PURSUIT DIFFERENTIAL GAMES OF INFINITE THREE-DIMENSIONAL SYSTEM OF DIFFERENTIAL EQUATIONS IN HILBERT SPACE

By

DIVIEKGA NAIR A/P MADHAVAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To my dear self
To my parents Mr Madhavan G.Gopal and Ms Suguna Ellappan
To my supervisor Associate Professor Dr Idham Arif Hj Alias

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chair: Idham Arif bin Hj Alias
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Early research on game problems described by a system of partial differential equations is followed by a reduction to those described by an infinite system of ordinary differential equations using the method of decomposition. Every infinite n-system of ordinary differential equations, $n \geq 2$ has a solution with a unique fundamental matrix which is then applied to study differential games in various perspectives. This thesis focuses in finding solutions to pursuit differential game problems of an infinite 3-system of first order ordinary differential equations in Hilbert space l_2. The model of the game is first formulated and then rewritten in a matrix form. The homogeneous solution of the model is obtained where a fundamental matrix is identified. Some notable properties of the fundamental matrix are proved and applied to find the particular solution of the model and simplify the calculations in the study of the differential game. The existence and uniqueness of the general solution of the game model in l_2 space are then proved.

The study of pursuit game begins with the problem of one pursuer and one evader where the pursuer aims to bring the state of the system from an initial state to the origin. On the other hand, the evader tries to prevent this from occurring as it moves freely. The game is studied separately with two different types of constraints on players' control functions, which are integral and geometric constraints. The control problem is studied where the control function is first constructed and then shown to be admissible. The control function is to transfer the state of the system into origin and to be applied in construction of admissible strategy for the pursuer. Sufficient conditions are obtained for pursuer to complete the pursuit in a finite time interval.

This thesis also examines pursuit differential games of both integral and geometric constraints where the pursuer's motive is to transfer an initial non zero state of the system into another non zero state. This investigation also requires the control problem to be solved so that it can be used to establish an admissible strategy for the pursuer to bring the system to another non zero state within a finite time interval.

A more refined study is carried out to solve optimal pursuit problem of the game with integral constraints where the evader moves with its own strategy rather than moving freely. In this investigation, an optimal control function is constructed and proven to be admissible. It is then utilised in establishing optimal strategies for both pursuer and evader to achieve the optimal pursuit time of the game.

The final part of this thesis is about a study of pursuit game that involve finitely many pursuers versus one evader with model of the game is similar to the model of the previous games. The control function of each player is subjected to integral constraint. It is assumed that the combined resources of all pursuers is greater than the resource of the evader. An admissible strategy for each pursuer is constructed where two cases are considered. The first case is to show that the game of pursuit can be terminated by one of the pursuers at some time in a finite time interval in which the evader moves freely. The second case is to find the optimal number of pursuers needed to terminate the game in which the evader moves with constructed admissible strategy.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PERMAINAN PEMBEZAAN PENGEJARAN SISTEM TIGA-DIMENSI TAK TERHINGGA PERSAMAAN PEMBEZAAN DI DALAM RUANG HILBERT

Oleh

DIVIEKGA NAIR A/P MADHAVAN

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Kajian awal mengenai masalah permainan yang diperihalkan oleh sistem persamaan pembezaan separa diikuti dengan penurunan kepada yang diperihalkan oleh sistem tak terhingga persamaan pembezaan biasa menggunakan kaedah penguraian. Setiap n-sistem tak terhingga persamaan pembezaan biasa, $n \geq 2$ mempunyai penyelesaian dengan matrik asasi unik yang kemudiannya diaplikasikan untuk mengkaji permainan pembezaan dalam pelbagai perspektif. Tesis ini memfokuskan dalam mencari penyelesaian kepada masalah permainan pembezaan pengejaran 3-sistem tak terhingga tertib pertama persamaan pembezaan biasa di dalam ruang Hilbert l_2. Model permainan telah diformulasi terlebih dahulu dan kemudiannya ditulis semula dalam bentuk matrik. Model penyelesaian homogen telah diperoleh di mana matrik asasi telah dikenal pasti. Beberapa sifat penting matrik asasi telah dibuktikan dan diaplikasikan untuk mencari penyelesaian khusus model dan mempermudah pengiraan dalam kajian permainan perbezaan. Kewujudan dan keunikan penyelesaian am model permainan di dalam ruang l_2 kemudiannya telah dibuktikan.

Kajian permainan pengejaran bermula dengan masalah satu pengejar dan satu pengelak di mana pengejar bermatlamat untuk membawa keadaan sistem dari keadaan awal ke asalan. Sebaliknya, pengelak cuba untuk menghalangnya daripada berlaku di mana ia bergerak bebas. Permainan ini telah dipelajari secara berasingan dengan dua jenis kekangan yang berbeza pada fungsi kawalan pemain, iaitu kekangan kamiran dan kekangan geometri. Masalah kawalan telah dipelajari di mana fungsi kawalan telah dibina terlebih dahulu dan kemudiannya ditunjukkan teraku. Fungsi kawalan tersebut ialah untuk memindahkan keadaan sistem ke asalan dan telah diap-

likasikan untuk membina strategi teraku pengejar. Syarat cukup telah diperoleh bagi pengejar melengkapkan pengejaran dalam suatu selang masa terhingga.

Tesis ini turut meneliti permainan perbezaan pengejaran bagi kedua-dua kekangan kamiran dan geometri di mana motif pengejar ialah untuk memindahkan sistem dalam keadaan awal bukan sifar ke keadaan bukan sifar yang lain. Kajian ini juga memerlukan masalah kawalan diselesaikan supaya ianya boleh digunakan untuk membina strategi teraku pengejar untuk membawa sistem ke keadaan bukan sifar yang lain dalam suatu selang masa terhingga.

Kajian yang lebih terperinci telah dilaksanakan untuk menyelesaikan masalah pengejaran optimal bagi permainan dengan kekangan kamiran di mana pengelak bergerak dengan strateginya yang tersendiri berbanding bergerak secara bebas. Dalam kajian ini, fungsi kawalan optimal telah dibina dan dibuktikan terakukan. Ia kemudiannya digunakan dalam membina strategi optimal untuk kedua-dua pengejar dan pengelak bagi mencapai masa pengejaran optimal permainan.

Bahagian terakhir tesis ini adalah mengenai kajian permainan pengejaran yang melibatkan pengejar yang terhingga banyaknya melawan satu pengelak dengan model permainan yang serupa dengan model permainan sebelumnya. Fungsi kawalan setiap pemain adalah tertakluk kepada kekangan kamiran. Telah diandaikan bahawa gabungan sumber semua pengejar adalah lebih besar daripada sumber pengelak. Strategi teraku setiap pengejar telah dibina di mana dua kes telah dipertimbangkan. Kes pertama adalah untuk menunjukkan bahawa permainan pengejaran boleh ditamatkan oleh satu pengejar pada suatu waktu dalam selang masa terhingga di mana pengelak bergerak bebas. Kes kedua adalah untuk mencari bilangan optimal pengejar yang diperlukan untuk menamatkan permainan di mana pengelak bergerak dengan strategi teraku yang dibina.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

l_2	$\{\rho = (\rho_1, \rho_2, ...) : \sum_{k=1}^{\infty} \rho_k^2 < \infty\}$
$C(0,T;l_2)$	Space of continuous functions space on $[0,T]$ in l_2 space
$L_2(0,T;l_2)$	Space of square integrable functions on $[0,T]$ in l_2 space
$\mu(\cdot)$	State of the system at time t for $t \in [0,T]$
$\boldsymbol{\phi}_k(t)$	Fundamental matrix of the solution at time t, $k = 1,2,...,$
μ^0	Initial non zero state of the system
μ^1	Another non zero state of the system
$w(\cdot)$	Control function at time t for $t \in [0,T]$
$u(\cdot)$	Control function of pursuer at time t for $t \in [0,T]$
$v(\cdot)$	Control function of evader at time t for $t \in [0,T]$
$\|\|\boldsymbol{A}\|\|$	Norm of matrix $\boldsymbol{A}$
$\boldsymbol{A}^*$	Transpose of matrix $\boldsymbol{A}$
$\boldsymbol{\varphi}_k(t_0,t_1)$	$\int_{t_0}^{t_1} \boldsymbol{\phi}_k(-s)\boldsymbol{\phi}_k^*(-s)ds, k = 1,2,...,$

CHAPTER 1

INTRODUCTION

Overview of the Chapter

This chapter describes the basic concepts of game theory followed by differential game theory and some fundamental concepts that are related to our study. Besides that, this chapter covers the method of decomposition that reduces a system of partial differential equations into an infinite system of ordinary differential equations. We also discuss the motivation, problem statement, objectives, scope, methodology and organisation of the thesis.

1.1 Game Theory

The development of knowledge in various areas or disciplines including game theory is due to the continuous efforts taken by researchers. These efforts include examination of real-life phenomena, investigation and finding solution of any problems that arise, implementation of reforms and improvement, and development of novel theories for universal use. Game theory serves as a decision-making tool in conflicting scenarios involving two parties of opposite objectives. It establishes some acceptable strategies to be utilised by the party that want to accomplish its goal.

Some of the well-known games in game theory are the Achilles and Tortoise race and the Lion and Man puzzle. The race between Achilles (pursuer) and a tortoise (evader) is a paradox introduced by Zeno (Sainsbury (2009)), in which both of them move on a straight line and in the same direction with the tortoise given a head start. Zeno asserted that no matter how fast Achilles runs, he will never be able to catch up with the tortoise because the tortoise will always be ahead. However, this problem is proved to be solvable by many researchers such as Ardourel (2015) and Driessen (2018).

The puzzle of a lion (pursuer) and a man (evader) takes place in a circle-shaped arena in which the lion wants to capture the man with the man avoids being captured. The game is considered in three different circumstances depending on the speed of the players. The lion seeks to catch the man at the shortest possible time but the man tries to keep the evasion at the longest possible time. In particular, if the speed of the lion equals to the speed of the man, it has been proven that evasion is possible with a smallest distance between both of them (Nahin (2007)).

1.2 Differential Game Theory

The theory of differential game is a cross between game theory and optimal The theory of optimal control deals with determination of control and state trajectories for dynamic system by considering the objective function to produce an optimized system. Differential game theory is representation of game theory involving two moving players which are governed by some differential equations but of contrasting objectives, in which optimal control theory is applied to develop strategized control for these dynamic players. Thus, the main focus of differential game theory is to describe the game and construct admissible strategy based on the control function to be used by the players to move accordingly in order to accomplish their respective objective and achieve the termination of the game.

The solution to a differential game problem depends on the design of the game which varies according to the equation of game, the space of game, the objective of players, the number of players, the type of constraints imposed on the control functions of players and the type of the game.

Differential game may occur in a finite dimensional space such as Euclidean space $\mathbb{R}^2, \mathbb{R}^3$ or $\mathbb{R}^n, n \in \mathbb{Z}$ or an infinite dimensional space like Hilbert space l_2 or l^2_{r+1}. Specifically, the game in an infinite dimensional space is described by an infinite system.

There are two kinds of moving players with contrasting objectives, that are:

1. Pursuer who intends to capture evader in which the position of the pursuer coincides with the position of the evader at some time on a finite time interval to complete the pursuit.

2. Evader who aims to avoid being caught by pursuer in which the state of the evader does not coincide with the state of the pursuer either on a finite time interval (definitely) or for all time (indefinitely) to ensure the evasion is possible.

In an infinite system, pursuer wants to transfer a state of the system into another state at some time to complete the pursuit. On the other hand, evader strives to avoid that from happening.

The players involved in the game could be in the form of one pursuer-one evader, one pursuer-many evaders, many pursuers-one evader or many pursuers-many evaders.

Generally, the control function of pursuer at time t is given by $u(t)$ and the control function of evader at time t is referred as $v(t)$. In every differential game, control function of the players are subjected to some types of constraints and the two most common are the following;

1. Geometric constraints is a norm-based constraints which usually limits the speed of players in the game. The general form of control functions $u(t)$ of pursuer and $v(t)$ of evader constrained by geometric constraints is given by

   $$||u(t)|| \leq \rho, \quad t \in [t_0, T],$$

   and

   $$||v(t)|| \leq \sigma, \quad t \in [t_0, T],$$

   respectively where ρ, σ are given positive numbers, $||u(t)||$ represents the speed of the pursuer and $||v(t)||$ is that of the evader.

2. Integral constraints is an integral-based constraints which restricts the total resources of players that can be exhausted due to consumption such as energy or fuel. The general form of control functions $u(t)$ of pursuer and $v(t)$ of evader constrained by integral constraints is given by

   $$\int_{t_0}^{T} ||u(t)||^2 dt \leq \rho^2$$

   and

   $$\int_{t_0}^{T} ||v(t)||^2 dt \leq \sigma^2$$

   respectively where ρ, σ are given positive numbers, $\int_{t_0}^{T} ||u(t)||^2 dt$ indicates the total resources of the pursuer and $\int_{t_0}^{T} ||v(t)||^2 dt$ represents the total resources of the evader.

Note that t_0 is an initial time and T is a finishing time of the game.

In order to win the game, players are equipped with some admissible strategies that adhere to some constraints. There are two types of differential game described as follows;

1. Pursuit differential game is a game where pursuer moves in accordance to constructed admissible strategy that guarantees the pursuit can be completed at a guaranteed pursuit time, while the evader moves at its discretion. However, the game could end at an earlier time known as optimal pursuit time. It is the earliest possible time that the pursuer can catch the evader as the evasion is still possible before that time, since the evader also moves with its own admissible strategy.

2. Evasion differential game is a game where evader's movement is based on constructed admissible strategy that ensures the evasion is possible in the game,

against the freely moving pursuer.

The game is called as pursuit-evasion differential game when both pursuit and evasion games are examined. However, the conditions set for these two games are different from one another and hence strategies are developed separately for pursuer and evader.

1.3 Fundamentals Concepts

In this section, we introduce some fundamental concepts that are related to our study.

1.3.1 Basic Definitions and Results

In this subsection, basic definitions and results are stated, together with some explanations wherever necessary.

Definition 1.3.1 *Homogeneous Linear Differential Equations (Arnold (1992))*
The equation of the form

$$\frac{dy}{dx} = f(x)y$$

is a first-order homogeneous linear differential equations.

(Cain and Reynolds (2010)) Suppose that $y_1, y_2, ..., y_n$ are dependent variable and t is an independent variable. The system of first order linear differential equations with constant coefficients is in the form of

$$\begin{aligned}
y_1'(t) &= a_{11}y_1(t) + a_{12}y_2(t) + ... + a_{1n}y_n(t) + f_1(t) \\
y_2'(t) &= a_{21}y_1(t) + a_{22}y_2(t) + ... + a_{2n}y_n(t) + f_2(t) \\
&\vdots \\
y_n'(t) &= a_{n1}y_1(t) + a_{n2}y_2(t) + ... + a_{nn}y_n(t) + f_n(t)
\end{aligned}$$

where a_{ij} is a constant, $1 \leq i \leq j \leq n$ and $f_i(t)$ for $i = 1, 2, ..., n$ are either constants or functions of t.

Lemma 1.3.1 *Unique solution of Initial Value Problem (Hartman (1982))*
The initial value problem

$$y' = \boldsymbol{A}(t)y + f(t)$$

where $\boldsymbol{A}(t)$ is a continuous $n \times n$ matrix and $f(t)$ a continuous vector on $t \in [a,b]$

and

$$y(t_0) = y_0, \quad t_0 \in [a,b]$$

has a unique solution $y = y(t)$ and $y(t)$ exists on $t \in [a,b]$.

Definition 1.3.2 *Eigenvalue and corresponding eigenvectors (Cain and Reynolds (2010))*
Suppose $\boldsymbol{A}$ is a square matrix. An eigenvalue for $\boldsymbol{A}$ is given by scalar λ if there exists a non-zero vector v such that $\boldsymbol{A}v = \lambda v$. Any non-zero vector which satisfy the equality is known as eigenvector corresponding to the eigenvalue λ.

Definition 1.3.3 *Characteristics polynomial of matrix (Cain and Reynolds (2010))*
Let $\boldsymbol{A}$ be a $n \times n$ matrix and $\boldsymbol{I}$ be a $n \times n$ identity matrix. The equation $det(\boldsymbol{A} - \boldsymbol{\lambda I}) = 0$ indicates the characteristics equation of matrix $\boldsymbol{A}$ whereas $det(\boldsymbol{A} - \boldsymbol{\lambda I})$ represents the characteristics polynomial of matrix $\boldsymbol{A}$.

Definition 1.3.4 *Partial differential equations (Cain and Reynolds (2010))*
Let $u = u(x,t)$, then equation defined by $F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},u_{tx},...) = 0$ is a partial differential equations. A partial differential equations is an equation bearing an unknown function made up of several variables and its partial derivatives with respect to the variables.

Definition 1.3.5 *Inner product space (Alabiso and Weiss (2014))*
Let V be a linear space over $\mathbb{R}$ or $\mathbb{C}$. The function $\langle \cdot, \cdot \rangle$ is said to be an inner product if for all $\alpha, \beta, \gamma \in V$ and $p, q \in \mathbb{R}$ or $\mathbb{C}$ and obeys the following;

i. *Symmetry:* $\langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}$,
ii. *Linearity:* $\langle \alpha, p\beta + q\gamma \rangle = p\langle \alpha, \beta \rangle + q\langle \alpha, \gamma \rangle$,
iii. *Positive Definite:* $\langle \alpha, \alpha \rangle \geq 0, \forall \alpha \in V$,
iv. $\langle \alpha, \alpha \rangle = 0$ *if and only if* $\alpha = 0$.

Definition 1.3.6 *Vector space (Muscat (2014))*
The vector space V over a field $\mathbb{F}$ is a set that is closed under addition and scalar multiplication. Let $\alpha, \beta, \gamma \in V$ and $p, q \in \mathbb{F}$, then the vector space V holds

i. *Additive associativity:* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
ii. *Additive commutativity:* $\alpha + \beta = \beta + \alpha$,
iii. *Additive identity:* $0 + \alpha = \alpha + 0 = \alpha$,
iv. *Addictive inverse:* $\alpha + (-\alpha) = 0$,
v. *Multiplicative associativity:* $p(q\alpha) = (pq)\alpha$,

vi. Scalar Sums Distributivity: $(p+q)\alpha = p\alpha + q\alpha$,

vii. Vector Sums Distributivity: $p(\alpha+\beta) = p\alpha + p\beta$,

viii. Multiplicative identity: $1\cdot\alpha = \alpha\cdot 1 = \alpha$.

(Cloud et al. (2014)) Let x, $y \in \mathbb{R}^n$ where $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R}\}$,

i. the inner product is given by

$$< x,y >= \sum_{k=1}^{n} x_k y_k,$$

ii. and norm is given by

$$\begin{aligned} x &= \sqrt{< x,x >} \\ &= \sqrt{\sum_{k=1}^{n} x_k^2}. \end{aligned}$$

Definition 1.3.7 *Normed space (Alabiso and Weiss (2014)), (Muscat (2014))*
A linear space V over $\mathbb{R}$ or $\mathbb{C}$ with the function called norm $||\cdot|| : V \longrightarrow \mathbb{R}$ or $\mathbb{C}$ for any $x, y \in \mathbb{V}$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$ which satisfies

i. Homogeneity: $||\lambda x|| = |\lambda|||x||$,

ii. Triangle Inequality: $||x+y|| \leq ||x|| + ||y||$,

iii. Positivity: $||x|| \geq 0$, $||x|| = 0$ *iff* $x = 0$,

iv. Linearity: $||-x|| = ||x||$,

v. Reverse Triangle Inequality: $||x-y|| \geq ||x|| - ||y||$,

vi. $||x_1 + x_2 + \ldots + x_n|| \leq ||x_1|| + ||x_2|| + \ldots + ||x_n||$.

Definition 1.3.8 *Matrix norm (Horn and Johnson (2012))*
A function $||\cdot|| : \boldsymbol{M_n} \rightarrow \mathbb{R}$ is a matrix norm, where $\boldsymbol{M_n}$ is a $n \times n$ dimension, if it satisfies

i. Non-negativity: $||\boldsymbol{A}|| \geq 0$

ii. Positivity: $||\boldsymbol{A}|| = 0$ *iff* $\boldsymbol{A} = 0$,

iii. Homogeneity: $||c\boldsymbol{A}|| = |c|||\boldsymbol{A}||, \quad \forall c \in \mathbb{R}$,

iv. Triangle Inequality: $||\boldsymbol{A}+\boldsymbol{B}|| \leq ||\boldsymbol{A}|| + ||\boldsymbol{B}||$,

v. *Submultiplicativity:* $||\boldsymbol{AB}|| \leq ||\boldsymbol{A}||||\boldsymbol{B}||$,

for all $\boldsymbol{A},\boldsymbol{B} \in \boldsymbol{M_n}$.

Definition 1.3.9 *Induced matrix norm (Horn and Johnson (2012)), (Ford (2014))*
Let $||\cdot||$ *is a vector norm,* $\boldsymbol{A}$ *is a* $m \times n$ *matrix and* x *is a* $n \times 1$ *vector. The matrix norm of* $\boldsymbol{A}$ *induced by* $||\cdot||$ *is*

$$||\boldsymbol{A}|| = \max_{||x||=1} ||\boldsymbol{A}x||.$$

Particularly, from Cloud et al. (2014), $||\boldsymbol{A}||$ is a number such that

$$||\boldsymbol{A}x|| \leq ||\boldsymbol{A}||||x||$$

for all $x \in \mathbb{R}^n$.

Definition 1.3.10 *Measurable criterion (Kadets (2018))*
Let $(\Omega_1,\sum_1)$ *and* $(\Omega_2,\sum_2)$ *be sets endowed with* σ*-algebras of its subsets. A map of* $f:\Omega_1 \to \Omega_2$ *is said to be measurable if* $f^{-1}(A) \in \sum_1$ *for all* $A \in \sum_1$.

(Kadets (2018)) Let $(\Omega,\sum)$ be a set endowed with a σ-algebra of its subsets and all the functions are assumed to be defined on Ω. The elements of the σ-algebra are referred as measurable sets.

Definition 1.3.11 *Measurable function (Kadets (2018))*
A function f *on* Ω *is said to be measurable with respect to the* σ*-algebra* $\sum$*, if for any Borel subset* $A \subset R$*, the set* $f^{-1}(A)$ *is measurable.*

(Kadets (2018)) A set $E \subset \Omega$ is said to be set of measure zero if for every positive ε, there exists a finite or countable number of open intervals $I_1,I_2,I_3,...$ such that $E \subset_j I$ and $\sum_j |I_j| < \varepsilon$. Set E is contained in a measurable set of measure zero.

(Kadets (2018)) The property P about points of the set Ω is said to hold almost everywhere if set of all points $t, t \in \Omega$ where P is false is negligible.

Definition 1.3.12 *Continuous function (Bartle and Sherbert (2000))*
Let $A \subseteq \mathbb{R}$ *and* $c \in A$*. The function* $f:A \to \mathbb{R}$ *is considered continuous at a point* $x=c$ *if* $\forall\varepsilon > 0$, $\exists\delta > 0$ *such that* $|f(x)-f(c)| < \varepsilon$ *whenever* $|x-c| < \delta$*. If f fails to be continuous at c, then* f *is discontinuous at c.*

(Hoffman (2011)) Every differentiable function is continuous, but some continuous functions are not differentiable. Every continuous function is integrable, but some integrable functions are not continuous. Differentiablity implies continuity and continuity implies integrability.

Definition 1.3.13 *Absolute continuous function (Berberian (2013))*
A function $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous if for all $\varepsilon > 0$, there exists a positive δ such that pairwise disjoint subintervals $I_1, I_2, ..., I_n$ of $[a,b]$ where $I_k, k = 1,2,...,n$, has endpoints a_k, b_k with $a_k \leq b_k, k = 1,2,...,n$, obeys

$$\sum_{k=1}^{n} b_k - a_k \leq d,$$

which then implies

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \varepsilon.$$

Definition 1.3.14 *Cluster point (Bartle and Sherbert (2000))*
Let $A \subseteq \mathbb{R}$. A point $a \in \mathbb{R}$ is called cluster point of A if $\forall \delta > 0, \exists x \in A$ such that $|x - c| < \delta$ where $x \neq c$.

Definition 1.3.15 *Limit of a point (Bartle and Sherbert (2000))*
Let $A \subseteq \mathbb{R}$ and c be cluster point of A. The function $f : A \to \mathbb{R}$ be defined on an open interval around c and limit of $f(x)$ as x approaches c is L, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

Theorem 1.3.1 *(Bartle and Sherbert (2000)) A Cauchy sequence of real numbers is bounded.*

Theorem 1.3.2 *Cauchy sequence (Bartle and Sherbert (2000))*
The sequence $X = \{X_n\}$ is called Cauchy sequence if $\forall \varepsilon > 0$, $\exists$ natural number N such that $||X_m - X_n|| < \varepsilon$ for every $m, n \geq N$.

Lemma 1.3.2 *Cauchy convergence criterion (Bartle and Sherbert (2000))*
A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Definition 1.3.16 *Bounded sequence (Bartle and Sherbert (2000))*
$S = \{S_n\}$ is a bounded sequence if $|S_n| < \beta$ for some $\beta \in \mathbb{R}$ for all $n \in \mathbb{Z}$.

Definition 1.3.17 *Null sequence (Knopp (1956))*
$S = \{S_n\}$ is a null sequence if $\forall \varepsilon > 0$, $\exists N \in \mathbb{Z}^+$ such that $|S_n| < \varepsilon$ whenever $n \geq N$. Null sequence is a sequence which converges to 0.

(Knopp (1956)) Infinite series is given by

$$S = \sum_{n=1}^{\infty} S_n = S_1 + S_2 + S_3 + ...$$

with its partial sums that is

$$s_k = \sum_{n=1}^{k} S_n = S_1 + S_2 + ... + S_k.$$

Convergence of sequence of partial sums s_k implies the convergence of infinite series S. That is, if $\lim_{k\to\infty} s_k = L$, then $\sum_{n=1}^{\infty} S_n = L$.

1.3.2 Basic Properties of Hilbert space l_2

In our study, the game takes place in Hilbert space l_2 and thus the basic information of l_2 space are given as follows.

Definition 1.3.18 *Hilbert space (Muscat (2014))*
A Hilbert space is a complete inner product space given by

$$l_2 = \{\rho = (\rho_1, \rho_2, ...) : \sum_{k=1}^{\infty} |\rho_k|^2 < \infty\}$$

where

i. *the inner product of $\rho, \varsigma \in l_2$ is defined as*

$$\begin{aligned} <\rho, \varsigma> &= \sum_{k=1}^{\infty} \rho_k \varsigma_k \\ &< \infty, \end{aligned}$$

ii. *and the norm of $\rho \in l_2$ is defined as*

$$\begin{aligned} ||\rho|| &= \sqrt{<\rho, \rho>} \\ &= \sqrt{\sum_{k=1}^{\infty} \rho_k^2} \\ &< \infty. \end{aligned}$$

(Cloud et al. (2014)) The space $C(0,T;l_2)$ is a space of continuous function in Hilbert space l_2 on time interval $[0,T]$ where

i. the inner product of $z(t)$:

$$\sum_{k=1}^{\infty} z_k^2(t) < \infty,$$

ii. and the norm of $z(t)$:

$$\sqrt{\sum_{k=1}^{\infty} z_k^2(t)} < \infty,$$

for $z(t) = (z_1(t), z_2(t), ...) \in l_2$.

(Cloud et al. (2014)) The space $L_2(0,T;l_2)$ is a space of square-integrable function in Hilbert space l_2 on time interval $[0,T]$ where

i. the inner product of $f(t)$:

$$< f(t), f(t) > = \sum_{k=1}^{\infty} \int_0^T f_k^2(t)dt$$
$$< \infty,$$

ii. and the norm of $f(t)$:

$$||f(t)|| = \sqrt{\sum_{k=1}^{\infty} \int_0^T f_k^2(t)dt}$$
$$< \infty,$$

for $f(t) = (f_1(t), f_2(t), ...) \in l_2$.

1.3.3 Basic Inequalities

The definitions listed in this subsection deals with some basic concepts of inequalities that are used in our study.

Corollary 1.3.1 *(Pachpatte (2005)) If $a_1, a_2, ..., a_n$ are real numbers, then the inequality*

$$(a_1 + a_2 + ... + a_n)^2 \leq n(a_1^2 + a_2^2 + ... + a_n^2)$$

holds.

Theorem 1.3.3 *Cauchy-Schwarz Inequality (Alabiso and Weiss (2014))*
If $\mathbb{V}$ is an inner product space, then for any $x, y \in \mathbb{V}$, the Cauchy-Schwarz inequality is as follows;

$$| < x, y > | \leq ||x|| \cdot ||y||.$$

Also, from Bernstein (2009), $|x^*y| \leq ||x||_2||y||_2$ where x^* is tranpose of x.

Theorem 1.3.4 *Cauchy-Schwarz summation inequality (Cloud et al. (2014))*
Let $a_k, b_k \in \mathbb{R}$ for $k = 1,2,...n$. Then,

$$\Big|\sum_{k=1}^{n} a_k b_k\Big| \leq \Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2}\Big(\sum_{k=1}^{n} b_k^2\Big)^{1/2},$$

equivalently,

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2.$$

The inequality is also valid for infinite series.

In Euclidean space $\mathbb{R}^n$, where $x =< x_1, x_2, ..., x_n >, y =< y_1, y_2, ..., y_n > \in \mathbb{R}^n$, we have

$$< x, y > = x_1 y_1 + x_2 y_2 + ... + x_n y_n,$$

and

$$< x, x >= x_1^2 + x_2^2 + ... + x_n^2,$$

also,

$$< y, y >= y_1^2 + y_2^2 + ... + y_n^2.$$

According to Cauchy- Schwarz inequality, we have

$$(x_1 y_1 + x_2 y_2 + ... + x_n y_n)^2 \leq (x_1^2 + x_2^2 + ... + x_n^2) \cdot (y_1^2 + y_2^2 + ... + y_n^2)$$

$$i.e. \qquad \big(\sum_{k=1}^{n} x_k y_k\big)^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2.$$

Theorem 1.3.5 *Cauchy-Schwarz integral inequality (Cloud et al. (2014))*
For $f(x), g(x) \in C[a,b]$, the Cauchy-Schwarz inequality is

$$\Big(\int_a^b f(x)g(x)dx\Big)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

Theorem 1.3.6 *Minkowski's Inequality (Alabiso and Weiss (2014))*
The inequality

$$||x+y||_p \leq ||x||_p + ||y||_p$$

holds for every $x, y \in l_p$ where $1 \leq p < \infty$.

Theorem 1.3.7 *Minkowski's summation inequality (Cloud et al. (2014))*
Let $a_k, b_k \in \mathbb{R}$ for $k = 1,2,...,n$. We have Minkowski's inequality in the form of

$$\Big(\sum_{k=1}^{n} |a_k + b_k|^p\Big)^{1/p} \leq \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{n} |b_k|^p\Big)^{1/p}.$$

The inequality is also valid for infinite series.

Theorem 1.3.8 *Minkowski's integral inequality (Cloud et al. (2014))*
Let $f(x), g(x) \in C[a,b]$. The Minkowski's inequality is

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}.$$

1.4 Reduction of Partial to Ordinary Differential Equations

A variety of real-world issues or time evolutionary problems where the model of the issue is characterised by partial differential equations. The state of the system depends on more than one independent variables. In the midst of the studies, the approach of decomposition technique is employed by the researchers to create an infinite system of ordinary differential equations from a system of partial differential equations.

This chapter discusses an overview of the method of decomposition (refer Ivanov and Avdonin (1995)).

1.4.1 System of Partial Differential Equations

A system of the game can be described by partial differential equations as follows;

$$\begin{aligned} \frac{\partial \bar{z}}{\partial t} &= B\bar{z} - u + v, \\ \bar{z}(x,0) &= \bar{z}_0(x), \quad x \in \psi, \end{aligned} \tag{1.4.1}$$

where

$$B\bar{z} = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(b_{jk}(x)\frac{\partial \bar{z}}{\partial x_k}\right), \quad x \in \psi. \tag{1.4.2}$$

The function $\bar{z}(x,t)$ which indicates the state of the system is made up of two variables that are coordinates vector $x = (x_1, x_2, ..., x_n) \in \psi \subset \mathbb{R}^n, n \geq 1$ and time t, while $\bar{z}(x,0)$ refers to the state of the system at initial time $t = 0$. In addition, functions $\bar{u}(x,t)$ and $\bar{v}(x,t)$ are control function of pursuer and control function of evader respectively. Let $\bar{w}(x,t) = -\bar{u}(x,t) + \bar{v}(x,t)$ be control function of the system. Notice that linear differential operator (1.4.2) is dependent of coordinates vector and independent of time t. The measurable bounded function $b_{jk}(x) = b_{kj}(x)$ obeys the conditions

$$c^2\sum_{j=1}^{n}\xi_j^2 \le \sum_{j,k=1}^{n} b_{jk}(x)\xi_j\xi_k, \qquad (1.4.3)$$
$$\forall(\xi_1,...,\xi_n)\in\mathbb{R}^n, \quad x\in\psi.$$

where c is a constant, $c \neq 0$ (Satimov and Tukhtasinov (2005a)). The notation $x \in \psi$ tells that the coordinate vector belongs to some bounded domain ψ for $t > 0$. Meanwhile, the boundary of the domain ψ is denoted by ψ' where ψ' is assumed to be piecewise smooth on finite time interval $[0,T]$ for $T > 0$.

First, as stated in Ibragimov (2002) and Satimov and Tukhtasinov (2005b), we define

$$l_r = \left\{\beta = (\beta_1,\beta_2,...) | \sum_{i=1}^{\infty}\mu_i{}^r\beta_i{}^2 < \infty\right\},$$

with inner product and norm given by:

$$<\beta,\gamma> = \sum_{i=1}^{\infty}|\mu_i|^r\beta_i\gamma_i, \quad \beta,\gamma\in l_r,$$
$$||\beta||_r = \sqrt{\sum_{i=1}^{\infty}\mu_i^r\beta_i{}^2}.$$

Also, space H_r as

$$H_r = H_r(\psi) = \left\{g\in L_2(\psi) | g = \sum_{i=1}^{\infty}\beta_iF_i, \beta_i\in l_r\right\}, \quad r\ge 0. \qquad (1.4.4)$$

It is clear that $||g|| = ||\beta||$ and $<g,h>=<\beta,\gamma>$ where $h = \sum_{i=1}^{\infty}\gamma_iF_i$. Observe that, (Ibragimov (2002)),

i. $C(0,T;H_r)$ is the space of continuous function in H_r on time interval $[0,T]$,

ii. $L_2(0,T;H_r)$ is the space of square integrable function in H_r on time interval $[0,T]$

where $T > 0$. When $r > 0$, $H_{r+1}(\psi)\subset H_r(\psi)$ and when $r = 0$, $H_0(\psi) = L_2(\psi)$.

Now, we are up to consider an eigenvalue problem stated as

$$BF(x) = -\mu(x)F(x), \quad x\in\psi,$$
$$JF(x) = 0, \quad x\in\psi'. \qquad (1.4.5)$$

From the solving process of eigenvalue problem (1.4.5), we can note that the linear operator (1.4.2) owns a spectrum of positive eigenvalues $\mu_i > 0$, that is $\mu_i \to +\infty$ as $i \to \infty$. The eigenfunction $F_i(x)$ corresponding to the obtained eigenvalues μ_i, $i = 1, 2, ...$, are said to be orthonormal and complete in the domain of $L_2(\psi)$ (Chernous'ko (1992); Tukhtasinov and Mamatov (2009)). The paper of Satimov and Tukhtasinov (2005a) and Satimov and Tukhtasinov (2006) discussed the case of linear operator bearing spectrum of negative eigenvalues of which $\mu_i \to -\infty$ as $i \to \infty$.

Next, we express all the involved functions in the form of Fourier series expansion that are,

$$\begin{aligned} \bar{z}(x,t) &= \sum_{i=1}^{\infty} z_i(t)F_i(x), \\ \bar{u}(x,t) &= \sum_{k=1}^{\infty} u_i(t)F_i(x), \\ \bar{v}(x,t) &= \sum_{i=1}^{\infty} v_i(t)F_i(x), \\ \bar{w}(x,t) &= \sum_{i=1}^{\infty} w_i(t)F_i(x). \end{aligned} \tag{1.4.6}$$

Also,

$$\begin{aligned} &\sum_{i=1}^{\infty} \lambda_i^{r+1} \int_0^{\theta} |z_i(t)|^2 dt < \infty, \\ &\sum_{i=1}^{\infty} \lambda_i^{r} \int_0^{\theta} |u_i(t)|^2 dt \le \rho^2, \\ &\sum_{i=1}^{\infty} \lambda_i^{r} \int_0^{\theta} |v_i(t)|^2 dt \le \sigma^2, \\ &\sum_{i=1}^{\infty} \lambda_i^{r} \int_0^{\theta} |w_i(t)|^2 dt < \infty. \end{aligned} \tag{1.4.7}$$

We note that functions $z(\cdot) \in L_2(0,T;H_{r+1})$ and $u(\cdot), v(\cdot), w(\cdot) \in L_2(0,T;H_r)$.

1.4.2 Decomposition Method

We substitute all the relevant expansions in (1.4.6) into the system (1.4.1) and get

$$\frac{\partial}{\partial t} \sum_{i=1}^{\infty} z_i(t)F_i(x) = B \sum_{i=1}^{\infty} z_i(t)F_i(x) - \sum_{i=1}^{\infty} u_i(t)F_i(x) + \sum_{i=1}^{\infty} v_i(t)F_i(x).$$

For simplicity, we use $\bar{w}(x,t)$ instead and ignore the summation,

$$\dot{z}_i(t)F(x) = Bz_i(t)F_i(x) + w_i(t)F_i(x).$$

By bearing the knowledge of orthogonality of $F(x)$ and comparing with (1.4.5), we obtain

$$\dot{z}_i(t)F_i(x) = -\mu_i(t)z_i(t)F_i(x) + w_i(t)F_i(x),$$
$$\dot{z}_i(t) = -\mu_i(t)z_i(t) + w_i(t).$$

Hence,

$$\dot{z}_i(t) + \mu_i z_i(t) = w_i(t).$$

The initial condition is as follows;

$$\bar{z}(x,0) = z(0)F(x) = z_0(x),$$

which means

$$z_i(0) = z_{i0} = \int_{\psi} z_{i0}(x)F_i(x).$$

The series $\bar{z}(x,t)$ converges uniformly in the space H_{r+1} on $[0,T]$. Thus, from any initial position $z_0 \in H_{r+1}$ and $\bar{w}(x,t) = L_2(0,T;H_r)$, there exists a unique solution $z(x,t)$ in the space $C(0,T;H_{r+1})$ for some $r \geq 0$ (Ivanov and Avdonin (1995), Chapter 3).

As such, the function $z_i(t), i = 1,2,...,$ on $[0,T]$ produces solution of Cauchy problem for infinite system of differential equations

$$z_i(t) + \mu_i z_i(t) = -u_i(t) + v_i(t), \quad z_i(0) = z_{i0}. \tag{1.4.8}$$

All these suggest that the game problem described by partial differential equations (1.4.1) can be reduced into the one described by an infinite system of differential equations.

1.5 Motivation and Problem

It is common that most real-life problems, including some problems in differential games, involve several factors which are usually represented as several variables. As a result, the problem is modelled as a system of partial differential equations. The problem-solving procedure is simplified with the use of the decomposition method which offers such system to be reduced into an infinite system of ordinary differential equations. A more complicated problem will have a more complex system of partial differential equations which corresponds to a higher level of an infinite system of ordinary differential equations. Consequently, differential game theorists have considered and investigated various games described by infinite one and two-system of ordinary differential equations. However, a differential game could also occur in a much higher level of system of differential equations. Therefore, in our thesis, we are motivated to investigate an infinite three-system of ordinary differential equations that can be associated with a more complex real-world problem.

In line with the general practice of differential games, the formulated system is then verified to be a valid game model. Every game model of any infinite system cannot be generalised to another and must be dealt with on its own. Thus, some pursuit game problems based on this three-system are studied and solved by figuring out an appropriate method, that suits the model, in establishing sufficient conditions and constructing players' strategies to terminate the game.

1.6 Objectives of the Thesis

The objectives of the thesis are as follows.

1. To find general solution $\mu(\cdot)$ of the three-system given by

$$\begin{aligned} \dot{x}_k &= -\alpha_k x_k + w_{k1}, \quad x_k(0) = x_k^0, \\ \dot{y}_k &= -\beta_k y_k - \gamma_k z_k + w_{k2}, \quad y_k(0) = y_k^0, \\ \dot{z}_k &= \gamma_k y_k - \beta_k z_k + w_{k3}, \quad z_k(0) = z_k^0 \end{aligned} \tag{1.6.1}$$

   where $\alpha_k, \beta_k \geq 0, \gamma_k \in \mathbb{R}$ and $w_{kj} = -u_{kj} + v_{kj} \in \mathbb{R}$ for $j = 1,2,3$ and $k = 1,2,...$, with $x^0 = (x_1^0, x_2^0, ...) \in l_2, y^0 = (y_1^0, y_2^0, ...) \in l_2, z^0 = (z_1^0, z_2^0, ...) \in l_2$, and prove its existence and uniqueness to prove the system can be used as a game model.

2. To determine sufficient conditions, and construct admissible control function in control problem and pursuer's strategy in pursuit problem defined by (1.6.1) where the pursuer aims to bring the initial state $\mu^0 = (\mu_1^0, \mu_2^0, ...) \in l_2$ of the system into origin of space l_2 that is $\mu(t) = 0$ at some time t for $t \in [0, T]$. The problems are considered for both integral and geometric constraints.

3. To determine sufficient conditions, and construct control and strategy of the pursuer needed in transferring the initial state $\mu^0 = (\mu_1^0, \mu_2^0, ...) \in l_2$ of system (1.6.1) into another non zero state $\mu^1 = (\mu_1^1, \mu_2^1, ...) \in l_2$ that is $\mu(t) = \mu^1$ at some time t for $t \in [0, T]$. The problems are also considered for both integral and geometric constraints.

4. To solve an optimal control problem followed by optimal pursuit differential game described by (1.6.1) with compliance to integral constraints.

5. To find solution for a differential game involving a finite number of pursuers and a single evader with players' movement described by

$$\begin{aligned} \dot{x}_k^i &= -\alpha_k x_k^i - u_{k1}^i + v_{k1}, \quad x_k^i(0) = x_k^{i0}, \\ \dot{y}_k^i &= -\beta_k y_k^i - \gamma_k z_k^i - u_{k2}^i + v_{k2}, \quad y_k^i(0) = y_k^{i0}, \\ \dot{z}_k^i &= \gamma_k y_k^i - \beta_k z_k^i - u_{k3}^i + v_{k3}, \quad z_k^i(0) = z_k^{i0} \end{aligned} \tag{1.6.2}$$

   where $\alpha_k, \beta_k \geq 0, \gamma_k \in \mathbb{R}$ and $u_{kj}^i, v_{kj} \in \mathbb{R}$ for $i = 1,2,...,m$, m is some positive integers, $j = 1,2,3$ and $k = 1,2,...$, with $x^{i0} = (x_1^{i0}, x_2^{i0}, ...) \in l_2, y^{i0} = (y_1^{i0}, y_2^{i0}, ...) \in l_2, z^{i0} = (z_1^{i0}, z_2^{i0}, ...) \in l_2$, with respect to integral constraints and then determine optimal number of pursuers to capture the evader.

1.7 Methodology

The following are methods carried out to solve problems of the thesis.

1. Formulate an infinite first order 3-system of differential equations (1.6.1), find solution $\mu(\cdot)$ of the system and show the solution $\mu(\cdot) \in C(0,T;l_2)$ where $C(0,T;l_2)$ is a space of continuous function in space l_2 on time interval $[0,T]$ that is to prove the solution exists and is unique. (Chapter 3)

2. Construct an admissible control function in the cases of steering the system (1.6.1) into origin (Chapter 4) or into another non zero state μ^1 (Chapter 5) at some time and thus solve the control problems by using the sufficient conditions.

3. Build an admissible strategy for the pursuer to terminate the pursuit game in accordance to respective cases (Chapter 4 and 5)

4. Develop an admissible time-optimal control function, an admissible pursuer's strategy followed by an admissible evader's strategy to obtain the optimal pursuit time for differential game (1.6.1). (Chapter 6)

5. Establish an admissible strategy for the pursuer together with an admissible strategy for the evader to prove the optimal number of pursuers for the completion of the game. (Chapter 7)

1.8 Organisation of the Thesis

We organise the thesis into eight chapters. The brief detail of every chapter is as follows.

In Chapter 1, we present some basic definitions, properties of Hilbert space l_2 and concepts of inequalities that are related to our study. Some common examples of games involved in the investigation of differential game theory are discussed.

Chapter 2 reviews some related past works in the field of differential game theory. The chapter begins with brief overview of the development of differential game theory followed by some previous studies on pursuit or evasion differential games including games in infinite system of differential equations.

In chapter 3, we formulate an infinite 3-system of differential equations. First, we provide introduction to the chapter and then examine the existence and uniqueness of the solution of the infinite 3-system of differential equations.

Chapter 4 begins with an introduction and deals with pursuit differential game based on the formulated model in Chapter 3. Both integral and geometric constraints are imposed on the players' control functions. We construct an admissible control function for the system to be steered into origin, which is to be applied for developing an admissible pursuer's strategy to complete the pursuit.

Next, in chapter 5, we discuss about pursuer's aim to shift the initial state of the system into another state of the system at some time by considering both integral and geometric constraints. The introduction of the chapter is followed by the establishment of sufficient conditions and admissible strategy for the pursuer in completing the game.

Chapter 6 starts with an introduction. Then, we propose an optimal pursuit time for differential game described by the system. Initially, we find solution for the optimal time control problem of the system. By referring to the obtained solution, we construct optimal strategies for the players to obtain the optimal pursuit time of the game.

We begin Chapter 7 with an introduction and study a differential game of a finite number of pursuers against one evader with integral constraints. We figure out strategy for the pursuers to ensure the pursuit is completed. We then consider the case where evader has a constructed strategy in order to find the the optimal number of pursuers to complete the pursuit.

Last but not least, in Chapter 8, we suggest and recommend some works that can be done in the future. General conclusion of the thesis is drawn too.

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