 : Let $\psi(\varsigma) = \varsigma^{\alpha}$, $\varsigma > 0$ and $\alpha > -1$.

From Definition (1.4), and for $\eta = \xi/\varsigma \implies \xi = \eta\varsigma,\ d\xi = \varsigma d\eta$. When $\xi \to 0, \eta \to 0$ and $\xi \to \varsigma, \eta \to 1$, it follows that

$$\begin{aligned}
{}^{RL}\mathcal{I}_{\varsigma}^{\mu}\varsigma^{\alpha} &= \frac{1}{\Gamma(\mu)}\int_0^{\varsigma} \xi^{\alpha}(\varsigma-\xi)^{\mu-1}d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_0^{\varsigma} \xi^{\alpha}\Big(\varsigma(1-\xi/\varsigma)\Big)^{\mu-1}d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_0^{1} \eta^{\alpha}(1-\eta)^{\mu-1}\varsigma^{\mu+\alpha}d\eta \\
&= \frac{\beta(\mu,\alpha+1)\varsigma^{\mu+\alpha}}{\Gamma(\mu)} \\
&= \frac{\Gamma(\mu)\Gamma(\alpha+1)\varsigma^{\mu+\alpha}}{\Gamma(\mu)\Gamma(\mu+\alpha+1)} \\
&= \frac{\Gamma(\alpha+1)\varsigma^{\mu+\alpha}}{\Gamma(\mu+\alpha+1)}.
\end{aligned}$$

Theorem 1.4 : *(Guo et al., 2015) Suppose that* $\psi(\varsigma)$ *is continuous on* $[0,T]$ *and* $\phi(\varsigma)$ *is analytical at* ς *for arbitrary* $\varsigma \in [0,T]$. *Then for any* $\mu > 0$ *and* $0 < \varsigma \leq T$, *there holds*

$$ {}^{RL}\mathcal{I}^{\mu}[\psi(\varsigma)\phi(\varsigma)] = \sum_{k=0}^{\infty} C_{\mu}^{k}[\mathcal{D}^{k}\phi(\varsigma)][\mathcal{D}^{\mu-k}\psi(\varsigma)]. \tag{1.60}$$

1.4.2 Riemann-Liouville Fractional Derivative

Definition 1.5 : *(Jiang et al., 2018) Let* $\mathrm{n}-1 < \mu \leq \mathrm{n}$ *for* $\mathrm{n} \in \mathbb{N}$ *and* $\psi(\nu, \varsigma) \in \mathrm{L}^1(\Omega)$. *The partial Riemann-Liouville fractional derivative of order* μ *of a function* $\psi(\nu, \varsigma)$ *with respect to* $\varsigma > 0$ *is defined as follows:*

$$\begin{aligned} {}_{\mathrm{a}}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma) &= \frac{\partial^{\mathrm{n}}}{\partial \varsigma^{\mathrm{n}}}\left[{}_{\mathrm{a}}{}^{RL}\mathcal{I}_{\varsigma}^{\mathrm{n}-\mu}\psi(\nu, \varsigma)\right] \\ &= \frac{1}{\Gamma(\mathrm{n}-\mu)}\frac{\partial^{\mathrm{n}}}{\partial \varsigma^{\mathrm{n}}}\int_{\mathrm{a}}^{\varsigma}\psi(\nu, \xi)(\varsigma-\xi)^{\mathrm{n}-\mu-1}\mathrm{d}\xi, \end{aligned} \tag{1.61}$$

for almost all $(\nu, \varsigma) \in \Omega$.

1.4.3 Caputo's Definitions of Fractional Derivative

Definition 1.6 : *(Dehghan et al., 2010) Let* n *be the smallest integer that exceeds* μ. *The Caputo time-fractional derivative operator of order* $\mu : \mathrm{n}-1 < \mu \leq \mathrm{n}$ *for* $\mathrm{n} \in \mathbb{N}$ *of a function* $\psi(\nu, \varsigma)$ *is defined as follows:*

$${}^{C}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma) = \begin{cases} \mathcal{I}_{\varsigma}^{\mathrm{n}-\mu}\Big(\dfrac{\partial^{\mathrm{n}}\psi(\nu, \varsigma)}{\partial \varsigma^{\mathrm{n}}}\Big) = \dfrac{1}{\Gamma(\mathrm{n}-\mu)}\displaystyle\int_{0}^{\varsigma}\dfrac{\partial^{\mathrm{n}}\psi(\nu, \xi)}{\partial \xi^{\mathrm{n}}}(\varsigma-\xi)^{\mathrm{n}-\mu-1}\mathrm{d}\xi, \\ \dfrac{\partial^{\mathrm{n}}\psi(\nu, \varsigma)}{\partial \varsigma^{\mathrm{n}}}, \quad \mu = \mathrm{n}. \end{cases} \tag{1.62}$$

Proposition 1.6 : *(Singh and Kumar, 2017) The operator* ${}^{C}\mathcal{D}_{\varsigma}^{\mu}$ *satisfies the following properties. Let* $\varsigma > 0$, $\mathrm{n}-1 < \mu \leq \mathrm{n}$ *for* $\mathrm{n} \in \mathbb{N}$, *then*

1. ${}^{C}\mathcal{D}_{\varsigma}^{\mu}[{}^{RL}\mathcal{I}_{\varsigma}^{\mu}\psi(\nu, \varsigma)] = \psi(\nu, \varsigma)$.
2. ${}^{C}\mathcal{D}_{\varsigma}^{\mu}[\alpha\psi(\nu, \varsigma) + \gamma\psi(\nu, \varsigma)] = \alpha\,{}^{C}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma) + \gamma\,{}^{C}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma), \quad \alpha, \gamma \in \mathbb{N}$.
3. ${}^{RL}\mathcal{I}_{\varsigma}^{\mu}[{}^{C}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma)] = \psi(\nu, \varsigma) - \displaystyle\sum_{\mathrm{k}=0}^{\mathrm{n}-1}\frac{\varsigma^{\mathrm{k}}}{\mathrm{k}!}\frac{\partial^{\mathrm{k}}\psi(\nu, \varsigma)}{\partial \varsigma^{\mathrm{k}}}\Big|_{\varsigma=0}$.

Remark 1.4 : (Kumar and Kumar, 2014; Bodkhe and Panchal, 2016; Khalouta and Kadem, 2019; Haroon et al., 2022; Sartanpara et al., 2022) In the following, we present the integral transform formulations of fractional integrals and derivatives.

Let $n = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ represents the integer part of α, then

1. Let $\mathcal{F}(\mathfrak{y})$ be the Laplace transform of $\psi(\varsigma)$, then the Laplace transform associated with the Riemann-Liouville fractional integral for a function $\psi(\varsigma)$ of order α is defined as follows:

$$\mathbb{L}_{\varsigma}[{}^{RL}\mathcal{I}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \mathfrak{y}^{-\alpha}\mathbb{L}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathfrak{y}^{-\alpha}\mathcal{F}(\mathfrak{y}). \tag{1.63}$$

2. The Shehu transform associated with the Riemann-Liouville fractional integral for a function $\psi(\varsigma)$ of order α is defined as follows:

$$\mathbb{SH}_{\varsigma}[{}^{RL}\mathcal{I}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \left(\frac{w}{\mathfrak{y}}\right)^{\alpha}\mathbb{SH}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \left(\frac{w}{\mathfrak{y}}\right)^{\alpha}\mathcal{U}(w,\mathfrak{y}), \tag{1.64}$$

where $\mathcal{U}(w,\mathfrak{y})$ is the Shehu transform of $\psi(\varsigma)$.

3. The Laplace transform associated with the Caputo fractional derivative for a function $\psi(\varsigma)$ of order $\alpha : n-1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{L}_{\varsigma}[{}^{C}\mathcal{D}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \mathfrak{y}^{\alpha}\mathcal{F}(\mathfrak{y}) - \sum_{k=0}^{n-1}\mathfrak{y}^{\alpha-k-1}\psi^{(k)}(\varsigma)\Big|_{\varsigma=0}, \tag{1.65}$$

where $\mathbb{L}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{F}(\mathfrak{y})$ is the Laplace transform of $\psi(\varsigma)$.

4. The Elzaki transform associated with the Caputo fractional derivative for a function $\psi(\varsigma)$ of order $\alpha : n-1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{E}_{\varsigma}[{}^{C}\mathcal{D}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \mathfrak{y}^{-\alpha}\mathcal{V}(\mathfrak{y}) - \sum_{k=0}^{n-1}\mathfrak{y}^{2-\alpha+k}\psi^{(k)}(\varsigma)\Big|_{\varsigma=0}, \tag{1.66}$$

where $\mathbb{E}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{V}(\mathfrak{y})$ is the Elzaki transform of $\psi(\varsigma)$.

5. The Sumudu transform associated with the Caputo fractional derivative for a function $\psi(\varsigma)$ of order $\alpha : n-1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{S}_{\varsigma}[{}^{C}\mathcal{D}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \mathfrak{y}^{-\alpha}\mathcal{G}(\mathfrak{y}) - \sum_{k=0}^{n-1}\mathfrak{y}^{k-\alpha}\psi^{(k)}(\varsigma)\Big|_{\varsigma=0}, \tag{1.67}$$

where $\mathbb{E}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{G}(\mathfrak{y})$ is the Elzaki transform of $\psi(\varsigma)$.

6. The Shehu transform associated with the Riemann-Liouville fractional derivative for a function $\psi(\varsigma)$ of order $\alpha : n-1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{SH}_{\varsigma}[{}^{RL}\mathcal{D}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \left(\frac{\mathfrak{y}}{w}\right)^{\alpha}\mathcal{U}(w,\mathfrak{y}) - \sum_{k=0}^{n-1}\left(\frac{\mathfrak{y}}{w}\right)^{k}\psi^{(\alpha-k-1)}(\varsigma)\Big|_{\varsigma=0}, \tag{1.68}$$

where $\mathbb{SH}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{U}(w,\mathfrak{y})$ is the Shehu transform of $\psi(\varsigma)$.

7. The Shehu transform associated with the Caputo fractional derivative for a function $\psi(\varsigma)$ of order $\alpha : n-1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{SH}_{\varsigma}[{}^{C}\mathcal{D}_{\varsigma}^{\alpha}\psi(\varsigma);\mathfrak{y}] = \left(\frac{\mathfrak{y}}{w}\right)^{\alpha}\mathcal{U}(w,\mathfrak{y}) - \sum_{k=0}^{n-1}\left(\frac{\mathfrak{y}}{w}\right)^{\alpha-k-1}\psi^{(k)}(\varsigma)\Big|_{\varsigma=0}, \tag{1.69}$$

where $\mathbb{SH}_{\varsigma}[\psi(\varsigma);\mathfrak{y}] = \mathcal{U}(w,\mathfrak{y})$ is the Shehu transform of $\psi(\varsigma)$.

Note: Since the applied problems demand the incorporation of fractional derivatives with appropriate utilization of initial conditions having established physical interpretations, particularly within the theory of viscoelasticity, the Caputo approach emerges as a more suitable methodological framework. This selection arises from the agreement of Caputo fractional derivatives to the similar initial conditions of integer ordered differential equations, thereby providing them with a real physical interpretation inherent to the associated problem.

1.4.4 Atangana-Baleanu Fractional Integral

Definition 1.7 : *(Haroon et al., 2022) The Atangana-Baleanu fractional integral operator of order $0 < \mu < 1$ and a function $\psi(\nu,\varsigma) \in L^1(\Omega)$ is presented by*

$${}_{a}^{AB}\mathcal{I}_{\varsigma}^{\mu}\psi(\nu,\varsigma) = \frac{1-\mu}{\mathcal{P}(\mu)}\psi(\nu,\varsigma) + \frac{\mu}{\Gamma(\mu)\mathcal{P}(\mu)}\int_{a}^{\varsigma}\psi(\nu,\xi)(\varsigma-\xi)^{\mu-1}d\xi, \tag{1.70}$$

where $\mathcal{P}(\mu)$ denotes a normalization function such that $\mathcal{P}(0) = \mathcal{P}(1) = 1$.

1.4.5 Atangana-Baleanu Fractional Derivative

Definition 1.8 : *(Atangana and Baleanu, 2016) The fractional order Atangana-Baleanu (AB) derivative of a function* $\psi(\nu,\varsigma) \in \mathrm{L}^1(\Omega)$ *and* $\mu \in (0,1)$ *in the Riemann-Liouville sense is presented as follows:*

$$ {}_{\mathrm{a}}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma) = \frac{\mathcal{P}(\mu)}{1-\mu}\frac{\mathrm{d}}{\mathrm{d}\varsigma}\int_{\mathrm{a}}^{\varsigma}\psi(\nu,\xi)\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi, \tag{1.71}$$

where $\mathrm{E}_{\mu}(\xi)$ *represents the one-parameter Mittag-Leffler function.*

Definition 1.9 : *(Atangana and Baleanu, 2016) The Atangana-Baleanu fractional derivative operator in the Caputo sense of order* $0 < \mu < 1$*, and* $\psi(\nu,\varsigma) \in \mathrm{L}^1(\Omega)$ *is defined by*

$$ {}_{\mathrm{a}}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma) = \frac{\mathcal{P}(\mu)}{1-\mu}\int_{\mathrm{a}}^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi. \tag{1.72}$$

Proposition 1.7 : *For* $\psi(\nu,\varsigma)$ *defined in* $[\mathrm{a},\mathrm{b}]$, $\varsigma > 0$, $\mathrm{n}-1 < \mu \leq \mathrm{n}$ *and* $\mathrm{n} \in \mathbb{N}$. *The Atangana-Baleanu fractional integral operator* ${}_{\mathrm{a}}^{AB}\mathcal{I}_{\varsigma}^{\mu}$ *satisfies the following properties which were verified by (Abdeljawad, 2017).*

1. ${}_{\mathrm{a}}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma) = 0$, *if* $\psi(\nu,\varsigma)$ *is a constant function.*
2. ${}_{\mathrm{a}}^{ABR}\mathcal{D}_{\varsigma}^{\mu}[{}_{\mathrm{a}}^{AB}\mathcal{I}_{\varsigma}^{\mu}\psi(\nu,\varsigma)] = \psi(\nu,\varsigma)$.
3. ${}_{\mathrm{a}}^{AB}\mathcal{I}_{\varsigma}^{\mu}[{}_{\mathrm{a}}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma)] = \psi(\nu,\varsigma) - \sum_{\mathrm{k}=0}^{\mathrm{n}}\frac{(\varsigma-\mathrm{a})^{\mathrm{k}}}{\mathrm{k}!}\frac{\partial^{\mathrm{k}}\psi(\nu,\mathrm{a})}{\partial\varsigma^{\mathrm{k}}}$.
4. ${}_{\mathrm{a}}^{AB}\mathcal{I}_{\varsigma}^{\mu}[{}_{\mathrm{a}}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma)] = \psi(\nu,\varsigma) - \sum_{\mathrm{k}=0}^{\mathrm{n}-1}\frac{(\varsigma-\mathrm{a})^{\mathrm{k}}}{\mathrm{k}!}\frac{\partial^{\mathrm{k}}\psi(\nu,\mathrm{a})}{\partial\varsigma^{\mathrm{k}}}$.

Remark 1.5 : (Atangana and Baleanu, 2016; Haroon et al., 2022) The relation between the noninteger constant order Atangana-Baleanu in the Caputo sense and Riemann-Liouville sense, respectively, with Laplace, Elzaki, Sumudu, and Shehu transform can be expressed as follows:

1. The Laplace transform operator connected with the Atangana-Baleanu-Caputo sense with respect to $\varsigma > 0$ is defined as follows:

$$\mathbb{L}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathfrak{y}^{\mu}\mathcal{P}(\mu)\Big\{\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] - \frac{\psi(\nu,0)}{\mathfrak{y}}\Big\}}{\mathfrak{y}^{\mu}(1-\mu)+\mu}. \tag{1.73}$$

Proof:

Let us observe that in the Definition (1.9), we have a convolution integral, it follows that

$$\int_0^{\varsigma} \frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi = \psi'(\nu,\varsigma) * \mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big],$$

then from Equation (1.34) and Proposition (1.2), one can has

$$\begin{aligned}
\mathbb{L}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{L}_{\varsigma}\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\int_0^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{L}_{\varsigma}\Big[\int_0^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{L}_{\varsigma}\Big[\psi'(\nu,\varsigma) * \mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{L}_{\varsigma}\Big[\psi'(\nu,\varsigma);\mathfrak{y}\Big]\mathbb{L}_{\varsigma}\Big[\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big];\mathfrak{y}\Big].
\end{aligned}$$

Using Equation (1.65) and applying Lemma (1.2), this implies that

$$\begin{aligned}
\mathbb{L}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \frac{\mathcal{P}(\mu)}{1-\mu}\{\mathfrak{y}\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] - \psi(\nu,0)\}\frac{\mathfrak{y}^{\mu-1}}{\mathfrak{y}^{\mu}+\frac{\mu}{1-\mu}} \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\{\mathfrak{y}^{\mu}\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] - \mathfrak{y}^{\mu-1}\psi(\nu,0)\}\frac{1}{\frac{\mathfrak{y}^{\mu}(1-\mu)+\mu}{1-\mu}} \\
&= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{(1-\mu)}{\mathfrak{y}^{\mu}(1-\mu)+\mu}\{\mathfrak{y}^{\mu}\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] - \mathfrak{y}^{\mu-1}\psi(\nu,0)\} \\
&= \frac{\mathcal{P}(\mu)}{\mathfrak{y}^{\mu}(1-\mu)+\mu}\{\mathfrak{y}^{\mu}\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] - \mathfrak{y}^{\mu-1}\psi(\nu,0)\} \\
&= \frac{\mathfrak{y}^{\mu}\mathcal{P}(\mu)\Big\{\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] - \frac{\psi(\nu,0)}{\mathfrak{y}}\Big\}}{\mathfrak{y}^{\mu}(1-\mu)+\mu}.
\end{aligned}$$

2. The Laplace transform operator connected with the Atangana-Baleanu-Riemann-Liouville sense with respect to $\varsigma > 0$ is defined as follows:

$$\mathbb{L}_{\varsigma}[{}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathfrak{y}^{\mu}\mathcal{P}(\mu)\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]}{\mathfrak{y}^{\mu}(1-\mu)+\mu}. \tag{1.74}$$

Proof:

From Definition (1.8) we have a convolution integral and using Equation (1.34) and Proposition (1.2), it follows that

$$\begin{aligned}
\mathbb{L}_{\varsigma}[{}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{L}_{\varsigma}\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\frac{\mathrm{d}}{\mathrm{d}\varsigma}\int_{0}^{\varsigma}\psi(\nu,\xi)\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{L}_{\varsigma}\Big[\frac{\mathrm{d}}{\mathrm{d}\varsigma}\Big(\psi(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big)\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\Big[\mathfrak{y}\mathbb{L}_{\varsigma}\Big((\psi(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big)-\mathbb{L}_{\varsigma}\Big((\psi(\nu,0)*\mathrm{E}_{\mu}[0]);\mathfrak{y}\Big)\Big] \\
&= \frac{\mathfrak{y}\mathcal{P}(\mu)}{1-\mu}\mathbb{L}_{\varsigma}\Big((\psi(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big) \\
&= \frac{\mathfrak{y}\mathcal{P}(\mu)}{(1-\mu)}\frac{\mathfrak{y}^{\mu-1}(1-\mu)}{\mathfrak{y}^{\mu}(1-\mu)+\mu}\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}] \\
&= \frac{\mathfrak{y}^{\mu}\mathcal{P}(\mu)\mathbb{L}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]}{\mathfrak{y}^{\mu}(1-\mu)+\mu}.
\end{aligned}$$

3. The Elzaki transform of the Atangana-Baleanu-Caputo fractional derivative of a function $\psi(\varsigma)$ of order $\mu : \mathrm{n}-1 < \mu \leq \mathrm{n}$ for $\mathrm{n} \in \mathbb{N}$ is defined as follows:

$$\mathbb{E}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathcal{P}(\mu)\{\mathbb{E}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\mathfrak{y}^{2}\psi(\nu,0)\}}{\mu\mathfrak{y}^{\mu}+1-\mu}. \tag{1.75}$$

Proof:

From Definition (1.9) we have a convolution integral, using Equation (1.40) and Proposition (1.3), it follows that

$$\begin{aligned}
\mathbb{E}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{E}_{\varsigma}\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\int_{0}^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{E}_{\varsigma}\Big[\int_{0}^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{E}_{\varsigma}\Big[\psi'(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\frac{1}{\mathfrak{y}}\mathbb{E}_{\varsigma}[\psi'(\nu,\varsigma);\mathfrak{y}]\mathbb{E}_{\varsigma}\Big[\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big];\mathfrak{y}\Big].
\end{aligned}$$

Using Equation (1.66) and applying Lemma (1.2), this implies that

$$\begin{aligned}
\mathbb{E}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{1}{\mathfrak{y}}\{\mathfrak{y}^{-1}\mathbb{E}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\mathfrak{y}\psi(\nu,0)\}\frac{\mathfrak{y}^{2}}{1+\frac{\mu}{1-\mu}\mathfrak{y}^{\mu}} \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\{\mathbb{E}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\mathfrak{y}^{2}\psi(\nu,0)\}\frac{1}{\frac{(1-\mu)+\mu\mathfrak{y}^{\mu}}{1-\mu}} \\
&= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{(1-\mu)}{(1-\mu)+\mu\mathfrak{y}^{\mu}}\{\mathbb{E}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\mathfrak{y}^{2}\psi(\nu,0)\} \\
&= \frac{\mathcal{P}(\mu)\{\mathbb{E}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\mathfrak{y}^{2}\psi(\nu,0)\}}{\mu\mathfrak{y}^{\mu}+1-\mu}.
\end{aligned}$$

4. The Elzaki transform of the Atangana-Baleanu-Riemann-Liouville fractional derivative of a function $\psi(\varsigma)$ of order $\mu : n-1 < \mu \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{E}_\varsigma[{}^{ABR}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathcal{P}(\mu)\mathbb{E}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]}{\mu\mathfrak{y}^{\mu}+1-\mu}. \tag{1.76}$$

Proof:

From Definition (1.8) we have a convolution integral and using Equation (1.40) and Proposition (1.3), it follows that

$$\begin{aligned}
\mathbb{E}_\varsigma[{}^{ABR}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{E}_\varsigma\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\frac{\mathrm{d}}{\mathrm{d}\varsigma}\int_0^{\varsigma}\psi(\nu,\xi)\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\Big]\\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{E}_\varsigma\Big[\frac{\mathrm{d}}{\mathrm{d}\varsigma}\Big(\psi(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big);\mathfrak{y}\Big]\\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\Big[\mathfrak{y}^{-1}\mathbb{E}_\varsigma\Big((\psi(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big)-\mathfrak{y}\mathbb{E}_\varsigma((\psi(\nu,0)*\mathrm{E}_{\mu}[0]);\mathfrak{y})\Big]\\
&= \frac{\mathfrak{y}^{-1}\mathcal{P}(\mu)}{1-\mu}\mathbb{E}_\varsigma\Big((\psi(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big)\\
&= \frac{\mathfrak{y}^{-1}\mathfrak{y}^{-1}\mathcal{P}(\mu)}{(1-\mu)}\mathbb{E}_\varsigma[(\psi(\nu,\varsigma);\mathfrak{y}]\mathbb{E}_\varsigma\Big[\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big];\mathfrak{y}\Big]\\
&= \frac{\mathfrak{y}^{-2}\mathcal{P}(\mu)}{(1-\mu)}\frac{\mathfrak{y}^{2}(1-\mu)}{\mu\mathfrak{y}^{\mu}+1-\mu}\mathbb{E}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]\\
&= \frac{\mathcal{P}(\mu)\mathbb{E}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]}{\mu\mathfrak{y}^{\mu}+1-\mu}.
\end{aligned}$$

5. The Sumudu transform of the Atangana-Baleanu-Caputo fractional derivative of a function $\psi(\varsigma)$ of order $\mu : n-1 < \mu \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{S}_\varsigma[{}^{ABC}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathcal{P}(\mu)\{\mathbb{S}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]-\psi(\nu,0)\}}{\mu\mathfrak{y}^{\mu}+1-\mu}. \tag{1.77}$$

Proof:

From Definition (1.9) we have a convolution integral, using Equation (1.38) and Proposition (1.4), it follows that

$$\begin{aligned}
\mathbb{S}_\varsigma[{}^{ABC}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{S}_\varsigma\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\int_0^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\Big]\\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{S}_\varsigma\Big[\int_0^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\Big]\\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{S}_\varsigma\Big[\psi'(\nu,\varsigma)*\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big]\\
&= \frac{\mathfrak{y}\mathcal{P}(\mu)}{1-\mu}\mathbb{S}_\varsigma\Big[\psi'(\nu,\varsigma);\mathfrak{y}\Big]\mathbb{S}_\varsigma\Big[\mathrm{E}_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big];\mathfrak{y}\Big].
\end{aligned}$$

Using Equation (1.67) and applying Lemma (1.2), this implies that

$$\begin{aligned}\mathbb{S}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \frac{\mathcal{P}(\mu)}{1-\mu}\mathfrak{y}\{\mathfrak{y}^{-1}\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\mathfrak{y}^{-1}\psi(\nu,0)\}\frac{1}{1+\frac{\mu}{1-\mu}\mathfrak{y}^{\mu}}\\ &= \frac{\mathcal{P}(\mu)}{1-\mu}\{\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\psi(\nu,0)\}\frac{1}{\frac{1-\mu+\mu\mathfrak{y}^{\mu}}{1-\mu}}\\ &= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{(1-\mu)}{(1-\mu)+\mu\mathfrak{y}^{\mu}}\{\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\psi(\nu,0)\}\\ &= \frac{\mathcal{P}(\mu)\{\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\psi(\nu,0)\}}{\mu\mathfrak{y}^{\mu}+1-\mu}.\end{aligned}$$

6. The Sumudu transform of the Atangana-Baleanu-Riemann-Liouville fractional derivative of a function $\psi(\varsigma)$ of order $\mu : n-1 < \mu \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{S}_{\varsigma}[{}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathcal{P}(\mu)\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]}{\mu\mathfrak{y}^{\mu}+1-\mu}. \tag{1.78}$$

Proof:

From Definition (1.8) we have a convolution integral, using Equation (1.38) and Proposition (1.4), this implies that

$$\begin{aligned}\mathbb{S}_{\varsigma}[{}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{S}_{\varsigma}\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\frac{d}{d\varsigma}\int_{0}^{\varsigma}\psi(\nu,\xi)E_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi\Big]\\ &= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{S}_{\varsigma}\Big[\frac{d}{d\varsigma}\Big(\psi(\nu,\varsigma)*E_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big);\mathfrak{y}\Big]\\ &= \frac{\mathcal{P}(\mu)}{1-\mu}\Big[\mathfrak{y}^{-1}\mathbb{S}_{\varsigma}\Big((\psi(\nu,\varsigma)*E_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big)-\mathfrak{y}^{-1}\mathbb{S}_{\varsigma}((\psi(\nu,0)*E_{\mu}[0]);\mathfrak{y})\Big]\\ &= \frac{\mathfrak{y}^{-1}\mathcal{P}(\mu)}{1-\mu}\mathbb{S}_{\varsigma}\Big((\psi(\nu,\varsigma)*E_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big)\\ &= \frac{\mathfrak{y}\mathfrak{y}^{-1}\mathcal{P}(\mu)}{1-\mu}\mathbb{S}_{\varsigma}[(\psi(\nu,\varsigma);\mathfrak{y}]\mathbb{S}_{\varsigma}\Big[E_{\mu}\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]);\mathfrak{y}\Big]\\ &= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{(1-\mu)}{\mu\mathfrak{y}^{\mu}+1-\mu}\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]\\ &= \frac{\mathcal{P}(\mu)\mathbb{S}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]}{\mu\mathfrak{y}^{\mu}+1-\mu}.\end{aligned}$$

7. The Shehu transform of the Atangana-Baleanu-Caputo fractional derivative of a function $\psi(\varsigma)$ of order $\mu : n-1 < \mu \leq n$ for $n \in \mathbb{N}$ is defined as follows:

$$\mathbb{SH}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathcal{P}(\mu)\{\mathbb{SH}_{\varsigma}[\psi(\nu,\varsigma);\mathfrak{y}]-\left(\frac{w}{\mathfrak{y}}\right)\psi(\nu,0)\}}{\mu\left(\frac{w}{\mathfrak{y}}\right)^{\mu}+1-\mu}. \tag{1.79}$$

Proof:

From Definition (1.9) we have a convolution integral and using Equation (1.41) and Proposition (1.4), this implies that

$$\begin{aligned}
\mathbb{SH}_\varsigma[{}^{ABC}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{SH}_\varsigma\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\int_0^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_\mu\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{SH}_\varsigma\Big[\int_0^{\varsigma}\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathrm{E}_\mu\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{SH}_\varsigma\Big[\psi'(\nu,\varsigma)*\mathrm{E}_\mu\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{SH}_\varsigma\Big[\psi'(\nu,\varsigma);\mathfrak{y}\Big]\mathbb{SH}_\varsigma\Big[\mathrm{E}_\mu\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big];\mathfrak{y}\Big].
\end{aligned}$$

Using Equation (1.69) and applying Lemma (1.2), this implies that

$$\begin{aligned}
\mathbb{SH}_\varsigma[{}^{ABC}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] &= \frac{\mathcal{P}(\mu)}{1-\mu}\Big\{\frac{\mathfrak{y}}{\mathrm{w}}\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]-\psi(\nu,0)\Big\}\frac{\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)}{1+\frac{\mu}{1-\mu}\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)^{\mu}} \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\Big\{\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]-\frac{\mathfrak{y}}{\mathrm{w}}\psi(\nu,0)\Big\}\frac{1}{\frac{(1-\mu)+\mu\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)^{\mu}}{1-\mu}} \\
&= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{(1-\mu)}{(1-\mu)+\mu(\frac{\mathrm{w}}{\mathfrak{y}})^{\mu}}\Big\{\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]-\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)\psi(\nu,0)\Big\} \\
&= \frac{\mathcal{P}(\mu)\{\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]-\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)\psi(\nu,0)\}}{\mu\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)^{\mu}+1-\mu}.
\end{aligned}$$

8. The Shehu transform of the Atangana-Baleanu-Riemann-Liouville fractional derivative of a function $\psi(\varsigma)$ of order $\mu : \mathrm{n}-1<\mu\leq \mathrm{n}$ for $\mathrm{n}\in\mathbb{N}$ is defined as follows:

$$\mathbb{SH}_\varsigma[{}^{ABR}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathcal{P}(\mu)\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]}{\mu\left(\frac{\mathrm{w}}{\mathfrak{y}}\right)^{\mu}+1-\mu}. \tag{1.80}$$

Proof:

From Definition (1.8) we have a convolution integral, using Equation (1.41) and Proposition (1.4), this implies that

$$\begin{aligned}
\mathbb{SH}_\varsigma[{}^{ABR}\mathcal{D}^{\mu}_{\varsigma}\psi(\nu,\varsigma);\mathfrak{y}] &= \mathbb{SH}_\varsigma\Big[\frac{\mathcal{P}(\mu)}{1-\mu}\frac{d}{d\varsigma}\int_0^{\varsigma}\psi(\nu,\xi)\mathrm{E}_\mu\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]d\xi\Big] \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\mathbb{SH}_\varsigma\Big[\frac{d}{d\varsigma}\Big(\psi(\nu,\varsigma)*\mathrm{E}_\mu\Big[\frac{\mu\varsigma^{\mu}}{\mu-1}\Big]\Big);\mathfrak{y}\Big]
\end{aligned}$$

$$= \frac{\mathcal{P}(\mu)}{1-\mu}\Big[\frac{\mathfrak{y}}{\mathrm{w}}\mathbb{SH}_\varsigma\left((\psi(\nu,\varsigma) * \mathrm{E}_\mu\Big[\frac{\mu\varsigma^\mu}{\mu-1}\Big]);\mathfrak{y}\right) - \frac{\mathfrak{y}}{\mathrm{w}}\mathbb{SH}_\varsigma\Big((\psi(\nu,0) * \mathrm{E}_\mu[0]);\mathfrak{y}\Big)\Big]$$
$$= \frac{\frac{\mathfrak{y}}{\mathrm{w}}\mathcal{P}(\mu)}{1-\mu}\mathbb{SH}_\varsigma\Big((\psi(\nu,\varsigma) * \mathrm{E}_\mu\Big[\frac{\mu\varsigma^\mu}{\mu-1}\Big]);\mathfrak{y}\Big)$$
$$= \frac{\frac{\mathfrak{y}}{\mathrm{w}}\mathcal{P}(\mu)}{1-\mu}\mathbb{SH}_\varsigma[(\psi(\nu,\varsigma);\mathfrak{y}]\mathbb{SH}_\varsigma\Big[\mathrm{E}_\mu\Big[\frac{\mu\varsigma^\mu}{\mu-1}\Big]);\mathfrak{y}\Big]$$
$$= \frac{\mathcal{P}(\mu)}{(1-\mu)}\frac{\frac{\mathfrak{y}}{\mathrm{w}}\frac{\mathrm{w}}{\mathfrak{y}}(1-\mu)}{\mu\Big(\frac{\mathrm{w}}{\mathfrak{y}}\Big)^\mu + 1 - \mu}\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]$$
$$= \frac{\mathcal{P}(\mu)\mathbb{SH}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]}{\mu\Big(\frac{\mathrm{w}}{\mathfrak{y}}\Big)^\mu + 1 - \mu}.$$

Lemma 1.3 : *The relation between the Atangana-Baleanu-Caputo and Atangana-Baleanu-Riemann-Liouville operators of a function* $\psi(\nu,\varsigma)$, $\varsigma > 0$ *and of the fractional order* μ *is given as follows:*

$$ {}_{\mathrm{a}}^{ABC}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma) = {}_{\mathrm{a}}^{ABR}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma) - \frac{\mathcal{P}(\mu)}{1-\mu}\psi(\nu,\mathrm{a})\mathrm{E}_\mu\Big(\frac{\mu(\varsigma-\mathrm{a})^\mu}{\mu-1}\Big). \tag{1.81}$$

Proof:

This can be shown by using Equation (1.73), it follows that

$$\mathbb{L}_\varsigma[{}^{ABC}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma);\mathfrak{y}] = \frac{\mathfrak{y}^\mu\mathcal{P}(\mu)\Big\{\mathbb{L}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}] - \frac{\psi(\nu,0)}{\mathfrak{y}}\Big\}}{\mathfrak{y}^\mu(1-\mu)+\mu}$$
$$= \frac{\mathfrak{y}^\mu\mathcal{P}(\mu)\{\mathbb{L}_\varsigma[\psi(\nu,\varsigma);\mathfrak{y}]\}}{\mathfrak{y}^\mu(1-\mu)+\mu} - \frac{\mathfrak{y}^\mu\mathcal{P}(\mu)\frac{\psi(\nu,0)}{\mathfrak{y}}}{\mathfrak{y}^\mu(1-\mu)+\mu}$$
$$= \mathbb{L}_\varsigma[{}^{ABR}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma);\mathfrak{y}] - \frac{\mathfrak{y}^\mu\mathcal{P}(\mu)\frac{\psi(\nu,0)}{\mathfrak{y}}}{\mathfrak{y}^\mu(1-\mu)+\mu}.$$

Using the inverse Laplace on both sides of the above equation we obtain the result

$$ {}_{\mathrm{a}}^{ABC}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma) = {}_{\mathrm{a}}^{ABR}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma) - \mathbb{L}_\varsigma^{-1}\Big[\frac{\mathfrak{y}^\mu\mathcal{P}(\mu)\frac{\psi(\nu,0)}{\mathfrak{y}}}{\mathfrak{y}^\mu(1-\mu)+\mu}\Big]$$
$$= {}_{\mathrm{a}}^{ABR}\mathcal{D}_\varsigma^\mu\psi(\nu,\varsigma) - \frac{\mathcal{P}(\mu)}{1-\mu}\psi(\nu,\mathrm{a})\mathrm{E}_\mu\Big(\frac{\mu(\varsigma-\mathrm{a})^\mu}{\mu-1}\Big),$$

which completed the proof.

Theorem 1.5 : *Assume that a continuous function* $\psi(\nu, \varsigma)$ *is defined on a closed interval* Ω. *Therefore, the Atangana-Baleanu fractional derivative is bounded, and the following inequality is derived over the domain* Ω

$$\|{}_{\mathrm{a}}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma)\| \leq \mathrm{K}\|\psi(\nu, \xi)\|, \tag{1.82}$$

where $\|\psi(\nu, \xi)\| = \max_{\mathrm{a}\leq\varsigma\leq\mathrm{b}} |\psi(\nu, \varsigma)|$.

Proof:

This can be easily proven by the Definition (1.8), it follows that

$$\begin{aligned}
\|{}_{\mathrm{a}}^{ABR}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma)\| &= \|\frac{\mathcal{P}(\mu)}{1-\mu}\frac{\mathrm{d}}{\mathrm{d}\varsigma}\int_{\mathrm{a}}^{\varsigma}\psi(\nu, \xi)\mathrm{E}_{\mu}\Big[\frac{\mu(\varsigma-\xi)^{\mu}}{\mu-1}\Big]\mathrm{d}\xi\| \\
&\leq \frac{\mathcal{P}(\mu)}{1-\mu}\|\frac{\mathrm{d}}{\mathrm{d}\varsigma}\int_{\mathrm{a}}^{\varsigma}\psi(\nu, \xi)\mathrm{d}\xi\| \\
&= \frac{\mathcal{P}(\mu)}{1-\mu}\|\psi(\nu, \xi)\| \\
&= \mathrm{K}\|\psi(\nu, \xi)\|,
\end{aligned}$$

where $\mathcal{P}(\mu)/(1-\mu) = \mathrm{K}$, which completed the proof. Similarly, under the same conditions, one can demonstrate that the ${}_{\mathrm{a}}^{ABC}\mathcal{D}_{\varsigma}^{\mu}\psi(\nu, \varsigma)$, remains bounded.

Remark 1.6 : Several differences between Riemann-Liouville and Caputo fractional derivatives can be observed.

1. The Riemann-Liouville fractional derivatives of a constant C is

$$_{\mathrm{a}}\mathcal{D}_{\varsigma}^{\mu}(\mathrm{C}) = \frac{\mathrm{C}(\varsigma-\mathrm{a})^{\mu}}{\Gamma(1-\mu)} \tag{1.83}$$

2. The interchange of integer and fractional order derivatives is allowed under different conditions.

   2.1 In the case of the Riemann-Liouville fractional derivative, for $\mathrm{k} = 0, 1, 2, \cdots$ and $\mathrm{n} - 1 < \mu < \mathrm{n}$ for $\mathrm{n} \in \mathbb{N}$ we have

$$^{RL}\mathcal{D}_{\varsigma}^{\mathrm{k}}[^{RL}\mathcal{D}_{\varsigma}^{\mu}\psi(\varsigma)] = {}^{RL}\mathcal{D}_{\varsigma}^{\mu+\mathrm{k}}\psi(\varsigma), \tag{1.84}$$

   which is allowed under the condition

$$\psi^{(\mathrm{p})}(0) = 0, \quad \mathrm{p} = \mathrm{n}, \mathrm{n}+1, \mathrm{n}+2, \cdots, \mathrm{k} \tag{1.85}$$

2.2 Whereas in the case of the Caputo fractional derivative, for $\mathrm{k} = 0, 1, 2, \cdots$ and $\mathrm{n} - 1 < \mu < \mathrm{n}$ for $\mathrm{n} \in \mathbb{N}$ we have

$$ {}^{C}\mathcal{D}_{\varsigma}^{\mu}[{}^{C}\mathcal{D}_{\varsigma}^{\mathrm{k}}\psi(\varsigma)] = {}^{C}\mathcal{D}_{\varsigma}^{\mu+\mathrm{k}}\psi(\varsigma), \tag{1.86} $$

which is allowed under the condition

$$ \psi^{(\mathrm{p})}(0) = 0, \quad \mathrm{p} = 0, 1, 2, \cdots, \mathrm{k} \tag{1.87} $$

Table 1.2: Some notable differences between fractional derivatives

	C	**RL**	**ABC**	**ABR**
Singular Kernel	✓	✓		
Nonsingular Kernel			✓	✓
$\mathcal{D}_{\varsigma}^{\mu}(\mathrm{K}) = 0$, K is a constant	✓		✓	
$\mathcal{D}_{\varsigma}^{\mu}(\mathrm{K}) \neq 0$, K is a constant		✓		✓

1.4.6 Some Applications of Fractional Differential Equations

In this section, we explore some of the significant fractional models and applications across various scientific domains and engineering fields that exist in current mathematical physics research. Applications range from physics, where they describe anomalous diffusion to biology, where they model population dynamics and neuronal activity.

Definition 1.10 : *A fractional differential equation is an equation that contains fractional derivatives of one or more independent variables. It can be classified as linear or nonlinear:*

1. *Fractional ordinary differential equation, where the unknown function depends only on one variable.*

2. *Fractional partial differential equation, where the unknown function depends on several variables.*

The space-time fractional partial diffusion equation of the following form

$$\begin{aligned} {}^{C}\mathcal{D}_{\varsigma}^{\alpha}\psi(\nu,\varsigma) &= {}^{RL}\mathcal{D}_{\nu}^{\beta}\psi(\nu,\varsigma), \quad 0<\varsigma\leq \mathrm{T} \\ \psi(\nu,0) &= \phi(\nu), \quad 0\leq\nu\leq \mathrm{L} \\ \psi(0,\varsigma) &= \psi(\mathrm{L},\varsigma)=0, \end{aligned} \tag{1.88}$$

where L,T $\in \mathbb{R}$, ${}^{C}\mathcal{D}_{\varsigma}^{\alpha}\psi(\nu,\varsigma)$ is the Caputo fractional derivative of order $0<\alpha\leq 1$ defined by

$$\frac{\partial^{\alpha}\psi(\nu,\varsigma)}{\partial\varsigma^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^{\varsigma}\frac{\partial\psi(\nu,\eta)}{\partial\eta}(\varsigma-\eta)^{-\alpha}d\eta, & 0<\alpha<1 \\ \dfrac{\partial\psi(\nu,\varsigma)}{\partial\varsigma}, & \alpha=1, \end{cases} \tag{1.89}$$

and ${}^{RL}\mathcal{D}_{\nu}^{\beta}\psi(\nu,\varsigma)$ is the Riemann-Liouville fractional derivative of order $1<\beta\leq 2$ defined by

$$\frac{\partial^{\beta}\psi(\nu,\varsigma)}{\partial\nu^{\beta}} = \begin{cases} \dfrac{1}{\Gamma(2-\beta)}\dfrac{\partial^2}{\partial\nu^2}\displaystyle\int_0^{\nu}\psi(\xi,\varsigma)(\nu-\xi)^{1-\beta}d\xi, & 1<\beta<2 \\ \dfrac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2}, & \beta=2. \end{cases} \tag{1.90}$$

When $\alpha=1$ and $\beta=2$ this equation (1.88) is the classical diffusion equation

$$\frac{\partial\psi(\nu,\varsigma)}{\partial\varsigma} = \frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2}. \tag{1.91}$$

Let us begin by providing a straightforward review of certain partial differential equations. Numerous nonlinear partial differential equations (NPDEs) exist within the domain of physics, notable among them being the Korteweg-de Vries (KdV) and Camassa-Holm (CH) equations. Investigating traveling wave solutions is particularly significant in the context of nonlinear partial differential equations, and these equations have been determined to exhibit various traveling wave solutions, yielding them of significant interest in the field (Nuseir, 2012; Camacho et al., 2017). Furthermore, (Clarkson et al., 1997) conducted an investigation into a category of third-order dispersive nonlinear equations represented in the following form

$$\begin{aligned} &\psi_{\varsigma}(\nu,\varsigma)-\alpha\psi_{\nu\nu\varsigma}(\nu,\varsigma)+2\kappa\psi_{\nu}(\nu,\varsigma)+\beta\psi(\nu,\varsigma)\psi_{\nu}(\nu,\varsigma) \\ &\quad=\gamma\psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma)+\omega\psi_{\nu}(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma), \end{aligned} \tag{1.92}$$

where α, κ, β, γ and ω are arbitrary parameters. It is very important to note that the above equation contains interesting different nonlinear equations such as the

Korteweg-de Vries equation, Degasperis-Procesi equation, Camassa-Holm equation, and Fornberg-Whitham equation. Therefore, by setting the coefficients for the nonlinear terms $\psi_\nu(\nu,\varsigma)$, $\psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma)$, $\psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma)$, $\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma)$ and $\psi_{\nu\nu\varsigma}(\nu,\varsigma)$ in Equation (1.92), we can obtain many different models formulations of nonlinear phenomena, which appear in the field of science and engineering. In the following, we present some special equations by considering specific values of these coefficients. For $\nu \in \mathbb{R}$ and $\varsigma > 0$, we observe

1. When $\beta = 6$, $\gamma = -1$ and $\alpha = \kappa = \omega = 0$ in Equation (1.92), then we obtain the well-known Korteweg-de Vries equation, which is a mathematical model of waves on shallow water surfaces that has smooth solitary wave solutions as given by (Lenells, 2004)

$$\psi_\varsigma(\nu,\varsigma) + 6\psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma) + \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) = 0. \tag{1.93}$$

2. When $\alpha = \gamma = 1$, $\kappa = 0$, $\beta = 4$ and $\omega = 3$ in Equation (1.92), then we obtain the Degasperis-Procesi equation, which is a mathematical model of nonlinear shallow water dynamics and has a variety of traveling wave solutions including solitary wave solutions as given by (Chen, 2015)

$$\begin{aligned} \psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + 4\psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma) \\ = \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) + 3\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma). \end{aligned} \tag{1.94}$$

3. When $\alpha = \omega = 1$, $\beta = 3$, $\gamma = 2$ in Equation (1.92) and $\kappa \in \mathbb{R}$ is a parameter related to the critical shallow water speed. Then we obtain the Camassa-Holm equation as given by (Lenells, 2004)

$$\begin{aligned} \psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + 2\kappa\psi_\nu(\nu,\varsigma) + 3\psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma) \\ = 2\psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) + \psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma), \end{aligned} \tag{1.95}$$

which is a model equation that describes the unidirectional propagation of shallow water waves over a flat bottom.

4. When $\alpha = \beta = \gamma = 1$, $\kappa = \frac{1}{2}$ and $\omega = 3$ in Equation (1.92), then we obtain the classical Fornberg-Whitham equation as given by (Whitham, 1967)

$$\begin{aligned} \psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + \psi_\nu(\nu,\varsigma) + \psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma) \\ = \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) + 3\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma), \end{aligned} \tag{1.96}$$

where $\psi(\nu,\varsigma)$ is the fluid velocity, ς is the time and ν is the spatial coordinate.

The fractional derivative is even more significant in modeling real-life situations; for example, the time-fractional partial nonlinear Fornberg-Whitham equation is a mathematical physics model that describes the evolution of nonlinear dispersive waves in fluid dynamics and the behavior of waves in plasma. Fornberg and Whitham obtained a peaked solution of the form $\psi(\nu,\varsigma) = \mathrm{K}\exp(-\frac{1}{2}|\nu - \frac{4}{3}\varsigma|)$, where K is an arbitrary constant. Among these models, we also mention that there is a wave-breaking model for the Fornberg-Whitham equation namely the Cauchy problem for the Fornberg-Whitham equation, which can be obtained, when $\alpha = 1$, $\kappa = -\frac{1}{2}$, $\beta = \gamma = \frac{3}{2}$ and $\omega = 3\beta$ in Equation (1.92), then it follows that

$$\begin{aligned} &\psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + \frac{3}{2}\psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma) - \psi_\nu(\nu,\varsigma) \\ &\quad = \frac{3}{2}\psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) + \frac{9}{2}\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma). \end{aligned} \tag{1.97}$$

The generalized Fornberg-Whitham equation is given by (Camacho et al., 2017)

$$\begin{aligned} &\psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + \alpha\psi^{\mathrm{n}}(\nu,\varsigma)\psi_\nu(\nu,\varsigma) + \beta\psi_\nu(\nu,\varsigma) \\ &\quad = \psi\psi_{\nu\nu\nu}(\nu,\varsigma) + 3\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma), \end{aligned} \tag{1.98}$$

where β, α and n are arbitrary parameters. If $\mathrm{n} = 2$ then we obtain the modified Fornberg-Whitham equation, which was proposed by (He et al., 2010). This can be observed by modifying the nonlinear term $\psi(\nu,\varsigma)\psi_\nu(\nu,\varsigma)$ in Equation (1.96) to $\psi^2(\nu,\varsigma)\psi_\nu(\nu,\varsigma)$ as follows:

$$\begin{aligned} &\psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + \psi_\nu(\nu,\varsigma) + \psi^2\psi_\nu(\nu,\varsigma) \\ &\quad = \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) + 3\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma). \end{aligned} \tag{1.99}$$

In addition to those already mentioned, (Wazwaz, 2006) studied the modified Camassa-Holm equation of the form

$$\begin{aligned} &\psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + 3\psi^2(\nu,\varsigma)\psi_\nu(\nu,\varsigma) \\ &\quad = 2\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma) + \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma), \end{aligned} \tag{1.100}$$

and the modified Degasperis-Procesi equation of the form

$$\begin{aligned} &\psi_\varsigma(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + 4\psi^2(\nu,\varsigma)\psi_\nu(\nu,\varsigma) \\ &\quad = 3\psi_\nu(\nu,\varsigma)\psi_{\nu\nu}(\nu,\varsigma) + \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma). \end{aligned} \tag{1.101}$$

1.5 Statement of the Problem

In the existing literature, many studies have indicated that fractional order models are more appropriate than the classical order models for providing applicable mathematical models and deriving approximate solutions, due to their accurate description of such nonlinear phenomena in various fields. Despite the significant progress made in fractional calculus and its application to diverse differential equations, many of the existing techniques for solving fractional partial differential equations lack efficiency, especially when applied to complex real-world systems with the absence of analytical solutions. While progress has been made, there remains a critical need for more robust semianalytical methods that can effectively solve such challenges, particularly in dealing with nonlinear models incorporating variable coefficients. Moreover, the classical numerical approaches often struggle with issues of convergence, stability, and computational cost, particularly when applied to noninteger order systems such as those modeled by incorporating variable coefficients with the Caputo fractional derivative or the Atangana-Baleanu-Caputo fractional derivative, which have not yet been thoroughly studied. The demand for more robust semianalytical methods that not only improve accuracy but also reduce computational overhead is evident. The current gap in literature lies in developing such methods, which can address these challenges while ensuring the theoretical rigor of existence and uniqueness of solutions. Thus, this research aims to explore approaches for solving fractional partial differential equations using innovative semianalytical methods, particularly focusing on the Caputo fractional derivative and the Atangana-Baleanu-Caputo fractional derivative. The goal is to improve the convergence of the approximate series solutions by identifying optimal intervals for the variable coefficients and demonstrating the consistency of the findings through different semianalytical methods on various test problems. This work also aims to provide practical tools for engineers and scientists working on mathematical models across various scientific fields, advancing both the theoretical understanding and practical application of fractional calculus.

1.6 Motivations of the Study

The motivation behind the selection of this research question arises from the fundamental assumption that fractional derivatives represent a broader range of physical phenomena compared to their integer order counterparts, thus offering a more accurate depiction or representation of the nonlinear behavior pervasive and common in diverse scientific and engineering fields. The significance of fractional derivatives in modeling real-life situations is pronounced across various scientific fields, offering deep insights into the behavior of complex systems. For example, the Fornberg-Whitham equation, a mathematical physics model, explains the evolution of nonlinear dispersive waves in fluid dynamics and the behavior of waves in plasma. This model's ability to incorporate fractional derivatives allows for capturing the complexities of wave propagation more accurately, enabling the simulation of a wider range of real-world scenarios and making more precise predictions about wave behavior under different conditions. In the context of wave propagation models, the incorporation of variable coefficients is critical to accommodate the spatial and temporal variations inherent in the medium through which the wave propagates. These variations can occur in various physical properties of the medium, such as density, elasticity, conductivity, and others, depending on the type of wave and the characteristics of the medium. The modified model with variable coefficients becomes a more accurate representation of real-world wave phenomena, thereby enhancing the investigation of fluid dynamics, plasma physics, and related fields. Moreover, including variable coefficients affords researchers and engineers the flexibility to adjust the wave propagation models to specific environments and study the effects of these variations. Further, the choice of fractional derivative definitions depends on the specific situation and the properties one aims to capture, allowing for a suitable selection to apply to the problem at hand. The cost of solving large linear or nonlinear systems can vary depending on factors such as the system's complexity and the solution method used. The different fractional definitions provided the opportunity to choose an appropriate one for the problem, further facilitating the accurate modeling of complex wave propagation phenomena.

1.7 Objectives of the Study

The objectives of this study are to:

1. derive approximate solutions for the classical Fornberg-Whitham equation to incorporate the Caputo fractional derivative with variable coefficients using the VIM, ADM, and HAM.
2. derive approximate solutions for the classical Fornberg-Whitham equation to incorporate the Atangana-Baleanu-Caputo fractional derivative with variable coefficients using the LVIM, LADM, and LHAM.
3. derive approximate solutions for the two-dimensional Helmholtz equation to incorporate the Caputo fractional derivative using the VIM.

1.8 Contributions of the Study

In our research, we engage in a multifaceted approach to generate and validate results. We begin by exploring and proving theorems related to this type of fractional partial differential equations, as well as solving complex problems relevant to our investigation. This active involvement in computation and proof emphasizes the depth of our research endeavor. Subsequently, to ensure the accuracy and reliability of our findings, we verified our results using Matlab, a powerful computational tool. By employing Matlab in this manner, we enhance the stringency and integrity of our research, leveraging its capabilities to confirm the validity of our theoretical and computational outcomes. The contributions of this thesis can be outlined as follows:

1. Establishment of sufficient conditions ensuring the existence of a singular solution to the fractional partial differential equations with variable coefficients, which have been expounded in Chapter 3.

2. Clarification of the considerations given to the convergence of these methods within the fractional partial differential equations with variable coefficients is detailed in Chapter 3.

3. Utilization of the previous condition (1) to estimate the maximum absolute truncated error of the series solution within numerical methods, expounded upon in Chapter 3.

4. Derivation of the approximate series solutions formulas relevant to the proposed methods for the problems is elaborated in Chapters 4, 5, and 6.

5. Presentation of several numerical examples illustrating the consistency of the obtained results through the proposed methods have been presented in Chapters 4, 5, and 6.

1.9 Outline of the Thesis

This thesis is devoted to achieving a semianalytical approximate solution of one dimensional time-dependent partial differential equations, with the considerations of two types of fractional derivatives, in the following outlines:

Chapter 1 offers an overview of the historical development of fractional calculus, and preliminary definitions and discusses specific properties. It revisits fundamental fractional concepts, integral transformations, outcomes derived from fractional calculus, and special functions of fractional differential equations. Special attention is given to the Mittag-Leffler function, Riemann-Liouville definition, and Caputo definition, recognizing their essential and pivotal roles in the theoretical framework of fractional differential equations. Moreover, this chapter presents the research objectives integral to this study, which will be elaborated upon in subsequent chapters.

Chapter 2 reviews the previous studies and provides a comprehensive review of the recent numerical methods for solving fractional partial differential equations with exact solutions. The presented methods aim to expand the scope of analytical or numerical solutions available for time-fractional partial differential equations and improve the accuracy and efficiency of the methods employed.

Chapter 3 deals with analyzing the convergence of variational iteration, Adomian decomposition, and homotopy analysis methods, along with their modified counterparts, as applied to nonlinear fractional partial differential equations. Specifically, the focus is on the fractional Fornberg-Whitham equation with variable coefficients. The chapter establishes sufficient conditions for convergence, provides error estimates to validate and quantify the accuracy of the obtained solutions, and outlines conditions for convergence based on the Banach's fixed point theorem. The utilization of a theorem, previously employed in other studies, serves as a sufficient condition for examining the convergence of the proposed methods across a broad scope of fractional order partial differential equations. Consequently, this theorem is extended within the context of this work.

Chapter 4 focuses on establishing a comprehensive analysis of the approximate solutions generated through the variational iteration, Adomian decomposition, and homotopy analysis methods. The primary objective is to utilize these three numerical techniques in approximating solutions for a one-dimensional time-dependent partial differential equation of fractional order characterized by variable coefficients. The differential equation under consideration is derived from the classical nonlinear Fornberg-Whitham equation by substituting the integer order derivative with the Caputo derivative of order $\mu \in (0, 1]$. This study considers homogeneous boundary conditions to establish an approximate series solution within the bounded space variable ν. To justify the efficacy of the proposed methods, two test problems are subjected to computational analysis. A comprehensive comparison between the results derived from the variational iteration method, Adomian decomposition method, and homotopy analysis method is presented through tables and graphs. The numerical findings illustrate the effectiveness of these numerical methods.

Chapter 5 is dedicated to establishing an analysis of the Laplace variational iteration method solutions, the Laplace Adomian decomposition method solutions, and the Laplace homotopy analysis method solutions. This chapter provides analytical and numerical solutions for partial differential equations involving time-fractional derivatives in the Atangana-Baleanu-Caputo sense. We implemented three powerful techniques to obtain an approximate solution for the bounded space variable ν. The Laplace transformation is used in the time-fractional derivative operator to enhance the performance and accuracy of the proposed numerical methods and find an approximate solution to fractional nonlinear Fornberg-Whitham equations. To confirm the accuracy of the proposed methods, we evaluate homogeneous fractional Fornberg-Whitham equations in terms of noninteger order and variable coefficients. The results obtained from the modified methods are shown through tables and graphs.

Chapter 6 introduces an efficient approach called the variational iteration method for solving linear and nonlinear differential equations of fractional order. This method represents a significant advancement in the field of fractional calculus. To demonstrate its effectiveness, we present two applications of the fractional variational iteration procedure to the linear Helmholtz equation. These applications showcase the simplicity and efficiency of the method in deriving analytical approximate solutions for fractional equations described by the Caputo derivative operator. Furthermore, the chapter explores the convergence analysis of the fractional variational iteration method using Banach's fixed point theorem. This analysis plays a crucial role in estimating the maximum absolute error values of the truncated approximate series solution. Additionally, we conduct a comparative study between the approximate solutions obtained through the fractional variational iteration method and the exact solution, presenting the results through graphical and tabular representations. This comparative analysis highlights the accuracy and reliability of the proposed method.

Chapter 7 concludes the thesis with a summary of some significant results and provides recommendations for further work.

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