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Equivalence of coercivity and mean coercivity in higher-order variational integrals with application to minimization

Xiaoying He[1,2] and Chuei Yee Chen[1*]

*Correspondence:
cychen@upm.edu.my
[1]Department of Mathematics and Statistics, Universiti Putra Malaysia, UPM, Serdang, 43400, Selangor Darul Ehsan, Malaysia
Full list of author information is available at the end of the article

Abstract

We consider a functional of the type $\mathcal{F}(u,\Omega) = \int_\Omega F(D^k u(x))dx$ on the Dirichlet class, where F is a continuous function and Ω is an open bounded set of $\mathbb{R}^n$ with a Lipschitz boundary. We prove that coercivity and mean coercivity are equivalent under growth conditions, and further we prove that mean coercivity and quasiconvexity are equivalent. Subsequently, we deduce that $\mathcal{F}(u,\Omega)$ has a minimum under the condition that the integrand F satisfies the growth condition and mean coercivity.

Keywords: Coercive; Mean coercive; Quasiconvex function; kth order partial derivative; Variational integral

1 Introduction

In recent years, there has been a renewed interest in higher-order variational problems, driven by their applications in diverse fields such as robotics, aviation, computer-aided design, and trajectory planning [1–5]. One such example is the uniform beam shape obtained by finding the minimum potential energy in mechanics is a second-order variational problem [6]. The higher variational integral considered in this paper takes the following form:

$$\mathcal{F}(u,\Omega) = \int_\Omega F\big(D^k u(x)\big)dx, \quad u \in W_g^{k,p}(\Omega,\mathbb{R}^m), \quad 1 < p < \infty, \tag{1}$$

for $n, m, k > 1$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary, $u : \Omega \to \mathbb{R}^m$, $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ is a continuous function, $\tau = \binom{n+k-1}{k}$, and $D^k u$ denotes the partial derivatives of order k. Here $W^{k,p}$ is the usual Sobolev space, and for a mapping $g \in W^{k,p}(\Omega,\mathbb{R}^m)$, we denote $W_g^{k,p}(\Omega,\mathbb{R}^m)$ to be a Dirichlet class defined by

$$W_g^{k,p}(\Omega,\mathbb{R}^m) = \big\{g + \varphi : \varphi \in W_0^{k,p}(\Omega,\mathbb{R}^m)\big\}. \tag{2}$$

Here F satisfies the growth condition

$$F(\xi) \le M(1 + |\xi|^p), \tag{3}$$

where $\xi \in \mathbb{R}^{m\tau}$ and $M > 0$.

The direct method in the calculus of variations is a powerful abstract method for proving the existence of minimizer for variational problems based on the assumptions that the integrand satisfies the growth and coercivity conditions, and the functional is sequentially weakly lower semicontinuous [7–9]. It turns out that convexity (scalar case) or quasiconvexity (vectorial case) of the integrand is a necessary and sufficient condition for the functional to be sequentially weakly lower semicontinuous [10–12].

For the case when $k = 1$, we say that F is coercive if

$$F\big(Du(x)\big) \geq \alpha_1 |Du(x)|^p + \beta_1,\ \alpha_1 > 0,\ \beta_1 \in \mathbb{R},\ u \in W_g^{1,p}(\Omega, \mathbb{R}^m).$$

In many cases, coercivity turns out to be restrictive and may not be satisfied [13]. However, they satisfy a weaker condition known as mean coercivity condition defined as follows:

$$\int_\Omega F\big(Du(x)\big)dx \geq \alpha_2 \int_\Omega |Du(x)|^p + \beta_2,\ \alpha_2 > 0,\ \beta_2 \in \mathbb{R},\ u \in W_g^{1,p}(\Omega, \mathbb{R}^m).$$

Considering functionals on the Dirichlet class $W_g^{1,p}(\Omega, \mathbb{R}^m) = \{g + \varphi : \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^m)\}$, where $\Omega \subset \mathbb{R}^n$ and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$, Chen and Kristensen [13] established the equivalence of coercivity and mean coercivity, which are also equivalent to a quasiconvexity condition. In the study by Gmeineder and Kristensen [14] on minimizing variational integrals of linear growth defined on the Dirichlet class of bounded variation maps, they showed that the existence of minimizer is based on the mean coercivity of the maps, which in turn is equivalent to the strong quasiconvexity of the variational integral. The strong quasiconvexity is essential for proving the partial regularity of the minimizer. Apart from that, coercivity and mean coercivity play a role in regularity theory, of which they are essential in establishing higher integrability of minimizers [15, 16] and partial regularity [17, 18].

At the same time, in the study of plate and shell bending in elasticity [19], nonlinear wave equations [20], and viscous effects in fluid mechanics [21], the mathematical models often involve second or higher order partial derivatives. With regards to this, we study the relation between coercivity, mean coercivity, and quasiconvexity to improve the variational principle for solving problems with functionals involving higher order derivatives.

In Sect. 2, we give some notation and preliminary definitions. In Sect. 3, we prove the equivalence of coercivity and mean coercivity under growth condition. Finally, we obtain the equivalence of mean coercivity and quasiconvexity, which leads to the existence theorem.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space $x = (x_1, \dots, x_n)$ with norm of $|x| = \big(\sum_{i=1}^n (x_i)^2\big)^{\frac{1}{2}}$. We denote by $u = (u_1, \dots, u_m)$ a vector valued function $u : \Omega \to \mathbb{R}^m$ and by $D^\alpha u = (D^{\alpha_1} u, \dots, D^{\alpha_n} u)$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\alpha_1 + \cdots + \alpha_n = |\alpha|$. For $k \in \mathbb{N}$, we define $D^k u = (D^\alpha u)_{|\alpha|=k}$.

In this study, $W^{k,p}(\Omega, \mathbb{R}^m)$ is the Sobolev space composed of functions $u : \Omega \to \mathbb{R}^m$ whose all weak partial derivatives of order $|\alpha| \leq k$ belong to $L^p(\Omega, \mathbb{R}^m)$. For $p \in (1, \infty)$, its norm is as follows:

$$\|u\|_{W^{k,p}(\Omega;\mathbb{R}^n)} = \Big(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u(x)\|^p_{L^p(\Omega, \mathbb{R}^m)} \Big)^{\frac{1}{p}}.$$

$W_0^{k,p}(\Omega,\mathbb{R}^m)$ is the closure of $C_0^\infty(\Omega,\mathbb{R}^m)$ in $W^{k,p}(\Omega,\mathbb{R}^m)$. In addition, we consider $W_g^{k,p}(\Omega,\mathbb{R}^m)$ to be the Dirichlet class defined by (2).

For $1 < p < \infty$, we say that the sequence u_h weakly converges to u if $u_h, u \in W^{k,p}(\Omega,\mathbb{R}^m)$ and

$$\lim_{h\to\infty}\int_\Omega \big[u_h(x) - u(x)\big]\varphi(x)dx = 0,$$

and

$$\lim_{h\to\infty}\int_\Omega \big[D^\alpha u_h(x) - D^\alpha u(x)\big]\varphi(x)dx = 0,$$

where $|\alpha| \le k$, $\frac{1}{p} + \frac{1}{p'} = 1$ and for all $\varphi \in L^{p'}(\Omega,\mathbb{R}^m)$. We denote this convergence by $u_h \rightharpoonup u$ in $W^{k,p}(\Omega,\mathbb{R}^m)$.

In the following, we extend the definition of coercivity and mean coercivity to $k > 1$.

Definition 1 Let $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ be a continuous function. We say that F is coercive if

$$F\big(D^k u(x)\big) \ge \alpha_1 |D^k u(x)|^p + \beta_1,$$

where $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$ and for all $u \in W_g^{k,p}(\Omega,\mathbb{R}^m)$.

Definition 2 Let $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ be a continuous function. We say that F is mean coercive if

$$\int_\Omega F\big(D^k u(x)\big)dx \ge \alpha_2 \int_\Omega |D^k u(x)|^p dx + \beta_2,$$

where $\alpha_2 > 0$, $\beta_2 \in \mathbb{R}$ and for all $u \in W_g^{k,p}(\Omega,\mathbb{R}^m)$.

Definition 3 Let $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ be a continuous function. We say that F is quasiconvex if

$$F(\xi) \le \frac{1}{\text{meas}(\Omega)} \int_\Omega F\big(\xi + D^k \varphi(x)\big)dx \tag{4}$$

for all $\xi \in \mathbb{R}^{m\tau}$ and for all $\varphi \in C_0^\infty(\Omega,\mathbb{R}^m)$.

Proposition 1 ([22, 23]) *The definition of quasiconvexity is independent of the choice of the bounded open set* Ω, *that is, if* (4) *holds for some bounded open set* $\Omega_0 \subset \mathbb{R}^n$, *then* (4) *holds for any bounded open set.*

3 Equivalence of coercivity and mean coercivity

In this section, we prove the equivalence of coercivity and mean coercivity. The following lemma is crucial for proving that mean coercivity implies coercivity.

Lemma 1 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $\xi_0 \in \mathbb{R}^{m\tau}$ *and* $u_j \rightharpoonup 0$ *on* $W^{k,p}(\Omega,\mathbb{R}^m)$. *Then, for* $1 < p < \infty$,

$$\lim_{j\to\infty}\int_\Omega \Big[\big|\xi_0 + D^k u_j\big|^p - |\xi_0|^p\Big]dx = 0. \tag{5}$$

Proof We take $F(z) = |z|^p$, $z \in \mathbb{R}^{m\tau}$ and $(\eta_\varepsilon)_{\varepsilon>0}$ to be a standard smooth mollifier on $\mathbb{R}^{m\tau}$. Let $F_\varepsilon(z) = (\eta_\varepsilon * F)(z) = \int_{\mathbb{R}^{m\tau}} \eta_\varepsilon(z-y)F(y)dy$. Then F_ε is smooth and convex, and $\lim_{\varepsilon\to 0} F_\varepsilon = F$. Applying the mean value theorem to F_ε and by the usual inner product of Euclidean space, we get

$$\begin{aligned} F_\varepsilon(\xi_0 + D^k u_j) - F_\varepsilon(\xi_0) &= F_\varepsilon'(t_0) \cdot D^k u_j \\ &= \left(\eta_\varepsilon * DF(t_0)\right) \cdot D^k u_j \\ &= (\eta_\varepsilon * p|t_0|^{p-1} t_0) \cdot D^k u_j, \end{aligned} \tag{6}$$

where t_0 is some number that lies between ξ_0 and $\xi_0 + D^k u_j$. Taking $\varepsilon \to 0$ from both sides of (6), we get

$$F(\xi_0 + D^k u_j) - F(\xi_0) = |\xi_0 + D^k u_j|^p - |\xi_0|^p = p|t_0|^{p-1} t_0 \cdot D^k u_j. \tag{7}$$

Due to $p|t_0|^{p-1}t_0 \in L^{p'}$ and $u_j \rightharpoonup 0$ on $W^{k,p}(\Omega, \mathbb{R}^m)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, invoking the definition of weak convergence directly leads us to conclude

$$\lim_{j\to\infty} \int_\Omega [|\xi_0 + D^k u_j|^p - |\xi_0|^p]dx = \lim_{j\to\infty} \int_\Omega p|t_0|^{p-1} t_0 \cdot D^k u_j dx = 0. \qquad \square$$

We may now state the theorem.

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary and $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ be a continuous function satisfying the growth condition*

$$F(\xi) \le M\left(1 + |\xi|^p\right),$$

where $M > 0$. Let

$$\mathcal{F}(u, \Omega) = \int_\Omega F\left(D^k u(x)\right)dx, \quad u \in W_g^{k,p}(\Omega, \mathbb{R}^m).$$

Then, for an exponent $1 < p < \infty$, the following two conditions are equivalent:

(*i*) *F is coercive.*

(*ii*) *F is mean coercive.*

Proof Coercivity implies mean coercivity is trivial, and the proof is omitted here. The converse will be proved in two steps. In the first step, we show that mean coercivity derives coercivity under a cube Q parallel to the axes. In the second step, we extend the cube Q to the general domain Ω.

Step 1. Suppose that $Q \subset \mathbb{R}^n$ is a cube parallel to the axes and sequence $u_j \rightharpoonup u$ in $W_g^{k,p}(Q, \mathbb{R}^m)$. Let $Q_0 \subset\subset Q$ be a cube and set $R = \frac{1}{2}\mathrm{dist}(Q_0, \partial Q)$. Let N be a positive integer and

$$Q_i = \left\{x \in Q : \mathrm{dist}(x, Q_0) < \frac{iR}{N}\right\}, \quad i = 1, \ldots, N.$$

Therefore, $Q_0 \subset Q_1 \subset \cdots \subset Q_N \subset Q$.

We introduce a cutoff function $\phi_i \in C_0^\infty(Q_i)$ such that

$$0 \le \phi_i \le 1;\ \phi_i = \begin{cases} 1 & \text{in } \mathrm{Q}_{i-1} \\ 0 & \text{in } \mathrm{Q}-\mathrm{Q_i} \end{cases};$$

$$|D^l\phi_i| \le c\Big(\frac{N+1}{R}\Big)^l,\ \ l = 1,\dots,k,$$

where $c > 0$ is a constant. Let

$$v_{ij} = \phi_i(u_j - u), \tag{8}$$

and we get

$$v_{ij} \rightharpoonup 0 \text{ in } W_g^{k,p}(Q,\mathbb{R}^m).$$

Assuming that ξ_0 is a constant vector in $\mathbb{R}^{m\tau}$, we have

$$\begin{aligned}\int_Q F(\xi_0 + D^k v_{ij})dx &= \int_{Q-Q_i} F(\xi_0)dx + \int_{Q_i-Q_{i-1}} F(\xi_0 + D^k v_{ij})dx \\ &\quad + \int_{Q_{i-1}} F\big(\xi_0 + D^k(u_j - u)\big)dx \\ &\le \int_{Q-Q_0} F(\xi_0)dx + \int_{Q_i-Q_{i-1}} F(\xi_0 + D^k v_{ij})dx \\ &\quad + \int_Q F\big(\xi_0 + D^k(u_j - u)\big)dx. \end{aligned} \tag{9}$$

Taking the summation from $i = 1$ to $i = N$ about (9), we get

$$\begin{aligned}\sum_{i=1}^N \int_Q F(\xi_0 + D^k v_{ij})dx &\le N\int_{Q-Q_0} F(\xi_0)dx + \sum_{i=1}^N \int_{Q_i-Q_{i-1}} F(\xi_0 + D^k v_{ij})dx \\ &\quad + N\int_Q F\big(\xi_0 + D^k(u_j - u)\big)dx. \end{aligned} \tag{10}$$

We now estimate for the terms on the right-hand side of (10), beginning with the second term:

$$\begin{aligned}&\sum_{i=1}^N \int_{Q_i-Q_{i-1}} F(\xi_0 + D^k v_{ij})dx \\ &\le \sum_{i=1}^N \int_{Q_i-Q_{i-1}} M(1 + |\xi_0 + D^k v_{ij}|^p)dx \\ &\le \sum_{i=1}^N \int_{Q_i-Q_{i-1}} M2^{p-1}(1 + |\xi_0|^p + |D^k v_{ij}|^p)dx \\ &\le \sum_{i=1}^N \int_{Q_i-Q_{i-1}} c\Big[1 + |\xi_0|^p + \sum_{l=1}^k |D^{k-l}(u_j - u)|^p\Big(\frac{N+1}{R}\Big)^{lp}\Big]dx \\ &\le \int_Q c\Big[1 + |\xi_0|^p + \sum_{l=1}^k |D^{k-l}(u_j - u)|^p\Big(\frac{N+1}{R}\Big)^{lp}\Big]dx, \end{aligned} \tag{11}$$

where we use the Leibniz formula [24] and Minkowski's inequality for $|D^k v_{ij}|^p$, that is,

$$\begin{aligned}|D^k v_{ij}|^p &\le \Big(\sum_{l=0}^{k}\binom{k}{l}|D^{k-l}(u_j-u)|\cdot|D^k\phi_i|\Big)^p\\ &\le \sum_{l=0}^{k}\binom{k}{l}^p|D^{k-l}(u_j-u)|^p\cdot|D^l\phi_i|^p\\ &\le \sum_{l=1}^{k}\binom{k}{l}^p|D^{k-l}(u_j-u)|^p\Big(\frac{N+1}{R}\Big)^{lp}\\ &\le c\sum_{l=1}^{k}|D^{k-l}(u_j-u)|^p\Big(\frac{N+1}{R}\Big)^{lp}. \end{aligned}\tag{12}$$

Here $\binom{k}{l}^p \le 2^{kp} \le c$. Substituting (11) into (9) and dividing both sides of the inequality by N, we obtain

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\int_Q F(\xi_0+D^k v_{ij})dx\\ &\le F(\xi_0)\mathrm{meas}(Q-Q_0)+\int_Q F\big(\xi_0+D^k(u_j-u)\big)dx\\ &\quad+\frac{c}{N}\int_Q\Big[1+|\xi_0|^p+\sum_{l=1}^{k}|D^{k-l}(u_j-u)|^p\Big(\frac{N+1}{R}\Big)^{lp}\Big]dx. \end{aligned}\tag{13}$$

We next estimate the second term on the right-hand side of (13). Since F is continuous and $u_j \rightharpoonup u$ in $W_g^{k,p}(Q,\mathbb{R}^m)$, we have

$$F\big(\xi_0+D^k(u_j-u)\big)\le \liminf_{j\to\infty}\big|F\big(\xi_0+D^k(u_j-u)\big)\big|.$$

By Fatou's lemma [25], we get

$$\begin{aligned}\int_Q F\big(\xi_0+D^k(u_j-u)\big)dx &\le \int_Q \liminf_{j\to\infty}\big|F\big(\xi_0+D^k(u_j-u)\big)\big|dx\\ &\le \liminf_{j\to\infty}\int_Q\big|F(\xi_0+D^k(u_j-u))\big|dx. \end{aligned}\tag{14}$$

Inequality (13) can be further rewritten as

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\int_Q F(\xi_0+D^k v_{ij})dx\\ &\le F(\xi_0)\mathrm{meas}(Q)+\liminf_{j\to\infty}\int_Q\big|F\big(\xi_0+D^k(u_j-u)\big)\big|dx\\ &\quad+\frac{c\,\mathrm{meas}(Q)(1+|\xi_0|^p)}{N}+\frac{c}{N}\int_Q\sum_{l=1}^{k}|D^{k-l}(u_j-u)|^p\Big(\frac{N+1}{R}\Big)^{lp}dx. \end{aligned}\tag{15}$$

On the other hand, F is mean coercive, we have

$$\begin{aligned}\int_Q F(\xi_0 + D^k v_{ij})dx &\geq \alpha_2 \int_Q |\xi_0 + D^k v_{ij}|^p dx + \beta_2 \\ &= \alpha_2 \int_{Q-Q_i} |\xi_0|^p dx + \alpha_2 \int_{Q_i - Q_{i-1}} |\xi_0 + D^k v_{ij}|^p dx \\ &\quad + \alpha_2 \int_{Q_{i-1}} |\xi_0 + D^k(u_j - u)|^p dx + \beta_2. \end{aligned} \tag{16}$$

Taking the summation from $i = 1$ to $i = N$, we get

$$\begin{aligned}&\sum_{i=1}^N \int_Q F(\xi_0 + D^k v_{ij})dx \\ &\geq \alpha_2 \sum_{i=1}^N \int_{Q-Q_i} |\xi_0|^p dx + \alpha_2 \sum_{i=1}^N \int_{Q_i - Q_{i-1}} |\xi_0 + D^k v_{ij}|^p dx \\ &\quad + \alpha_2 \sum_{i=1}^N \int_{Q_{i-1}} |\xi_0 + D^k(u_j - u)|^p dx + N\beta_2. \end{aligned} \tag{17}$$

The first term on the right-hand side of (16) is rewritten as

$$\begin{aligned}&\sum_{i=1}^N \int_{Q-Q_i} |\xi_0|^p dx \\ &= \sum_{i=1}^N \int_Q |\xi_0|^p dx - \sum_{i=1}^N \int_{Q_i} |\xi_0|^p dx \\ &= \sum_{i=1}^N \int_Q |\xi_0|^p dx - \sum_{i=1}^N \int_{Q_i - Q_{i-1}} |\xi_0|^p dx - \sum_{i=1}^N \int_{Q_{i-1}} |\xi_0|^p dx. \end{aligned} \tag{18}$$

Substituting (18) into (17) and dividing both sides by N, we have

$$\begin{aligned}&\frac{1}{N} \sum_{i=1}^N \int_Q F(\xi_0 + D^k v_{ij})dx \\ &\geq \alpha_2 |\xi_0|^p \operatorname{meas}(Q) + \frac{\alpha_2}{N} \sum_{i=1}^N \int_{Q_i - Q_{i-1}} \left[|\xi_0 + D^k v_{ij}|^p - |\xi_0|^p\right] dx \\ &\quad + \frac{\alpha_2}{N} \sum_{i=1}^N \int_{Q_{i-1}} \left[|\xi_0 + D^k(u_j - u)|^p - |\xi_0|^p\right] dx + \beta_2. \end{aligned} \tag{19}$$

Combining (15) and (19), we get

$$\begin{aligned}&\alpha_2 |\xi_0|^p \operatorname{meas}(Q) + \frac{\alpha_2}{N} \sum_{i=1}^N \int_{Q_i - Q_{i-1}} \left[|\xi_0 + D^k v_{ij}|^p - |\xi_0|^p\right] dx \\ &\quad + \frac{\alpha_2}{N} \sum_{i=1}^N \int_{Q_{i-1}} \left[|\xi_0 + D^k(u_j - u)|^p - |\xi_0|^p\right] dx + \beta_2 \end{aligned}$$

$$\leq F(\xi_0)\text{meas}(Q) + \liminf_{j\to\infty}\int_Q \big|F\big(\xi_0 + D^k(u_j - u)\big)\big|dx$$
$$+ \frac{c\,\text{meas}(Q)(1+|\xi_0|^p)}{N} + \frac{c}{N}\int_Q \sum_{l=1}^{k} |D^{k-l}(u_j - u)|^p\Big(\frac{N+1}{R}\Big)^{lp}dx. \tag{20}$$

We first estimate the second term on left-hand side of (20). By Lemma 1, we get

$$\lim_{j\to\infty}\int_{Q_i - Q_{i-1}}\big[|\xi_0 + D^k v_{ij}|^p - |\xi_0|^p\big]dx = 0 \tag{21}$$

and

$$\lim_{j\to\infty}\int_{Q_{i-1}}\big[|\xi_0 + D^k(u_j - u)|^p - |\xi_0|^p\big]dx = 0. \tag{22}$$

It follows from the Rellich–Kondrachov theorem that $\|D^l u_j - D^l u\|_{L^p(\Omega,\mathbb{R}^m)} \to 0$ for $l = 0,\dots,k-1$. Letting $j\to\infty$ and $N\to\infty$ on both sides of (20) and dividing both sides of the inequality by meas(Q), we get

$$F(\xi_0) \geq \alpha_1|\xi_0|^p + \beta_1,$$

where $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2 - \liminf_{j\to\infty}\int_Q \big|F\big(\xi_0 + D^k(u_j - u)\big)\big|dx$.

Step 2. We let $\delta > 0$ and h be an integer and approximate Ω with a union of cubes Q_s parallel to the axes in $\mathbb{R}^n$ and whose edge length is $1/h$. We denote this union by H_h and choose h large enough so that

$$\text{meas}(\Omega - H_h) \leq \delta \ \text{ where } \ H_h := \bigcup Q_s.$$

We set

$$\xi_s := \frac{1}{\text{meas}(Q_s)}\int_{Q_s} D^k u(x)dx.$$

Denote by ξ equal to ξ_s on the cube Q_s. For fixed $\varepsilon > 0$, we choose small δ such that

$$\Big(\sum_s \int_{Q_s} |D^k u(x) - \xi_s|^p dx\Big)^{\frac{1}{p}} < \varepsilon.$$

Since F satisfies the mean coercivity, we have

$$\int_{Q_s} F(\xi_s + D^k v_{ij})dx \geq \alpha_2 \int_{Q_s} |\xi_s + D^k v_{ij}|^p dx + \beta_2,$$

where $v_{ij} = \phi_i(u_j - u)$ as in (8). From step 1, we can deduce that

$$F(\xi_s) \geq \alpha_1|\xi_s|^p + \beta_1. \tag{23}$$

Taking the integral of both sides of (23) over Q_s and summing it up, we have

$$\sum_s \int_{Q_s} F(\xi_s)dx \geq \alpha_1 \sum_s \int_{Q_s} |\xi_s|^p dx + \sum_s \int_{Q_s} \beta_1 dx.$$

So we obtain

$$\int_{H_h} F(\xi)dx \geq \alpha_1 \int_{H_h} |\xi|^p dx + \int_{H_h} \beta_1 dx. \tag{24}$$

Dividing both sides of inequality (24) by meas(H_h), we have

$$F(\xi) \geq \alpha_1 |\xi|^p + \beta_1.$$

It follows that F is coercive. □

4 Equivalence of mean coercivity and quasiconvexity

In this section, we prove the equivalence of mean coercivity and quasiconvexity.

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary and $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ be a continuous function satisfying the growth condition*

$$F(\xi) \leq M(1 + |\xi|^p), \tag{25}$$

and

$$|F(\xi) - F(\eta)| \leq L(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|, \tag{26}$$

where $\xi, \eta \in \mathbb{R}^{m\tau}$, $M > 0$, and $0 < L < \infty$. Let

$$\mathcal{F}(u, \Omega) = \int_\Omega F\big(D^k u(x)\big)dx, \; u \in W_g^{k,p}(\Omega, \mathbb{R}^m). \tag{27}$$

Then, for an exponent $1 < p < \infty$, the following two conditions are equivalent:
(i) *F is mean coercive.*
(ii) *F is quasiconvex.*

Remark 1 Replacing condition (26) with Lipschitz continuity or local Lipschitz continuity still leads to the same conclusion. However, condition (26) does not necessitate the function to be differentiable, thereby extending its applicability.

Proof **(i)** $\Rightarrow$ **(ii)**. It is noted that if inequality (4) holds for a bounded open set $Q \subset \Omega$, then it holds for any bounded open set. We therefore discuss one particular cube Q defined below. We will show that the mean coercivity in Ω implies quasiconvexity in Q.

Let $Q \subset \mathbb{R}^n$ be a cube whose planes are parallel to the axes, and $\xi \in \mathbb{R}^{m\tau}$. Let $\varphi \in W_g^{k,p}(Q, \mathbb{R}^m)$ be extended by periodicity from Q to $\mathbb{R}^n$, that is, if the edge length of Q is d, then

$$\varphi(x + dz) = \varphi(x) \;\text{ for every } x \in Q \text{ and } z \in \mathbb{Z}^n.$$

Let ν be an integer and let

$$\varphi_\nu(x) := \frac{1}{\nu^k}\varphi(\nu x).$$

We have

$$\varphi_\nu \rightharpoonup 0 \text{ in } W_g^{k,p}(Q,\mathbb{R}^m).$$

Defining $u_s(x) := \frac{1}{k!}\xi_0 x^k$, $D^k u_s = \xi_0$, ξ_0 is a constant vector in $\mathbb{R}^{m\tau}$, and letting

$$u_\nu(x) := \begin{cases} u_s(x), & x \in \Omega - Q, \\ u_s(x) + \dfrac{1}{\nu^k}\varphi(\nu x), & x \in Q, \end{cases}$$

we have

$$u_\nu \rightharpoonup u_s \text{ in } W_g^{k,p}(\Omega,\mathbb{R}^m).$$

Since F satisfies the growth condition and mean coercivity, there exists $u_0 \in W_g^{k,p}(\Omega,\mathbb{R}^m)$ such that

$$|\mathcal{F}(u_0,\mathbb{R}^m)| < \infty.$$

We observe that

$$\begin{aligned}\mathcal{F}(u_\nu,\Omega) &= \int_\Omega F\big(D^k u_\nu(x)\big)dx \\ &= \int_{\Omega-Q} F(\xi_0)dx + \int_Q F\big(\xi_0 + D^k\varphi(\nu x)\big)dx \\ &= F(\xi_0)\text{ meas}(\Omega - Q) + \frac{1}{\nu^n}\int_{\nu Q} F\big(\xi_0 + D^k\varphi(y)\big)dy \\ &= F(\xi_0)\text{ meas}(\Omega - Q) + \int_Q F\big(\xi_0 + D^k\varphi(y)\big)dy, \end{aligned} \tag{28}$$

where we have used in the last equality the periodicity of φ. We can rewrite (28) as

$$\int_Q F\big(\xi_0 + D^k\varphi(y)\big)dy - F(\xi_0)\text{ meas}(Q) = \int_\Omega F\big(D^k u_\nu(x)\big)dx - \int_\Omega F(\xi_0)dx. \tag{29}$$

Using (26), we get

$$\begin{aligned}&\Big|\int_\Omega \big(F(D^k u_\nu(x)\big)dx - \int_\Omega F(\xi_0)dx\Big| \\ &\le \int_\Omega \big|F\big(D^k u_\nu(x)\big) - F(\xi_0)\big|dx \\ &\le c\int_\Omega (1 + |\xi_0|^{p-1} + |D^k u_\nu(x)|^{p-1})|D^k u_\nu(x) - \xi_0|dx. \end{aligned} \tag{30}$$

We estimate the right-hand side of (30). By applying Hölder's inequality, $u_\nu \in W_g^{k,p}(\Omega,\mathbb{R}^m)$, and Ω is bounded, we have

$$\begin{aligned}&\int_\Omega \big(1 + |\xi_0|^{p-1} + |D^k u_\nu(x)|^{p-1}\big)\,dx \\ &\le \Big(\int_\Omega \big(1 + |\xi_0|^{p-1} + |D^k u_\nu(x)|^{p-1}\big)^{\frac{p}{p-1}}\,dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega 1^p dx\Big)^{\frac{1}{p}}\end{aligned}$$

$$\le \left(\int_\Omega \left(1 + |\xi_0|^p + |D^k u_\nu(x)|^p\right) dx \right)^{\frac{p-1}{p}} \left(\text{meas}(\Omega)\right)^{\frac{1}{p}}$$
$$< \infty. \tag{31}$$

Thus we obtain that $(1 + |\xi_0|^{p-1} + |D^k u_\nu(x)|^{p-1}) \in L^{p'}(\Omega, \mathbb{R}^m)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Due to $u_\nu \rightharpoonup u_s$ in $W_g^{k,p}(\Omega, \mathbb{R}^m)$, invoking the definition of weak convergence leads us to conclusion

$$\int_\Omega \left(1 + |\xi_0|^{p-1} + |D^k u_\nu(x)|^{p-1}\right) |D^k u_\nu(x) - \xi_0| dx \le \varepsilon \text{ as } \upsilon \to \infty. \tag{32}$$

Substituting (32) and (30) into (29) and letting $\upsilon \to \infty$, we can finally obtain

$$\int_Q F\left(\xi_0 + D^k \varphi(y)\right) dy - F(\xi_0) \text{ meas}(Q) \ge -\varepsilon.$$

Taking into account the fact that ε is arbitrarily small, we get

$$\int_Q F\left(\xi_0 + D^k \varphi(y)\right) dy \ge F(\xi_0) \text{ meas}(Q).$$

(ii) $\Rightarrow$ **(i)**. We assume that F is quasiconvex and let

$$G(D^k u) = F(D^k u) - \gamma |D^k u|^p, \tag{33}$$

where γ is to be determined later. We next show that G is also quasiconvex. For $\xi \in \mathbb{R}^{m\tau}$, $\varphi \in W_g^{k,p}(\Omega, \mathbb{R}^m)$, we have

$$\begin{aligned} \int_\Omega G\left(\xi + D^k \varphi(x)\right) dx &= \int_\Omega F\left(\xi + D^k \varphi(x)\right) dx - \gamma \int_\Omega |\xi + D^k \varphi(x)|^p dx \\ &\ge \int_\Omega F(\xi) dx - \gamma \int_\Omega |\xi|^p dx + \gamma \int_\Omega [|\xi|^p - |\xi + D^k \varphi(x)|^p] dx, \end{aligned} \tag{34}$$

where we have invoked the quasiconvexity of F. Take appropriate $\gamma < \infty$ such that

$$\gamma \int_\Omega \left[|\xi|^p - |\xi + D^k \varphi(x)|^p\right] dx > 0. \tag{35}$$

From (34), we have

$$\int_\Omega G\left(\xi + D^k \varphi(x)\right) dx \ge \int_\Omega F(\xi) dx - \gamma \int_\Omega |\xi|^p dx = \int_\Omega G(\xi) dx. \tag{36}$$

Thus G is quasiconvex. By (33), we get

$$\int_\Omega F\left(\xi + D^k \varphi(x)\right) dx \ge \gamma \int_\Omega |\xi + D^k \varphi(x)|^p dx + \int_\Omega F(\xi) dx - \gamma \int_\Omega |\xi|^p dx.$$

Taking $\alpha_2 = \gamma$, $\beta_2 = \int_\Omega [F(\xi) - \gamma |\xi|^p] dx$, we have

$$\int_\Omega F\left(\xi + D^k \varphi(x)\right) dx \ge \alpha_2 \int_\Omega |\xi + D^k \varphi(x)|^p dx + \beta_2.$$

This completes the proof. □

From Theorem 2 and Theorem 3, we can obtain an existence theorem.

Proposition 4 *Let* $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ *be a bounded open set with a Lipschitz boundary, and let* $F : \mathbb{R}^{m\tau} \to \mathbb{R}$ *be a continuous function satisfying the growth condition*

$$F(\xi) \le M(1 + |\xi|^p)$$

and mean coercivity

$$\int_\Omega F\big(D^k u(x)\big)dx \ge \alpha_2 \int_\Omega |D^k u(x)|^p dx + \beta_2.$$

Further, assume that

$$|F(\xi) - F(\eta)| \le L(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|,$$

where $\alpha_2 > 0$, $\beta_2 \in \mathbb{R}$, $\xi, \eta, u \in W_g^{k,p}(\Omega)$, $M > 0$, *and* $0 < L < \infty$. *Then*

$$\mathcal{F}(u, \Omega) = \int_\Omega F\big(D^k u(x)\big)dx$$

admits at least one minimizer.

Proof The proof follows from Theorem 2, Theorem 3, and the general existence theorem. □

5 Conclusion

In this paper, we studied the equivalence of coercivity, mean coercivity, and quasiconvexity of integrands for functionals of the type $\mathcal{F}(u, \Omega) = \int_\Omega F\big(D^k u(x)\big)dx$ on the Dirichlet class, thus improving the variational principle for solving problems with functionals involving higher order derivatives and integrands satisfying the standard growth condition. This equivalence is significant as it allows the use of mean coercivity, which is often easier to verify in practical applications. Consequently, this result supports the existence of minimizers for variational problems under broader conditions, making it a powerful tool in the study of higher-order variational integrals.

For future work, one can consider other growth conditions such as nonpolynomial growth and controlled growth (for functionals involving lower order terms) to establish the equivalence between the corresponding mean coercivity condition and quasiconvexity. Following this, further study can also be conducted to investigate the regularity and stability of solutions when the growth rate varies.

Author contributions
Both authors have made equal contributions. The first draft of the manuscript was written by X. He and both authors commented on previous versions of the manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics and Statistics, Universiti Putra Malaysia, UPM, Serdang, 43400, Selangor Darul Ehsan, Malaysia. [2]Faculty of Science, Guangxi University of Science and Technology, WenChang Road, Liuzhou, 545006, Guangxi, China.

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