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Robust estimation of the three parameter Weibull distribution for addressing outliers in reliability analysis

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Accurate estimation techniques are crucial in statistical modeling and reliability analysis, which have significant applications across various industries. The three-parameter Weibull distribution is a widely used tool in this context, but traditional estimation methods often struggle with outliers, resulting in unreliable parameter estimates. To address this issue, our study introduces a robust estimation technique for the three-parameter Weibull distribution, leveraging the probability integral transform and specifically employing the Weibull survival function for the transformation, with a focus on complete data. This method is designed to enhance robustness while maintaining computational simplicity, making it easy to implement. Through extensive simulation studies, we demonstrate the effectiveness and resilience of our proposed estimator in the presence of outliers. The findings indicate that this new technique significantly improves the accuracy of Weibull parameter estimates, thereby expanding the toolkit available to researchers and practitioners in reliability data analysis. Furthermore, we apply the proposed method to real-world reliability datasets, confirming its practical utility and effectiveness in overcoming the limitations of existing estimation methodologies in the presence of outliers.

Keywords Probability integral transform, Robust estimation, Weibull distribution, Monte Carlo simulation, Outliers

The Weibull distribution has consistently proved itself as a vital and versatile tool in reliability analysis[1]. Its flexibility in accommodating and modeling a wide array of data types makes it a practical choice in numerous scenarios[2]. The distinct characteristics of the three-parameter Weibull distribution, determined by its shape, scale, and location parameters, are fundamental to delineating its shape and overall behavior. Using a parametric model like the three-parameter Weibull distribution to model failure time data offers significant advantages. It condenses the description to a few key parameters instead of detailing an entire curve and ensures smooth estimates of the failure time distributions[3]. In previous studies, there has been substantial interest among researchers in developing new models within the Weibull distribution family[4–12]. The main goal of extending the Weibull distribution is to create more flexible distribution structures that can better fit various data sets.

The application of the three-parameter Weibull distribution in real-world data, particularly for reliability analysis, has been explored extensively in previous studies. For example, Sürücü and Sazak[13] utilized this distribution to monitor reliability through control charts, demonstrating its effectiveness in detecting process deterioration by modeling the cumulative time elapsed between failures. Curtis and Juszczyk[14] applied the three-parameter Weibull model to analyze the strength of chemically toughened glass and phosphate-bonded investment, highlighting its ability to provide reliable estimates of Weibull modulus and characteristic strength. Nohut[15] emphasized the superiority of the three-parameter Weibull distribution over the two-parameter version for ceramics showing R-curve behavior, where the three-parameter model better fits the strength distribution due to the increase in fracture resistance with crack extension. Alqam et al.[16] compared two-parameter and three-parameter Weibull distributions for fiber-reinforced polymeric composites and recommended the

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two-parameter model due to small differences in design values and computational efficiency, although they acknowledged scenarios where the three-parameter model provided a more accurate fit. Deng et al.[17] conducted systematic Monte Carlo simulations and concluded that the three-parameter Weibull distribution is not always necessary for characterizing the statistical variation of the fracture strength of brittle ceramics, but it can be valuable for specific cases. Han et al.[18] conducted a detailed reliability analysis of Zr-based bulk metallic glasses using the three-parameter Weibull model, illustrating its ability to provide interpretable and accurate reliability assessments. These studies collectively underscore the practical utility and effectiveness of the three-parameter Weibull distribution in reliability analysis, validating its importance and highlighting ongoing efforts to refine and enhance Weibull modeling for better data fitting and reliability assessment.

Accurately estimating the parameters of the three-parameter Weibull distribution is crucial as they play a critical role in shaping the form of the distribution and, consequently, the outcomes of any reliability analysis. The precise estimation of these parameters underpins the integrity and reliability of any analysis involving the Weibull distribution. Traditional techniques such as maximum likelihood estimator (MLE) and method of moments (MOM) have been widely used for estimating the parameters of the Weibull distribution[19,20]. Despite their theoretical simplicity and universal applicability, these methods present significant challenges, particularly their heightened sensitivity to outliers[21]. Outliers can distort the parameter estimates, leading to models that do not accurately reflect the underlying data, thereby producing potentially misleading results[22].

Robust parameter estimation for the two-parameter Weibull distribution has been a topic of interest in several studies. For instance, various robust techniques have been proposed to enhance estimation accuracy in the presence of outliers, yet each comes with its own set of limitations. Seki and Yokoyama[23] introduced a robust method using two percentiles, Zhang et al.[21] evaluated robust regression techniques, and Boudt et al.[24] introduced several robust estimators including the quantile least squares and repeated median. However, these advancements primarily address the two-parameter model, which, while useful, does not accommodate datasets with non-zero thresholds or those requiring adjustment for location shifts. Notably, there has been a significant gap in robust estimation techniques specifically designed for the three-parameter Weibull distribution. This distribution is crucial for more accurately modeling real-world scenarios where data behavior begins beyond a natural zero point or where adjustments for baseline shifts are necessary. Addressing this gap, our study introduces a new robust estimation technique for the three-parameter Weibull distribution, aiming to enhance estimation accuracy while maintaining robustness against outliers. This development is particularly vital given the three-parameter model's ability to provide a more comprehensive fit across a broader range of data conditions, crucial for fields requiring precise reliability analysis.

In response to the identified gaps in existing estimation methodologies, our study introduces a new approach, the probability integral transform estimator (PITE), specifically tailored for the three-parameter Weibull distribution. This technique uniquely leverages the probability integral transform not just to achieve robustness, but to fundamentally reshape the estimation process. By converting the distribution of the random variable to a uniform distribution, PITE dramatically enhances its resistance to outliers and simplifies computational demands. This application of the probability integral transform in PITE is particularly innovative, as it directly addresses the challenges posed by datasets with shifts and non-zero origins, which are typically not well-served by traditional methods. To validate the effectiveness of PITE, we conduct extensive Monte Carlo simulations comparing its performance with traditional estimators, including MLE, MOM, weighted MLE (WMLE), and maximum product of spacing estimator (MPSE). These simulations assess the robustness and efficiency of PITE under various scenarios, particularly focusing on datasets with outliers. Additionally, we apply the proposed method to real-world reliability datasets to demonstrate its practical utility and effectiveness in overcoming the limitations of existing estimation methodologies. This study contributes to the field of reliability data analysis by:

1. Introducing a robust estimation technique for the three-parameter Weibull distribution that is resilient to outliers, specifically designed for complete data.
2. Conducting a thorough assessment of PITE via simulation studies, demonstrating its enhanced performance over traditional estimators when dealing with outliers.
3. Demonstrating the practical applicability of PITE using real-world reliability datasets, thereby reinforcing its potential as a valuable tool for researchers and practitioners.

The structure of this paper is as follows: Section "Three-parameter Weibull distribution" presents the three-parameter Weibull distribution. Section "Robust Estimation Technique for the Three-Parameter Weibull Distribution" discusses the basis of the PITE and its characteristics. Section "Monte Carlo simulation study" outlines the design and outcomes of the Monte Carlo simulation study. Section "Application: Strength of glass and strength of carbon fibers" applies PITE to real-world reliability datasets in comparison to other estimators. Finally, Section "Conclusion" concludes the study with a summary of key insights.

Three-parameter Weibull distribution

This paper focuses on robust estimation of the parameters for the three-parameter Weibull distribution: θ (shape), β (scale), and γ (location). The three-parameter Weibull probability density function (PDF) is given by:

$$f\left(x;\theta,\beta,\gamma\right)=\frac{\theta}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\theta-1}exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\theta}\right],x>\gamma,\theta>0,\beta>0. \tag{1}$$

The cumulative distribution function (CDF) corresponding to this is defined as:

$$F(x;\theta,\beta,\gamma)=1-exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\theta}\right],x>\gamma,\theta>0,\beta>0. \tag{2}$$

In this model, the shape parameter θ determines the profile of distribution. Varying θ alters the form of the distribution function: for $\theta<1$, the density function is J-shaped due to the dominance of the exponential part. For $\theta>1$, the polynomial part becomes more pronounced, resulting in a skewed unimodal density curve[25]. The scale parameter β, measured in the same unit as x, influences the spread of the distribution. Increasing β compresses or reduces the density, while decreasing β magnifies or stretches it. Therefore, a larger β causes greater variation in x, while a smaller β results in less variation[25]. The location parameter γ shifts the distribution along the x-axis, providing a flexible model for various real-world scenarios[26].

In reliability data analysis, essential metrics such as the mean time to failure (MTTF), the survival function, and the failure rate function are crucial. For the three-parameter Weibull distribution, these are given by:

$$MTTF(\theta,\beta,\gamma)=\gamma+\beta\Gamma(1+1/\theta), \tag{3}$$

$$S(x;\theta,\beta,\gamma)=\exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\theta}\right],x>\gamma,\theta>0,\beta>0, \tag{4}$$

$$h(x;\theta,\beta,\gamma)=\frac{\theta}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\theta-1},x>\gamma,\theta>0,\beta>0. \tag{5}$$

Here, $\Gamma(\cdot)$ denotes the gamma function. The θ parameter is pivotal for representing the aging characteristics of a system or product. A $\theta<1$ indicates early-life failures (negative aging), $\theta=1$ signifies failures occurring randomly (non-aging), and $\theta>1$ represents wear-out failures (positive aging)[27]. The introduction of the γ parameter allows for the inclusion of a time delay before the onset of aging, enhancing the model's applicability, particularly in scenarios where wear and tear do not commence immediately[28,29]. The failure rate function in Eq. (5) provides a dynamic view of the failure rates over time, offering practitioners a powerful tool to assess the appropriateness of the model in reliability studies. By analyzing how the failure rate varies with θ, practitioners can assess whether the model accurately reflects the aging or wear-out patterns observed in empirical data, thereby ensuring the model's effectiveness in predicting lifecycle and planning maintenance strategies.

Robust estimation technique for the three-parameter Weibull distribution

Utilizing the probability integral transform statistic as a foundation, this section introduces a novel robust estimator specifically designed for the three-parameter Weibull distribution. We examine the critical attributes of this estimator, including its efficiency and sensitivity curve. Furthermore, this section outlines the methodology for computing the confidence intervals for the Weibull parameters, offering a comprehensive approach to parameter estimation.

Proposed estimator based on probability integral transform

In the context of the three-parameter Weibull distribution, consider $x_1,x_2,\ldots,x_n$ as a random sample. Each sample is described by a CDF as specified in Eq. (2). This CDF is both continuous and strictly increasing, a fundamental property that allows us to apply the probability integral transform. According to this principle, if X is a continuous random variable with a CDF F, then the transformed variable $F(X)$ follows a standard uniform distribution $U(0,1)$. Therefore, the transformed random variables $F(x_1),F(x_2),\ldots,F(x_n)$ are uniformly distributed on $[0,1]$ as $F(x_i)\sim U(0{,}1)$ for each i. Similarly, the survival function $S(x)$ for the Weibull distribution given in Eq. (4) also transforms the variables $S(x_1),S(x_2),\ldots,S(x_n)$ such that they follow a standard uniform distribution. This uniformity arises because $S(X)$ is a complementary CDF, fulfilling the conditions of the probability integral transform, and thereby minimizing the effects of outliers as the data is transformed to values between 0 and 1. By utilizing this property, we introduce the PITE for the three-parameter Weibull distribution. It is defined as:

$$T_{n,\rho}(\theta,\beta,\gamma)=n^{-1}\sum_{i=1}^{n}exp\left[-\rho\left(\frac{x_i-\gamma}{\beta}\right)^{\theta}\right]. \tag{6}$$

This expression is essentially an average of the transformed survival functions, where the transformation is applied to each observation x_i through the survival function $S(x)$. The parameter ρ is a tuning factor that scales this transformation, influencing the weight and decay rate of the exponential term, which enhances the estimator's robustness and sensitivity to outliers.

When $\rho=1$, the transformation $\exp\left[-((x_i-\gamma)/\beta)^{\theta}\right]$ simplifies to $S(x_i)$, reflecting the survival function for each observation. Since $S(x_i)=1-F(x_i)$, and given our earlier discussion that the survival function $S(X)$ transforms such that $S(x_i)$ uniformly distributes over $[0,1]$, this directly demonstrates the application of the probability integral transform under these specific conditions. Let us consider $U_1,U_2,\ldots,U_n$ as independent and identically distributed standard uniform random variables. According to the strong law of large numbers, the average of these variables raised to any power ρ will almost surely converge to their expected value as the sample size n increases. The expected value of U^{ρ} is given by:

$$E\left[U^{\rho}\right]=\int_{0}^{1}udu=\frac{1}{\rho+1}.$$

Thus, the expression

$$\frac{1}{n}\sum\nolimits_{i=1}^{n}U_{i}^{\rho}$$

Converges almost surely to $1/(\rho+1)$. This establishes that the transformed variables, when averaged and raised to the power ρ, will adhere to the expected statistical pattern of convergence as outlined by the strong law of large numbers. Similarly, for powers 2ρ and 3 ρ, the expected values are:

$$E\left[U^{2\rho}\right]=\frac{1}{2\rho+1}\text{and}E\left[U^{3\rho}\right]=\frac{1}{3\rho+1}.$$

Utilizing the MOM concept, we define the PITE as the solution to θ, β, and γ in the equations:

$$T_{n,\rho}\left(\theta,\beta,\gamma\right)=n^{-1}\sum_{i=1}^{n}exp\left[-\rho\left(\frac{x_i-\gamma}{\beta}\right)^{\theta}\right]=\frac{1}{\rho+1}, \tag{7}$$

$$T_{n,2\rho}\left(\theta,\beta,\gamma\right)=n^{-1}\sum_{i=1}^{n}exp\left[-2\rho\left(\frac{x_i-\gamma}{\beta}\right)^{\theta}\right]=\frac{1}{2\rho+1}, \tag{8}$$

$$T_{n,3\rho}\left(\theta,\beta,\gamma\right)=n^{-1}\sum_{i=1}^{n}exp\left[-3\rho\left(\frac{x_i-\gamma}{\beta}\right)^{\theta}\right]=\frac{1}{3\rho+1}, \tag{9}$$

The estimates for Weibull parameters are obtained by solving Eqs. (7), (8), and (9) simultaneously through a numerical method.

Asymptotic relative efficiency of PITE

The efficiency of the MLE is well-established, serving as a benchmark for efficiency comparisons. For the three-parameter Weibull distribution, we assess the asymptotic relative efficiency (ARE) of the PITE for parameters θ, β and γ in comparison to the MLE. Let the covariance matrices of PITE and MLE be denoted as $\sum_{PITE}$ and $\sum_{MLE}$, respectively. The ARE of PITE can be defined as:

$$ARE\left(\hat{\gamma}_{PITE},\hat{\beta}_{PITE},\hat{\theta}_{PITE}\right)=\lim_{n\to\infty}\left(\frac{\left|\sum_{MLE}\right|}{\left|\sum_{PITE}\right|}\right)^{1/3}, \tag{10}$$

where $\left|\sum\right|$ represents the determinant of the covariance matrix.

Owing to the inherent difficulty in obtaining explicit forms of $\sum_{MLE}$ and $\sum_{PITE}$ for the three-parameter Weibull distribution, we utilize Monte Carlo simulations to estimate these matrices. This methodological choice facilitates the computation of ARE, as stated in Eq. (10), across a spectrum of the tuning parameter ρ.

Adjusting ρ allows us to manage the trade-off between efficiency and robustness in the PITE. Table 1 presents the ARE values across a range of ρ and θ values, as obtained from simulation. It shows that for $\theta<4$, the maximum ARE is generally achieved when ρ ranges between 0.68 and 1.00. However, for $\theta\geq4$, an increase in ρ generally leads to a decline in ARE. These findings underscore the importance of selecting an appropriate ρ value based on the distribution's shape characteristics, highlighting the need for careful tuning to achieve an optimal trade-off between robustness and efficiency in PITE applications.

	ARE (%)					
ρ	θ= 1.5	θ= 2	θ= 3	θ= 4	θ= 5	θ= 6
0.2	40	45	55	80	83	92
0.32	42	46	58	75	80	89
0.42	44	47	59	69	77	87
0.68	48	51	61	68	71	82
1.00	50	52	60	65	67	72
1.35	49	51	58	62	62	66
1.75	48	49	55	58	57	60

Table 1. The ARE of PITE for several values of θ and ρ.

Consistency of PITE

A fundamental aspect of statistical estimators is their consistency, which ensures that as the sample size increases, the estimator converges to the true parameter value. To explore the consistency of the PITE in the presence of outliers, we simulate Weibull-distributed data with parameters $\theta = 2, 4, 6$, $\beta = 3$, and $\gamma = 1$, progressively increasing the sample size from 10 to 5000. Within each dataset, we include three outliers to test the robustness of PITE under non-ideal conditions. Each simulation round calculates PITE estimates for increasing sample sizes, examining whether the estimates approach the true parameter values despite the presence of outliers. As the sample size increases, our results illustrate a clear convergence of the PITE estimates towards the true values of the Weibull parameters, demonstrating the consistency of PITE even in the presence of outliers. This convergence is visually represented in Fig. 1, each depicting how the estimates stabilize as the sample size grows, across different tuning parameters ($\rho = 0.32, 0.42, 0.68, 1.00$).

Sensitivity curve of PITE

A desirable robustness criterion for an estimator is that it has bounded sensitivity curve (SC). The sensitivity curve (SC) in the context of a random sample, measures how an estimator responds to the presence of an outlier, denoted as x_τ[30]. Specifically, the SC of the PITE estimates for a random sample $x_1, x_2, \ldots, x_n$ is influenced by the position of an outlier x_τ, as shown in the following expressions:

$$SC\left(x_\tau, \hat{\mu}\right) = \frac{\hat{\mu}\left(x_1, x_2, \ldots, x_n, x_\tau\right) - \hat{\mu}\left(x_1, x_2, \ldots, x_n\right)}{1/(n+1)}, \tag{11}$$

where, $\hat{\mu}\left(x_1, x_2, \ldots, x_n\right)$ represents the PITE estimates for each Weibull parameters corresponding to the sample $x_1, x_2, \ldots, x_n$.

To determine the SC of the PITE, we initially generate a random sample of 50 observations the Weibull distribution with parameters $\theta = 2, 4, 6$, $\beta = 3$, and $\gamma = 1$. A smaller sample size is chosen to make the impact of a single outlier more noticeable. We then introduce an outlier x_τ, varying its value from 1 to 50 in increments of 0.5, and compute the SC according to Eq. (11). Figure 2 display the SC of the PITE for tuning parameters $\rho = 0.32, 0.42, 0.68, 1.00$. As the value of x_τ increases, the influence of the outlier becomes more significant. However, the SC curves for different tuning parameters converge toward a bounded limit, even as the outlier grows more extreme. This demonstrates that PITE maintains a bounded SC, confirming its robustness in the presence of a single large outlier.

Using the same setting, we further investigate the SC of the PITE for multiple outliers, specifically three outliers. The results are presented in Fig. 3. Similar to the case of a single outlier, it can be observed from these figure that the SC of the PITE remains bounded, even as the number of outliers increases.

Confidence interval of PITE

From Eqs. (7)–(9), we observe that $T_{n,\rho}(\theta, \beta, \gamma)$, $T_{n,2\rho}(\theta, \beta, \gamma)$, and $T_{n,3\rho}(\theta, \beta, \gamma)$ display probabilistic behavior similar to $n^{-1}\sum_{i=1}^{n} U_i^{\rho}$, $n^{-1}\sum_{i=1}^{n} U_i^{2\rho}$, and $n^{-1}\sum_{i=1}^{n} U_i^{3\rho}$, respectively, where U denotes a standard uniform random variable. These sums show probabilistic behavior independently of any unknown parameters. To

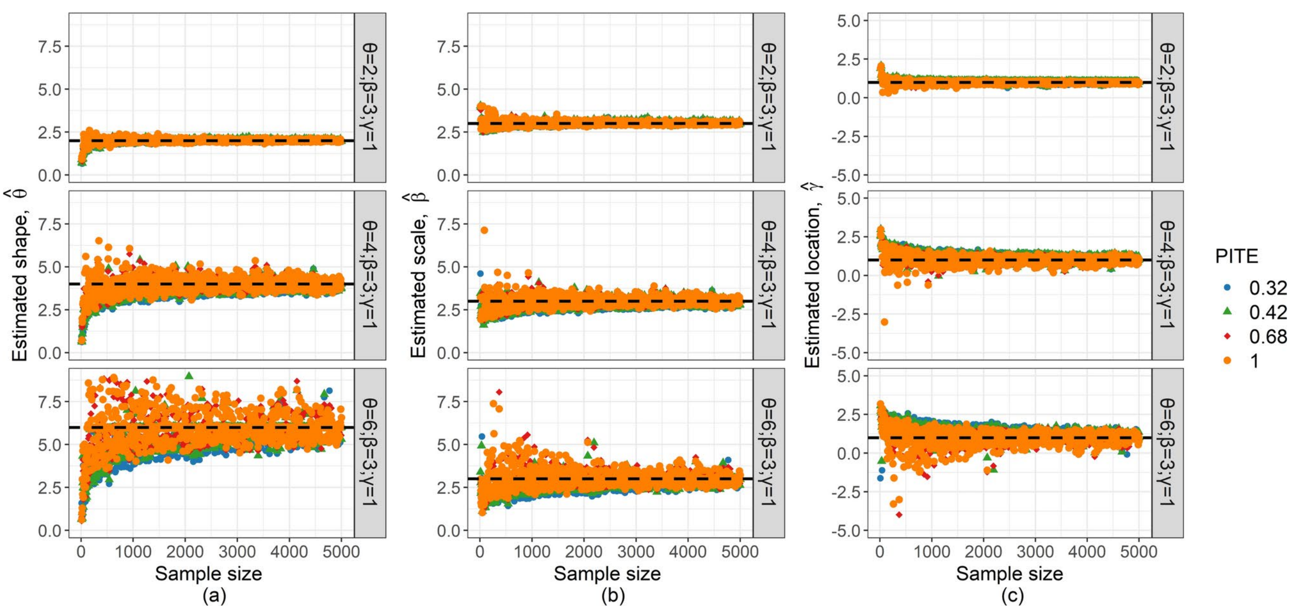

Fig. 1. Convergence of PITE for different values of ρ for (**a**) the shape parameter ($\theta = 2, 4, 6$), (**b**) the scale parameter ($\beta = 3$), and (**c**) the location parameter ($\gamma = 1$). The dashed lines represent the true parameter values. The results demonstrate that as the sample size increases, the PITE estimates converge towards the true values, indicating the consistency of the estimator.

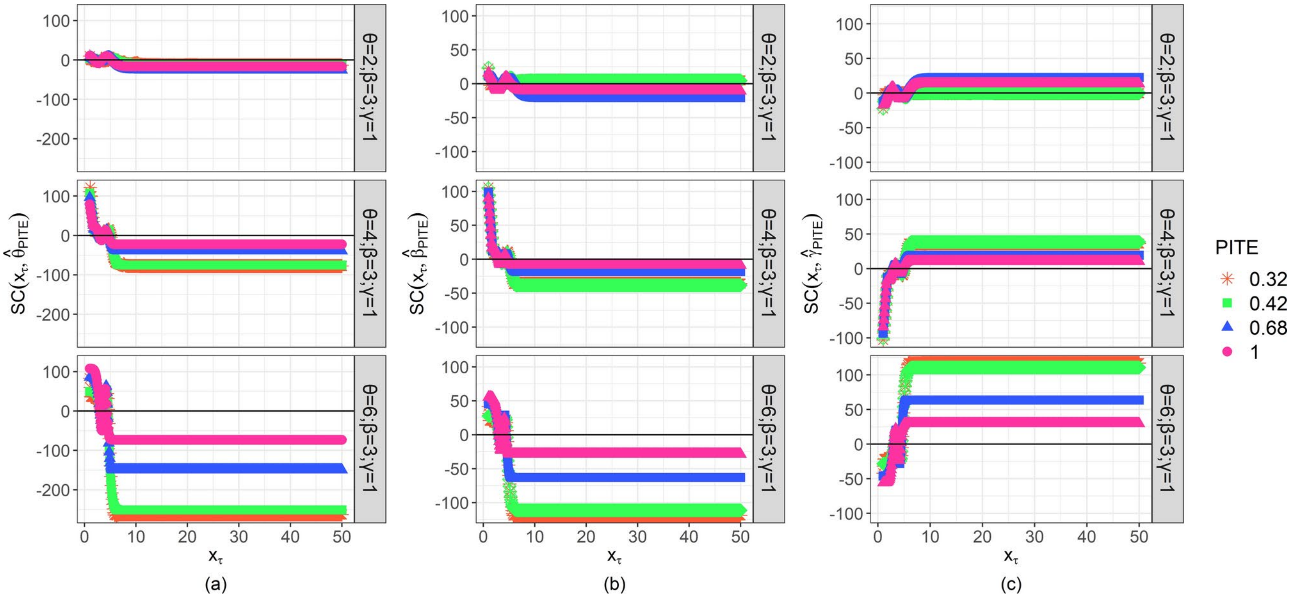

Fig. 2. Graphs of (**a**) $SC\left(x_\tau, \hat{\theta}_{PITE}\right)$, (**b**) $SC\left(x_\tau, \hat{\beta}_{PITE}\right)$, and (**c**) $SC\left(x_\tau, \hat{\gamma}_{PITE}\right)$ for several different ρ, with parameters $\theta = 2, 4, 6$, $\beta = 3$, and $\gamma = 1$. The plots illustrate how the PITE estimates respond to an outlier x_τ, where different values of ρ influence the robustness of the estimator.

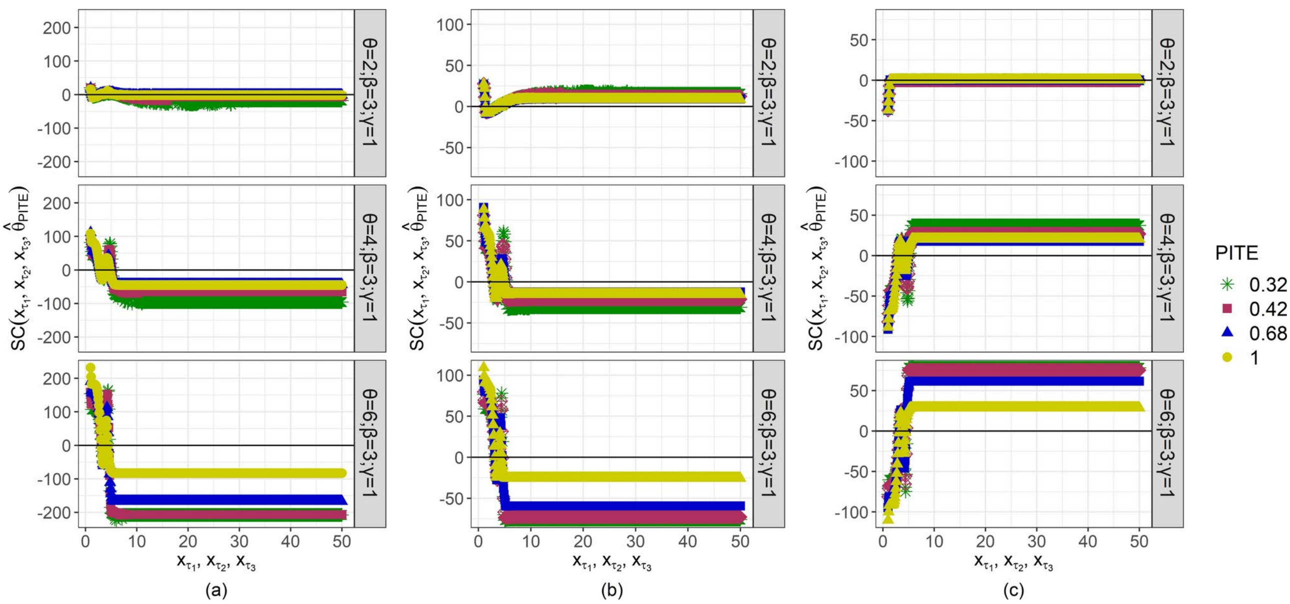

Fig. 3. Graphs of (**a**) $SC\left(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, \hat{\theta}_{PITE}\right)$, (**b**) $SC\left(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, \hat{\beta}_{PITE}\right)$, and (**c**) $SC\left(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, \hat{\gamma}_{PITE}\right)$ for several different ρ, with parameters $\theta = 2, 4, 6$, $\beta = 3$, and $\gamma = 1$. The plots illustrate how the PITE estimates respond to outliers $x_{\tau_1}, x_{\tau_2}, x_{\tau_3}$, where different values of ρ influence the robustness of the estimator.

formulate a two-sided $(1-\alpha)$ 100% confidence interval (CI) for $\hat{\theta}_{PITE}$, $\hat{\beta}_{PITE}$, and $\hat{\gamma}_{PITE}$, begin by identifying the quantiles $\alpha/2$ and $1-\alpha/2$ from the distribution of $n^{-1}\sum_{i=1}^{n} U_i^{\rho}$, $n^{-1}\sum_{i=1}^{n} U_i^{2\rho}$, and $n^{-1}\sum_{i=1}^{n} U_i^{3\rho}$. These quantiles, labeled as $q_{\theta;\alpha/2}$ and $q_{\theta;1-\alpha/2}$ for $\hat{\theta}_{PITE}$, $q_{\beta;\alpha/2}$ and $q_{\beta;1-\alpha/2}$ for $\hat{\beta}_{PITE}$, and $q_{\gamma;\alpha/2}$ and $q_{\gamma;1-\alpha/2}$ for $\hat{\gamma}_{PITE}$, can be precisely determined through simulation because the distribution is not dependent on any hidden parameters. Subsequently, the $(1-\alpha)$ 100% CI for $\hat{\theta}_{PITE}$ is constructed as $[l_\theta, u_\theta]$, where l_θ satisfies $T_{n,\tau}\left(l_\theta\right) = q_{\theta;1-\alpha/2}$ and u_θ satisfies $T_{n,\tau}\left(u_\theta\right) = q_{\theta;\alpha/2}$. Similarly, the $(1-\alpha)$ 100% CI for $\hat{\beta}_{PITE}$ is $[l_\beta, u_\beta]$, where l_β satisfies $T_{n,2\tau}\left(l_\beta\right) = q_{\beta;1-\alpha/2}$ and u_β satisfies $T_{n,2\tau}\left(u_\beta\right) = q_{\beta;\alpha/2}$. Finally, the $(1-\alpha)$ 100% CI for $\hat{\gamma}_{PITE}$ is $[l_\gamma, u_\gamma]$, where l_γ satisfies $T_{n,3\tau}\left(l_\gamma\right) = q_{\gamma;1-\alpha/2}$ and u_γ satisfies $T_{n,3\tau}\left(u_\gamma\right) = q_{\gamma;\alpha/2}$.

Monte Carlo simulation study

This section presents an assessment of PITE with ρ =0.32, 0.42, 0.68, 1.00, compared to MLE, WMLE, MPSE, and MOM through a Monte Carlo simulation. The simulations are executed using R statistical software version 4.4.1[31]. Note that the R commands for the other estimators used in this study are accessible in the ForestFit package[32].

Simulation framework

The Monte Carlo simulation procedure includes the following steps:

Step 1. Generate random samples from a Weibull distribution with sample sizes of $n = 50$, 100, and 500. The shape parameter θ varies among 2, 4, or 6, with the scale parameter β and the location parameter γ are fixed at 3 and at 1, respectively, throughout the simulation study.
Step 2. Introduce outliers by replacing some random observations. These outliers are generated from another Weibull distribution with the same shape and location parameters but with the scale parameter adjusted by a factor $\omega = 5$ for outliers. The outliers are added in fixed proportions $\varepsilon = 2\%$, 4%, 6%, 8%.
Step 3. Estimate the parameters θ, β, and γ using PITE ($\rho = 0.32$, 0.42, 0.68, 1.00), MLE, WMLE, MPSE, and MOM.
Step 4. Repeat steps 1–3 for a total of 5,000 iterations.
Step 5. Assess the performance of each estimation method by calculating the root mean square error (RMSE) and the deficiency (Def) criterion. These metrics are defined as follows:

$$RMSE(\hat{\alpha}) = \sqrt{\frac{1}{N}\sum\nolimits_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2}, \tag{12}$$

$$Def\left(\hat{\theta}, \hat{\beta}, \hat{\gamma}\right) = \frac{RMSE\left(\hat{\theta}\right) + RMSE\left(\hat{\beta}\right) + RMSE\left(\hat{\gamma}\right)}{3}. \tag{13}$$

In Eq. (12), $\hat{\alpha}_i$ represents the estimated Weibull parameters for the i-th simulated dataset (where $i = 1, 2, \ldots, N$), α is the true value of each Weibull parameter and N is the total number of iterations. Lower values of RMSE and Def signify greater accuracy and precision.

Simulation results

Figure 4 displays the results of the Monte Carlo simulations, showcasing the computed RMSEs and Defs. A summary of the key findings is provided below:

1 A decrease in both RMSE and Def values is observed across all estimators with an increase in sample size. This signifies an improvement in the performance of all estimators, indicating that larger sample sizes lead to more accurate parameter estimates and better overall estimator performance.
2 As the shape parameter (θ) increases, there is a general decline in the performance of most estimators, as indicated by increased RMSE and Def values. This trend suggests that higher shape parameters make the estimation more challenging, impacting the accuracy of the parameter estimates in the presence of outliers.
3 RMSE for shape parameter (Fig. 4(a))

- PITEs, especially PITE ($\rho = 0.68$) and PITE ($\rho = 1.00$) consistently show lower RMSE values compared to MLE, WMLE, MPSE, and MOM, demonstrating their robustness to outliers.
- MLE and MPSE show relatively stable performance, but their RMSE values increase with higher outlier percentages.
- WMLE exhibits erratic behavior with very high RMSE values for smaller sample sizes, indicating high sensitivity to outliers.
- MOM exhibits the highest RMSE values across different levels of outliers and sample sizes, indicating poorer performance in estimating the shape parameter compared to other estimators.

4 RMSE for scale parameter (Fig. 4(b))

- PITEs show relatively stable performance across different percentages of outliers and sample sizes. PITEs demonstrate the best performance in estimating the scale parameter when the shape parameter value is small.
- MLE offers a more balanced performance with moderate sensitivity to outliers and better results in larger sample sizes.
- MPSE is considered the best estimator for the scale parameter, particularly when the shape parameter is large.
- WMLE is highly sensitive to outliers, showing the worst performance in smaller sample sizes. However, its performance improves significantly with larger sample sizes.
- MOM consistently underperforms compared to other estimators, with the highest RMSE values

5 RMSE for location parameter (Fig. 4(c))

- PITEs, especially PITE ($\rho = 1.00$), demonstrate slightly better performance compared to MLE and MPSE for the location parameter.

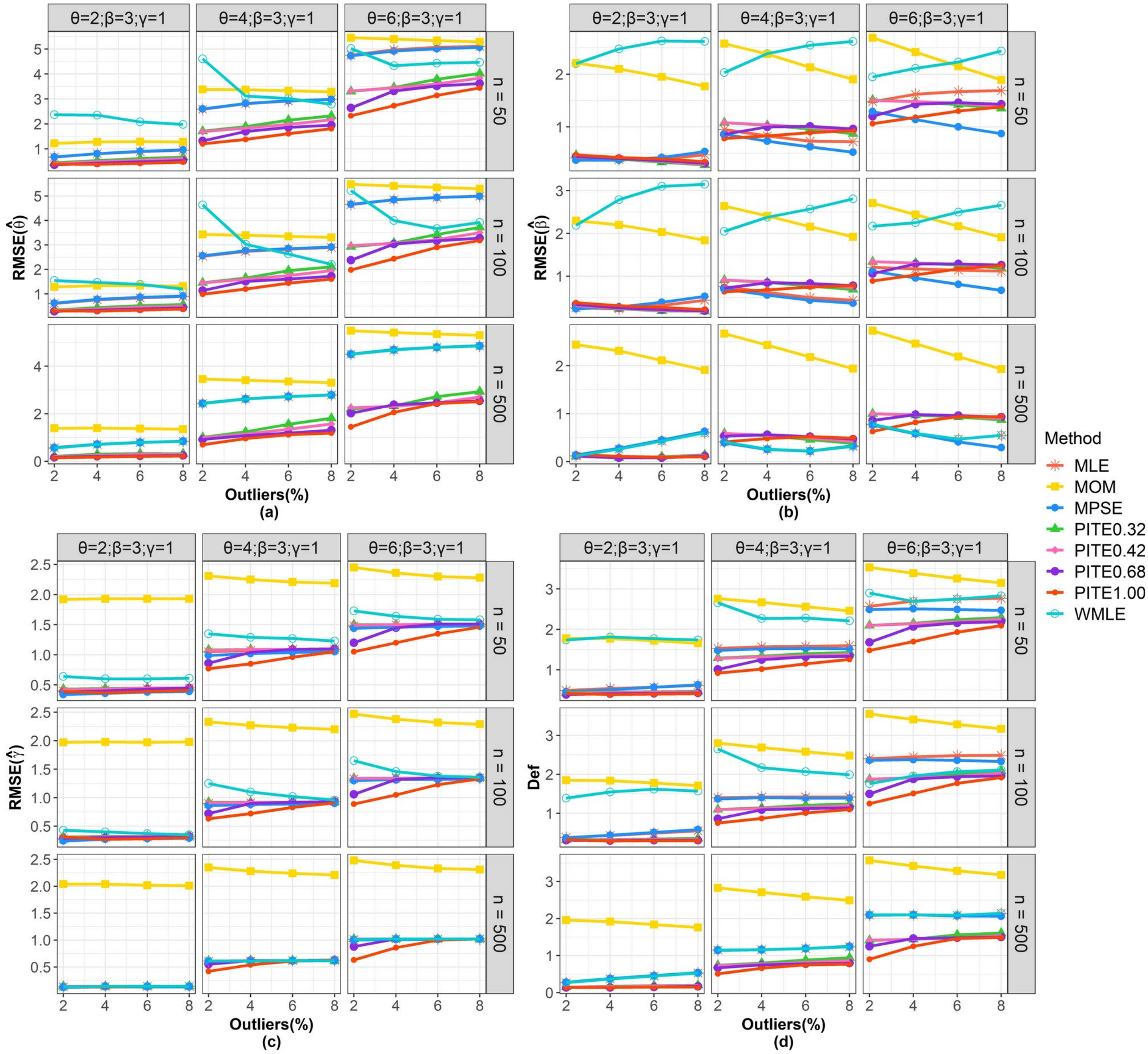

Fig. 4. RMSE values for different estimation methods under varying levels of contamination. The figure presents (**a**) RMSE of the shape parameter $\hat{\theta}$, (**b**) RMSE of the scale parameter $\hat{\beta}$, (**c**) RMSE of the location parameter $\hat{\gamma}$, and (**d**) the overall deficiency measure $Def\left(\hat{\theta}, \hat{\beta}, \hat{\gamma}\right)$. Results are shown for three shape parameters (θ = 2, 4, 6), with fixed scale (β = 3) and location ($\gamma = 1$) parameters, across different sample sizes (n = 50, 100, 500) and contamination levels (and ε = 2%, 4%, 6%, 8%).

- WMLE performs well when the sample size is large, but its performance is poor with smaller sample sizes.
- MOM shows the highest RMSE values, indicating the least accurate estimates for the location parameter among the estimators.

6 Overall performance based on Def criterion (Fig. 4 (d))

- PITEs show the best overall performance, with PITE ($\rho = 1.00$) consistently having the lowest values of Def across different sample sizes and outlier percentages.
- MLE, WMLE and MPSE are sensitive to outliers, displaying higher Def values. WMLE, in particular, shows notably poor performance when the sample size is small.
- MOM consistently shows the highest Def values, indicating the poorest overall performance among the estimators.

Dataset	Sample size	Mean	Median	Min	Max	Standard deviation	Skewness
1	18	38.368	36.150	24.120	61.720	10.934	0.577
2	217	4223.086	4141.953	1832.558	8083.980	1169.280	0.250
3	63	3.059	2.996	1.901	5.020	0.621	0.618

Table 2. Summary statistics for all datasets.

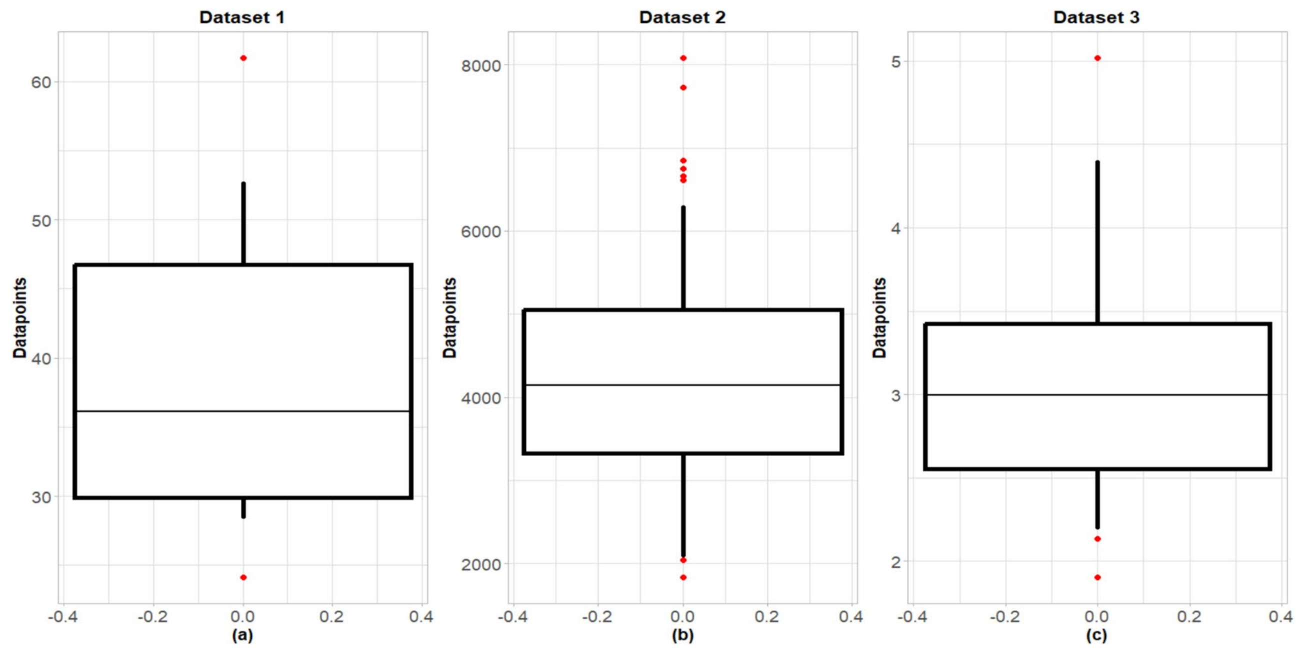

Fig. 5. Detection of outliers in the datasets using generalized boxplots. The figure illustrates the presence of potential outliers (red dots) in (**a**) Dataset 1, (**b**) Dataset 2, and (**c**) Dataset 3.

Application: Strength of glass and strength of carbon fibers

In this section, we perform statistical analyses on three reliability datasets to demonstrate the performance of PITE in comparison to other estimation techniques. The first dataset, named Dataset 1, originates from research by Datsiou and Overend[33] and consists of strength measurements of naturally aged glass surfaces, expressed in megapascals (MPa). The second dataset, known as Dataset 2, is derived from studies by Mesquita et al.[34,35] and includes measurements of the strength of T700S carbon fibers, also reported in MPa. The third dataset, Dataset 3, includes data from Bader and Priest[36], featuring the tensile strength of 69 carbon fibers measured in gigapascals (GPa), tested under tension at gauge lengths of 20 mm.

To begin, we calculated descriptive statistics for each dataset and identified outliers using the generalized boxplot method. This approach is especially effective for distributions that are skewed or have heavy tails, characteristics often associated with the Weibull distribution[37]. Table 2 provides the summary statistics for all datasets, offering crucial details like sample size, mean, median, minimum and maximum values, standard deviation, and skewness. These statistics are fundamental for understanding the central tendencies, variability, and asymmetry of datasets. which are important for the subsequent analyses. Additionally, Fig. 5 visually highlights the presence of outliers in each dataset, underscoring the importance of employing a robust method such as PITE for fitting the Weibull distribution.

To assess the suitability of the three-parameter Weibull distribution for data modeling, we employ three goodness of fit (GoF) tests: the Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramer-Von Mises (CVM) tests. These tests provide benchmarks for comparing the effectiveness of various estimation methods. The adequacy of the Weibull distribution is determined by low-test statistic or a p-value greater than 0.05. Table 3 summarizes the parameter estimates, standard errors (SE), 95% confidence intervals (CI), and goodness-of-fit (GoF) assessments for Datasets 1 – 3 based on the Weibull distribution.

Table 3 shows that the p-values from the GoF tests for the Weibull model, when using the PITE method across various ρ values, are all above the 0.05 significance level. This suggests that the Weibull model is appropriately fitting the strength data. Among the estimators, the PITE with $\rho = 0.68$ and $\rho = 1.00$ consistently show the smallest test statistics and highest p-values, indicating superior GoF. This demonstrates that PITE is a robust estimator for the Weibull distribution, effectively handling outliers. In comparison, the MLE also shows adequate performance but is generally less effective than PITE, especially in the presence of outliers. The WMLE

Dataset	Estimator	Estimates with (SE) and (95% CI)			KS (p-value)	AD (p-value)	CVM (p-value)
		$\hat{\theta}$	$\hat{\beta}$	$\hat{\gamma}$			
1	PITE ($\rho = 0.32$)	2.294; (0.958); (1.809, 5.246)	27.616; (12.214); (19.888, 63.849)	13.650; (10.990); (−19.993, 20.021)	0.1494 (0.7629)	0.4501 (0.7959)	0.0702 (0.7565)
	PITE ($\rho = 0.42$)	1.839; (0.901); (1.434, 4.814)	22.192; (11.474); (14.968, 56.449)	18.603; (10.120); (−12.955, 24.131)	0.1350 (0.8559)	0.3622 (0.8838)	0.0565 (0.8425)
	PITE ($\rho = 0.68$)	1.459; (0.886); (1.158, 4.511)	18.072; (10.551); (12.533, 54.828)	22.199; (8.723); (−9.055, 24.143)	0.1162 (0.9449)	0.3256* (0.9168)	0.0472 (0.8987)
	PITE ($\rho = 1.00$)	1.376; (0.828); (0.972, 3.719)	17.404; (7.232); (11.110, 39.896)	22.794; (5.668); (4.085, 27.628)	0.1124* (0.9575)	0.3436 (0.9010)	0.0465* (0.9026)
	MLE	1.279; (0.719); (0.689, 2.848)	15.907; (5.729); (9.676, 31.603)	23.524; (3.572); (16.248, 29.180)	0.1273 (0.8974)	0.4592 (0.7865)	0.0517 (0.8720)
	WMLE	17.091;(49.963); (21.331, 172.242)	31.823; (3.002); (26.965, 37.017)	24.130; (1.961); (24.130, 28.520)	0.7167 (< 0.0001)	Inf (< 0.0001)	2.5471 (< 0.0001)
	MPSE	1.395; (0.810); (0.857, 3.467)	18.992; (8.571); (10.852, 39.637)	21.662; (6.668); (6.825, 27.892)	0.1206 (0.9284)	0.3273 (0.9153)	0.0496 (0.8847)
	MOM	1.071; (0.058); (0.951, 1.182)	6.778; (1.122); (4.247, 8.550)	31.768; (1.946); (28.384, 35.993)	0.3333 (0.0277)	Inf (< 0.0001)	0.3507 (0.0971)
2	PITE ($\rho = 0.32$)	3.173 (0.522); (2.638, 4.487)	3788.629 (546.287); (3226.187, 5123.579)	824.855 (533.310); (−510.255, 1356.619)	0.0428 (0.8214)	0.5082* (0.7383)	0.0698 (0.7537)
	PITE ($\rho = 0.42$)	3.563 (0.471); (3.034, 4.666)	4232.474 (496.515); (3687.430, 5403.038)	394.713 (485.552); (−750.638, 926.379)	0.0394 (0.8885)	0.5514 (0.6949)	0.0642 (0.7878)
	PITE ($\rho = 0.68$)	3.976 (0.441); (3.367, 4.870)	4679.615 (457.028); (4030.859, 5613.114)	−42.571 (446.595); (−954.398, 578.046)	0.0373* (0.9233)	0.6210 (0.6283)	0.0653 (0.7811)
	PITE ($\rho = 1.00$)	3.496 (0.378); (2.970, 4.303)	4184.131 (386.606); (3638.721, 4982.672)	443.864 (365.319); (−304.327, 961.585)	0.0387 (0.9012)	0.5473 (0.699)	0.0625* (0.7985)
	MLE	2.494; (0.345); (2.036, 3.503)	3053.745; (290.165); (2586.595, 3863.940)	1514.003; (248.717); (763.469, 1921.688)	0.0552 (0.5232)	0.6476 (0.604)	0.1076 (0.5491)
	WMLE	2.494; (0.356); (2.036, 3.397)	3054.591; (297.257); (2587.647, 3752.728)	1513.764; (256.380); (880.644, 1921.832)	0.0550 (0.5276)	0.6458 (0.6056)	0.1070 (0.5518)
	MPSE	3.094; (0.170); (2.839, 3.497)	3788.629; (33.257); (3726.338, 3873.329)	824.855; (35.460); (773.147, 927.249)	0.0505 (0.6385)	0.5737 (0.673)	0.0890 (0.6418)
	MOM	1.127; (0.027); (1.079, 1.183)	760.111; (30.795); (697.168, 819.308)	3495.268; (77.486); (3347.862, 3658.431)	0.2913 (< 0.0001)	Inf (< 0.0001)	4.5392 (< 0.0001)
3	PITE ($\rho = 0.32$)	2.033; (0.696); (1.650, 4.237)	1.348; (0.387); (1.123, 2.616)	1.865; (0.360); (0.672, 2.041)	0.0660 (0.9464)	0.2636 (0.9626)	0.0411 (0.9286)
	PITE ($\rho = 0.42$)	2.089; (0.576); (1.691, 3.831)	1.383; (0.316); (1.164, 2.337)	1.833; (0.287); (0.930, 1.997)	0.0662 (0.9452)	0.2559* (0.9671)	0.0408 (0.9304)
	PITE ($\rho = 0.68$)	2.152; (0.526); (1.672, 3.607)	1.419; (0.271); (1.154, 2.151)	1.799; (0.235); (1.124, 2.005)	0.0669 (0.9408)	0.2582 (0.9658)	0.0409 (0.9299)
	PITE ($\rho = 1.00$)	2.057; (0.521); (1.547, 3.568)	1.369; (0.259); (1.090, 2.083)	1.846; (0.218); (1.204, 2.081)	0.0653 (0.9512)	0.2569 (0.9665)	0.0405* (0.9318)
	MLE	2.123; (0.453); (1.266, 3.042)	1.405; (0.231); (0.888, 1.740)	1.815; (0.188); (1.548, 2.205)	0.0680 (0.9328)	0.2567 (0.9666)	0.0409 (0.9298)
	WMLE	1.858; (1.265); (1.922, 6.768)	0.788; (0.377); (0.834, 2.078)	2.532; (0.400); (1.662, 2.532)	0.3240 (< 0.0001)	Inf (< 0.0001)	1.4999 (0.0002)
	MPSE	2.154; (0.502); (1.312, 3.285)	1.513; (0.266); (0.949, 1.928)	1.728; (0.222); (1.370, 2.169)	0.0612* (0.9722)	0.3414 (0.9038)	0.0462 (0.9003)
	MOM	1.064; (0.049); (0.981, 1.167)	0.383; (0.034); (0.318, 0.450)	2.686; (0.068); (2.565, 2.826)	0.3651 (< 0.0001)	Inf (< 0.0001)	1.5735 (0.0001)

Table 3. Parameter estimates with SE and 95% CI for all considered estimators, along with GoF assessments for the Weibull distribution across all datasets. *Indicates the smallest test statistic and the highest p-value.

and MOM estimators exhibit poorer performance, particularly indicated by higher test statistics and lower p-values. The MPSE performs moderately well but still does not match the robustness of PITE.

Based on these results, PITE with $\rho = 1.00$ is identified as the most optimal estimator for the Weibull parameters for both Datasets 1–3. If we average the rankings in terms of the GoF measures, PITE with $\rho = 1.00$ consistently ranks the highest across all criteria. Figure 6 presents the Weibull density functions fitted to the histograms of each dataset using the estimates obtained from all considered estimators. The figure provides a visual comparison of the model fits, demonstrating that the Weibull model estimated using PITE with $\rho = 1.00$ offers the best fit to the underlying data distribution across all datasets.

To further evaluate the performance of PITE, we compare the fitted three-parameter Weibull distribution with three modifications and extensions of the Weibull distribution: the Exponential Weibull (Exp Weibull)[38], Extended Weibull (Ext Weibull)[39], and Gamma Weibull (Gam Weibull)[40]. These distributions are extensions of the two-parameter Weibull distribution, each incorporating an additional parameter, making them part of the same class as the three-parameter Weibull distribution (see Table 4). The parameters of these distributions are estimated using MLE, and their GoF is assessed through the KS, AD, and CVM tests. As shown in Table 5, the three-parameter Weibull distribution fitted using PITE with $\rho = 1.00$, consistently outperforms the alternative

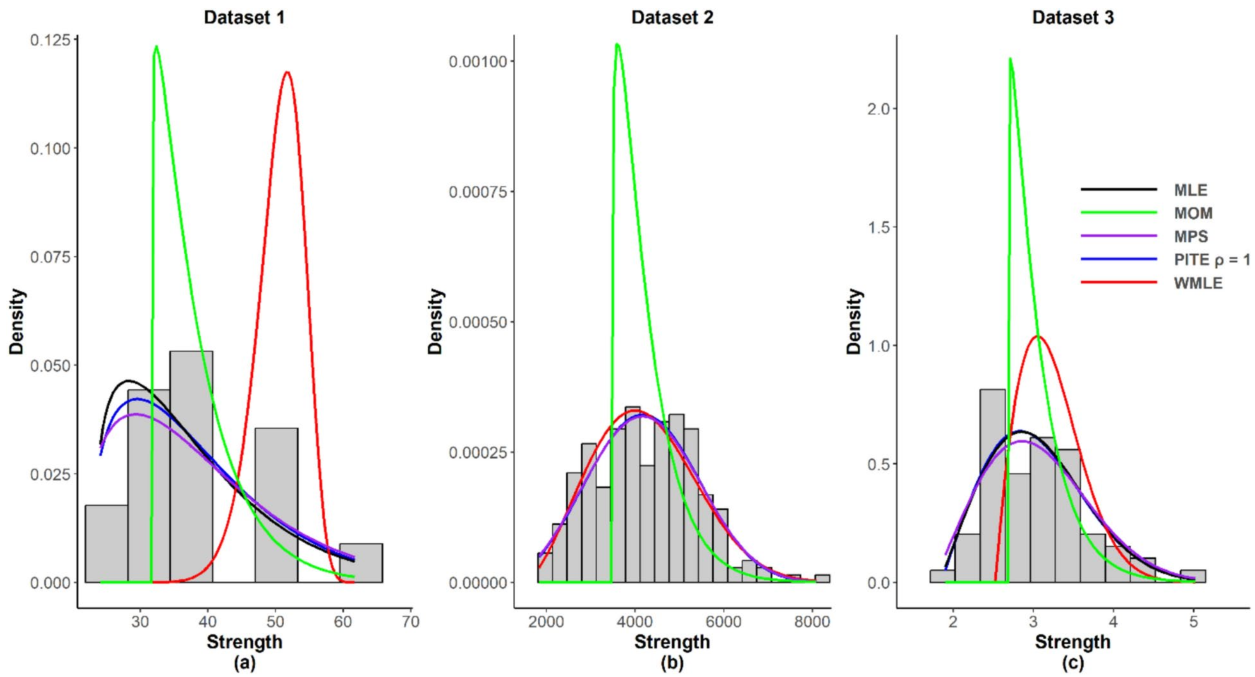

Fig. 6. Fitted Weibull density on the histogram of (**a**) Dataset 1, (**b**) Dataset 2, and (**c**) Dataset 3 using different estimators.

Distribution	CDF
Exp Weibull	$F(x) = [1 - \exp(-\mu x^{\sigma})]^{\nu}, x > 0, \mu > 0, \sigma > 0, \nu > 0$
Ext Weibull	$F(x) = \frac{1-\exp(-\mu x^{\sigma})}{1-(1-\nu)\exp(-\mu x^{\sigma})}, x > 0, \mu > 0, \sigma > 0, \nu > 0$
Gam Weibull	$F(x) = \frac{\gamma(\nu, \mu x^{\sigma})}{\Gamma(\nu)}, x > 0, \mu > 0, \sigma \geq 0, \nu > 0$ $\Gamma(\cdot)$ is gamma function and $\gamma(\cdot,\cdot)$ is incomplete beta function

Table 4. The distributions considered for comparison with the three-parameter Weibull distribution.

Dataset	Distribution	Estimates	KS (*p*-value)	AD (*p*-value)	CVM (*p*-value)
1	3p-Weibull PITE ($\rho = 1.00$)	$\hat{\theta} = 1.376; \hat{\beta} = 17.404; \hat{\gamma} = 22.794$	0.1124 (0.9575)	0.3436 (0.9010)	0.0465 (0.9026)
	Exp Weibull	$\hat{\mu} = 0.066; \hat{\sigma} = 1.125; \hat{\nu} = 0.821$	0.1331 (0.8668)	0.3894 (0.8575)	0.2175 (0.2368)
	Ext Weibull	$\hat{\mu} = 1.813 \times 10^{-5} - 05; \hat{\sigma} = 3.026; \hat{\nu} = 1.959$	0.1739 (0.5881)	0.6656 (0.586)	0.10502 (0.5656)
	Gamma Weibull	$\hat{\mu} = 0.111; \hat{\sigma} = 1.209; \hat{\nu} = 9.202$	0.24915 (0.1802)	1.0491 (0.3315)	0.2175 (0.2368)
2	3p-Weibull PITE ($\rho = 1.00$)	$\hat{\theta} = 3.496; \hat{\beta} = 4184.131; \hat{\gamma} = 443.864$	0.0387 (0.9012)	0.5473 (0.699)	0.0625 (0.7985)
	Exp Weibull	$\hat{\mu} = 0.003; \hat{\sigma} = 0.870; \hat{\nu} = 50.790$	0.08428 (0.09163)	2.3233 (0.06147)	0.38479 (0.0792)
	Ext Weibull	$\hat{\mu} = 1.929 \times 10^{-12}; \hat{\sigma} = 3.241; \hat{\nu} = 2.011$	0.04934 (0.6662)	0.847 (0.4482)	0.07557 (0.7184)
	Gamma Weibull	$\hat{\mu} = 0.029; \hat{\sigma} = 0.787; \hat{\nu} = 20.256$	0.07196 (0.2112)	1.8837 (0.1066)	0.25672 (0.18)
3	3p-Weibull PITE ($\rho = 1.00$)	$\hat{\theta} = 2.057; \hat{\beta} = 1.369; \hat{\gamma} = 1.846$	0.0653 (0.9512)	0.2569 (0.9665)	0.0405 (0.9318)
	Exp Weibull	$\hat{\mu} = 0.554; \hat{\sigma} = 1.672; \hat{\nu} = 21.510$	0.0824 (0.7862)	0.3351 (0.9093)	0.0623 (0.8016)
	Ext Weibull	$\hat{\mu} = 0.010; \hat{\sigma} = 4.201; \hat{\nu} = 1.954$	0.1064 (0.4731)	1.0133 (0.35)	0.1179 (0.5054)
	Gamma Weibull	$\hat{\mu} = 0.746; \hat{\sigma} = 1.940; \hat{\nu} = 6.779$	0.1687 (0.0555)	2.6583 (0.0411)	0.5101 (0.0372)

Table 5. Parameter estimates for all distributions, along with GoF assessments across all datasets.

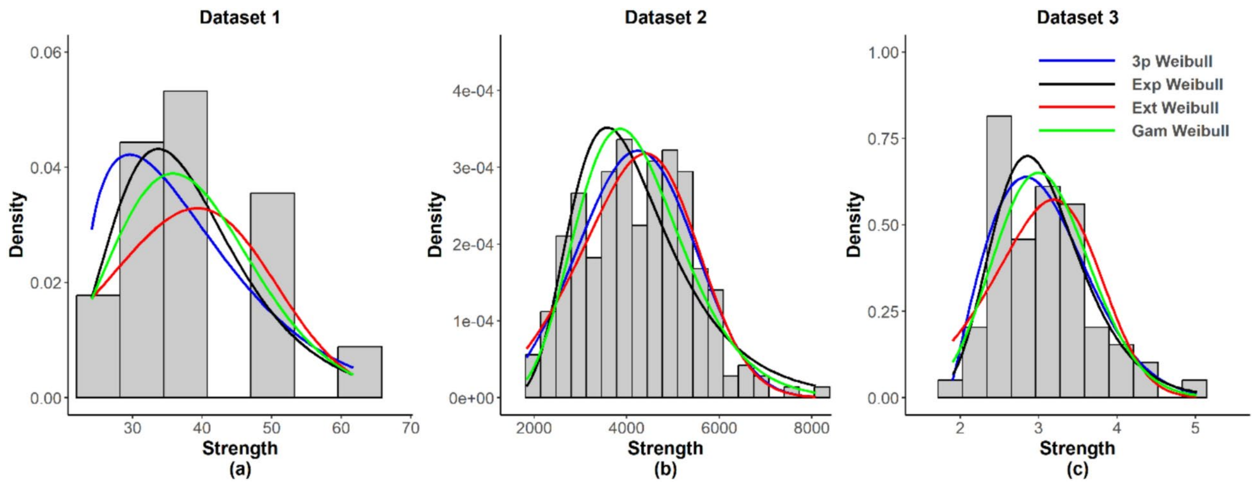

Fig. 7. Fitted density curves for the three-parameter Weibull, Exp Weibull, Ext Weibull, and Gam Weibull distributions across (**a**) Dataset 1, (**b**) Dataset 2, and (**c**) Dataset 3.

models across all datasets, demonstrating lower test statistics and higher p-values. Figure 7 visually confirms that the three-parameter Weibull distribution estimated using the PITE method with $\rho = 1.00$ provides the best fit to the observed data.

Given that the reliability metrics derived from the three-parameter Weibull distribution depend heavily on the parameters θ, β, and γ, choosing the most suitable estimation method is essential. Our study demonstrates that by applying the proposed PITE to real-world datasets, it serves as an effective alternative in determining the parameters of the three-parameter Weibull distribution, particularly in the presence of outliers. The results from both simulations and real data applications consistently highlight that PITE with $\rho = 1.00$ offers the best overall performance in outlier-prone datasets. However, for cleaner datasets with no or minimal outliers, traditional methods such as MLE may provide better performance. This suggests that while PITE is robust to outliers, careful selection of ρ is necessary depending on the data characteristics to achieve the optimal balance between robustness and efficiency. To this end, it is recommended to explore multiple values of ρ to identify the most suitable setting for a given dataset.

Conclusion

This study introduces the PITE, a new method for estimating the parameters of the Weibull distribution, particularly effective in addressing datasets with outliers. The robustness of PITE, attributed to its bounded SC, makes it a reliable choice in such scenarios. However, its efficiency, quantified through ARE, diminishes in datasets with minimal or no outliers, especially when the Weibull distribution's shape parameter is small. This limitation suggests that PITE may not be the optimal choice for cleaner datasets. Despite this, PITE's computational simplicity offers significant practical advantages. It consistently outperforms traditional estimators, including MLE, WMLE, MPSE, and MOM, in handling outlier-prone data.

In addition to comparing different estimation methods for the three-parameter Weibull distribution, this study also evaluates the performance of alternative three-parameter distributions, namely the Exp Weibull, Ext Weibull, and Gamma Weibull distributions. These distributions, which extend the traditional two-parameter Weibull by incorporating an additional parameter, offer greater flexibility in data modeling. However, the results indicate that the three-parameter Weibull distribution estimated using PITE ($\rho = 1.00$) provides the best overall fit to the datasets, demonstrating superior GoF performance compared to the alternative distributions. This underscores the effectiveness of PITE in modeling reliability data while maintaining robustness in the presence of outliers.

Future research opportunities for PITE are extensive. Future work will focus on extending its applicability to a broader range of statistical models and enhancing its capability to handle censored data. These advancements aim to overcome current limitations and expand its practical utility across diverse data scenarios.

Data availability

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

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Muhammad Aslam Mohd Safari: Conceptualization, Methodology, Software, Formal analysis, Writing—original draft, Project administration. Nurulkamal Masseran: Supervision, Writing – review & editing, Validation. Muhammad Hilmi Abdul Majid: Software, Writing – review & editing, Investigation. Razik Ridzuan Mohd Tajuddin: Writing – review & editing, Validation.

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Competing interests
The authors declare no competing interests.

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