CANONICAL GROUP QUANTIZATION ON NON-COTANGENT BUNDLE PHASE SPACE AND ITS APPLICATION IN QUANTUM INFORMATION THEORY

By

AHMAD HAZAZI BIN AHMAD SUMADI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

Dedicated with sincere gratitude this humble opus to Rahmah, ‘Atif and Ibrahim who inspired me to seek knowledge...

Rabbi zidnī ‘ilman!

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Spin quantization has always been an interesting intrinsic feature in quantum mechanics. This thesis discussed the holomorphic polarization method, motivated by geometric quantization, in a new formulation of canonical group quantization on non-cotangent bundle phase space to produce spin quantization. The first part focuses on determining the one-dimensional complex projective space $\mathbb{C}\mathrm{P}^1$ as a compact phase space and a special unitary group of degree two $SU(2)$ as its canonical group that is not in the semi-direct product form. The emergence of the hidden discrete symmetry which is not deducible from the Lie algebraic structure of $SU(2)$ indicates that it is the double-covering group. Thus its global structure is determined through the lifting $SU(2)$ action on the fibre bundle over phase space. The second part focuses on the quantization process with the holomorphic wavefunction determined through the holomorphic local section of the fibre bundle and the natural polarization arises through the unitary irreducible representation of $SU(2)$ that does not follow Mackey's induced representation theory. From the representation operators, a set of spin angular momentum operators is generated as complex differential operators associated with a connection-type term l from action on holomorphic wavefunctions. Such representation operators' matrix elements and characters are determined as Jacobi polynomials and its application in describing the single-qubit pure state is discussed. In conclusion, it is shown that the holomorphic polarization naturally emerged in the canonical group quantization on non-cotangent bundle phase space and has its application in quantum information theory which arises geometrically from the quantization problem.

Keywords: Canonical group quantization, holomorphic wavefunction, non-cotangent bundle, qubit, spin quantization.

SDG: GOAL 9: Industry, Innovation and Infrastructure

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGKUANTUMAN KUMPULAN BERKANONIK KE ATAS RUANG FASA BERKAS TAK-KOTANGEN DAN APLIKASINYA DALAM TEORI MAKLUMAT KUANTUM

Oleh

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Pengkuantuman spin sentiasa menjadi ciri intrinsik yang menarik dalam mekanik kuantum. Tesis ini mengkaji kaedah pengkutuban holomorfik, yang termotivasi oleh pengkuantuman geometri, dalam formulasi baharu pengkuantuman kumpulan berkanonik pada ruang fasa berkas tak-kotangen untuk menghasilkan pengkuantuman spin. Bahagian pertama memfokuskan kepada penentuan ruang unjuran geometri kompleks berdimensi satu $\mathbb{C}\mathrm{P}^1$ sebagai ruang fasa klasik mampat dan kumpulan unitari istimewa berdimensi dua $SU(2)$ sebagai kumpulan berkanonik tepat baginya yang bukan dalam bentuk hasil darab separa langsung. Kemunculan simetri diskrit tersembunyi yang tidak terdeduksi daripada struktur aljabar Lie $SU(2)$ menunjukkan bahawa ia adalah kumpulan litupan berganda. Oleh itu struktur sejagatnya diperoleh melalui tindakan angkatan $SU(2)$ ke atas ruang berkas gentian. Bahagian kedua memfokuskan kepada proses pengkuantuman dengan fungsi gelombang holomorfik ditentukan melalui keratan setempat berkas gentian atas ruang fasa dan seterusnya menunjukkan bahawa pengkutuban holomorfik muncul secara tabii melalui perwakilan unitari tak turunkan $SU(2)$ yang tidak mematuhi teori perwakilan teraruh Mackey. Seterusnya, set

pengoperasi momentum sudut spin dijanakan dalam bentuk pengoperasi pembeza kompleks bersama dengan sebutan bak-kaitan l daripada tindakan pada fungsi gelombang holomorfik. Unsur-unsur dan ciri-ciri matriks pengoperasi perwakilannya ditentukan sebagai polinomial Jakobi dan aplikasinya dalam menjelaskan keadaan tulen qubit tunggal turut dibincangkan. Kesimpulannya, ditunjukkan bahawa pengkutuban holomorfik muncul secara tabii dalam pengkuantuman kumpulan berkanonik pada ruang fasa berkas tak-kotangen dan mempunyai aplikasinya dalam teori maklumat kuantum yang muncul secara geometri daripada masalah pengkuantuman.

Kata Kunci: Pengkuantuman kumpulan berkanonik, fungsi gelombang holomorfik, berkas tak-kotangen, qubit, pengkuantuman spin

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As the saying attributed to the Prophet Muhammad goes, *"He who does not thank the people does not thank God!"*[1]

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[1] *Narrated from Abū Hurayra, as reported in the Kitāb al-Sunan of Abū Dawūd, The Book of Manners, Chapter: Regarding gratitude for acts of kindness* (`https://sunnah.com/abudawud:4811`)

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

CCR	Canonical Commutation Relation
C-R	Cauchy- Riemann relation
CGQ	Canonical group quantization
GQ	Geometric quantization
GQM	Geometric quantum mechanics
UIR	Unitary irreducible representation
FLT	Fractional linear transformation
RHS	Right hand side
LHS	Left hand side
OPS	One-parameter subgroup
PB	Poisson bracket
ON	Orthonormal basis
i	Imaginary unit
$\mathcal{H}$	Hilbert space
$\mathbb{C}$	Complex vector space/set of complex numbers depending on the context
$\mathbb{R}$	Real vector space/set of real numbers depending on the context
$\mathbb{Z}$	Integers
$\mathbb{C}^*$	Set of complex numbers without zero
$C^\times$	An extended complex plane
$\|\psi\rangle$, $\|\varphi\rangle$	State vector
ψ	Holomorphic wavefunction
$\mathcal{M}$, $\tilde{\mathcal{M}}$	Symplectic manifold/classical phase space
Q	Configuration space
$\overline{\text{Overhead bar}}$	Complex conjugate
$\rtimes$, $\times$, $\otimes$	Semidirect product, Cartesian product, tensor product
$GL(n,\mathbb{C})$	General linear group of n-dimensional complex matrices

$\mathcal{U}$	Unitary irreducible representation
$\hbar$	reduced Planck constant
$\mathcal{G}$,$\tilde{\mathcal{G}}$	Canonical group
G	Lie group/Riemannian metric
θ	One-form/symplectic potential
π	Projection mapping from total space to its base space
$Aut(V)$	Automorphism of vector space V
$\mathcal{Q}$	Quantization map
P^*	Stereographic projection
$L(\mathcal{G})$, $L(\tilde{\mathcal{G}})$	Lie algebra of canonical groups $\mathcal{G}, \tilde{\mathcal{G}}$
$\mathfrak{su}(2)$	Lie algebra of $SU(2)$
$\mathfrak{so}(3)$	Lie algebra of $SO(3)$
$\mathfrak{Re}$	Real part
$\mathfrak{Im}$	Imaginary part
$\mathcal{D}$	Matrix elements of $\mathcal{U}$
pr	Natural projection from the total space of its base space
$P_{l+k}^{(j-k;-(k+j))}(\cos\theta_2)$	Jacobi polynomial terms
$Riem\Sigma$	The space of smooth Riemannian three-compact manifold
$Z \equiv [Z] = [Z_1 : Z_2 :, \cdots, Z_n]$	Homogenous coordinates on $\mathbb{C}\mathrm{P}^{n-1}$
z_N, z_S, z^1, z^2	Inhomogeneous coordinates on Riemann sphere, $\mathbb{C}\mathrm{P}^1$
$w = (w_1, w_2, \cdots w_n)$	Complex coordinates on $\mathbb{C}^n$
$d\mu$, $d\mu_{\tilde{g}}$	Measure, Haar measure
$\widehat{\text{hat}}$	Quantum operators
HamVF($\mathcal{M}/\tilde{\mathcal{M}}$)	Hamiltonian vector fields on $\mathcal{M}/\tilde{\mathcal{M}}$
Diff$(-)$	A diffeomorphism group of space
T	Matrix transpose
g, $\tilde{g}$	Elements of canonical group
$\Psi(x)$	Cross-section of $\mathcal{L}$ bundle

ξ, ξ_f, ξ_H	Vector field, Hamiltonian vector field
ω	Symplectic 2-form
Ω	Symplectic 2-form on $\mathbb{C}\mathrm{P}^1$
$\mathcal{F}$	Fibre on the bundle
$\mathcal{E}$	Total space of the bundle
$\mathcal{P}$	Principal bundle
$\mathcal{L}$	Associated vector bundle
$\langle \cdot, \cdot \rangle$	The Hermitian vector bundle is antilinear in its first argument following physics notation
Q	Configuration space
T^*Q	Cotangent bundle phase space
$\mathcal{X}(\mathcal{M})$	Set of vector fields
$\Lambda^1(\mathcal{M})$	Set of one-form

CHAPTER 1

INTRODUCTION

1.1 A brief historical development of quantum theory

The quantum theory has undergone significant conceptual and mathematical advancements over time. For more than a century, it has demonstrated remarkable success in predicting a wide range of experimental outcomes with high accuracy. The first quantum revolution in the 20th century led to the birth of groundbreaking technologies such as the transistor, laser, spectrometer, atomic clock *etc*. In the 21st century, continued progress has positioned quantum theory as the foundation of the cutting-edge science and technology essential to modern society. The second quantum revolution is emerging, defined by advancements in quantum technologies such as quantum computing, quantum sensing and metrology, quantum communication and security *etc* (Dowling and Milburn, 2003). But underlying all of these deep technologies at the very foundation level the quantum theory is based on the deep mathematical and conceptual formulations that are full of polemics (Barret, 2019).

Historically, in 1900, Max Planck discovered that black body radiation could not be explained by the previously existing classical physics. He proposed that energy is emitted in *discrete packets* called *quanta*, introducing the idea that energy levels are quantized. This discovery laid the foundation for quantum theory (Jammer, 1966). After that, Werner Heisenberg made the first attempt to formulate the mathematical foundation of quantum theory in his historic 1925 paper (where the year of 2025 is its centenary and also the year that the United Nations proclaimed to be the International Year of Quantum Science and Technology), then followed by Dirac (1925), Schrödinger (1926), Bohr (1976), de Broglie (1987), Born and Jordan as discussed in Fedak and Prentis (2009), and others as discussed in Waerden (1967); Mehra and Rechenberg (1987). Essentially, the newly formulated quantum theory especially on the idea of

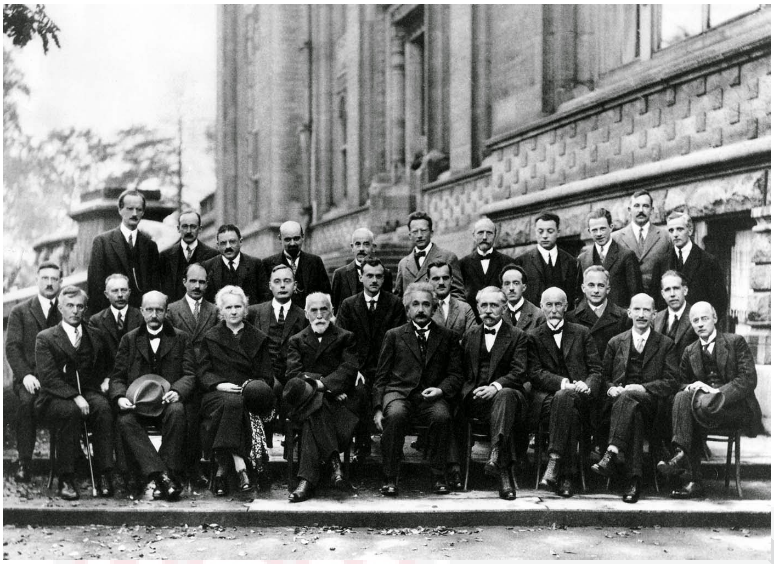

Figure 1.1: The Solvay Conference on Electrons and Photons 1927

electrons and photons was discussed by all of those physicists, including others, at the Solvay Conference 1927 (see Figure 1.1) (Clary, 2022). Furthermore, the development of the mathematical foundation of quantum theory for a more axiomatic mathematically rigorous and conceptually coherent was developed by John von Neumann based on algebra and analysis approaches (Neumann and Wheeler, 2018). Years later, Dirac made a more concise algebraic approach known as Dirac (or *canonical*) quantization (Dirac, 1958). Until now, his approaches played a pivotal role in developing other quantization formulations including the one discussed in this study.

In brief, many quantization programs are proposed to formulate a quantum theory *e.g.* Feynman-path integral quantization, Moyal quantization, stochastic quantization, geometric quantization, Berezin quantization *etc*. From all those quantization programs mentioned, they differ in the fundamental structures assumed on the classical phase space for example its canonical commutation relation (CCR) of position q and momentum p (Ali and Englis, 2005). There is no one *unique* quantization program that produces a well-defined quantum formulation by quantizing its classical structures.

However, there are other mathematical formulations of quantum theory based on geometrical language not concerned with the quantization process. It is formulated in the language of Hamiltonian phase-space dynamics for instance the works done by Kibble (1979); Ashtekar and Schilling (1995); Bengtsson et al. (2002); Brody and Hughston (2001); Sanborn (2011) *etc*. Alternatively, there are also other formulations of a quantum theory that are not based on the Hilbert space approach, *e.g.* quantum logic schemes as proposed by Birkhoff and von Neumann (1936), C^*- and W^*- algebras as discussed in Strocchi (2005); David (2014), categorical quantum mechanics as proposed by Abramsky and Coecke (2009), topos-theoretic approaches as proposed by Doering and Isham (2010) *etc*. The progress of the mathematical foundations of quantum theory is still an interesting topic of discussion among both physicists and mathematicians.

Therefore, in this study, a particular quantization method developed by Isham (1984) known as *canonical group quantization* (CGQ) is studied to obtain a spin quantization through a non-cotangent bundle physical phase space.

1.2 Motivation

Spin quantization is always an interesting idea to deeply explore in quantum mechanics. The notion of spin is an intrinsic form of angular momentum, which is different from the idea of orbital angular momentum operators, that gives a radical departure from classical physics (Sakurai and Napolitano, 2011). The Stern–Gerlach experiment proves the existence of electron spin angular momentum in 1922 in which silver atoms were observed to possess two possible discrete angular momenta despite having no orbital angular momentum (Estermann, 1959; Friedrich et al., 2003). The space of angular momentum states of the spin electron, with the spin number $s = \frac{1}{2}$, has the space in two-dimensional due to the spin-up $|0\rangle \equiv \frac{1}{2}$ and spin-down $|1\rangle \equiv -\frac{1}{2}$ properties (see Figure 1.2).

This experimental result shows that the electron spin is mathematically formulated as

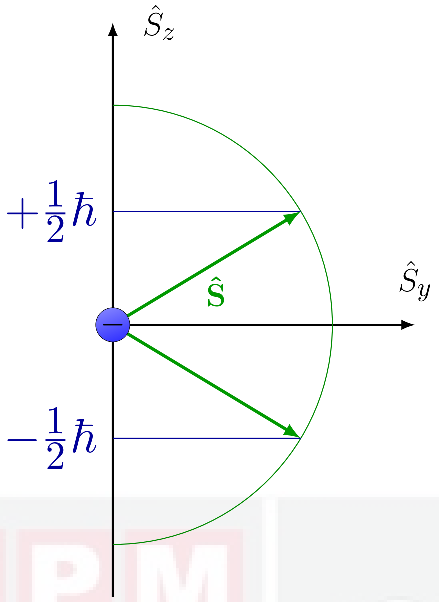

Figure 1.2: Electron spin

a general spinor spin state in a linear combination form (Isham, 1995),

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{1.1}$$

such that α, β are complex coefficients, $|0\rangle, |1\rangle$ are correspond to spin-up and spin-down respectively, and $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$. The observables are found by the spin operators,

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x, \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y, \quad \hat{S}_z = \frac{\hbar}{2}\sigma_z; \tag{1.2}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. In compact form,

$$\hat{S}_j = \frac{\hbar}{2}\sigma_j \quad (j = 1, 2, 3), \tag{1.3}$$

where $\{1, 2, 3\}$ respectively correspond to $\{x, y, z\}$ axes. It is called *Pauli two-*

component formalism and satisfies the commutator algebra

$$[\hat{S}_j, \hat{S}_k] = i\varepsilon_{jkl}\hat{S}_l \tag{1.4}$$

where ε_{jkl} is a total antisymmetric cyclic permutation

$$\varepsilon_{jkl} = \begin{cases} 1; \;\; ijk \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1; \;\; ijk \in \{(3,2,1),(2,1,3),(1,3,2)\} \\ 0 \;\; \text{otherwise.} \end{cases} \tag{1.5}$$

The ladder operators $\hat{S}_\pm = \hat{S}_1 \pm i\hat{S}_2$ of the spin are

$$\hat{S}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \hat{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.6}$$

then together with $\hat{S}_3$ satisfies

$$\begin{gathered} [\hat{S}_+, \hat{S}_-] = 2\hat{S}_3 \;\; ; \;\; [\hat{S}_3, \hat{S}_+] = \hat{S}_+, \\ [\hat{S}_3, \hat{S}_-] = -\hat{S}_- \;\; ; \;\; [\hat{S}^2, \hat{S}_\pm] = [\hat{S}^2, \hat{S}_3] = 0. \end{gathered} \tag{1.7}$$

From the definition of spinor, take z-component of spin represented as

$$\hat{S}_z = \frac{\hbar}{2}\sigma_z; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.8}$$

acts on spinors $|\psi\rangle$, it produces 2×2 matrices. $\hat{S}_z$ has eigenvalues $\pm\frac{\hbar}{2}$ corresponding to vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.9}$$

respectively. Moreover, with the advent of the field of quantum information theory and quantum computation, the idea of intrinsic spin-$\frac{1}{2}$ degree of freedom of individual electrons is expressed as a basic unit of quantum information known as a *quantum bit*

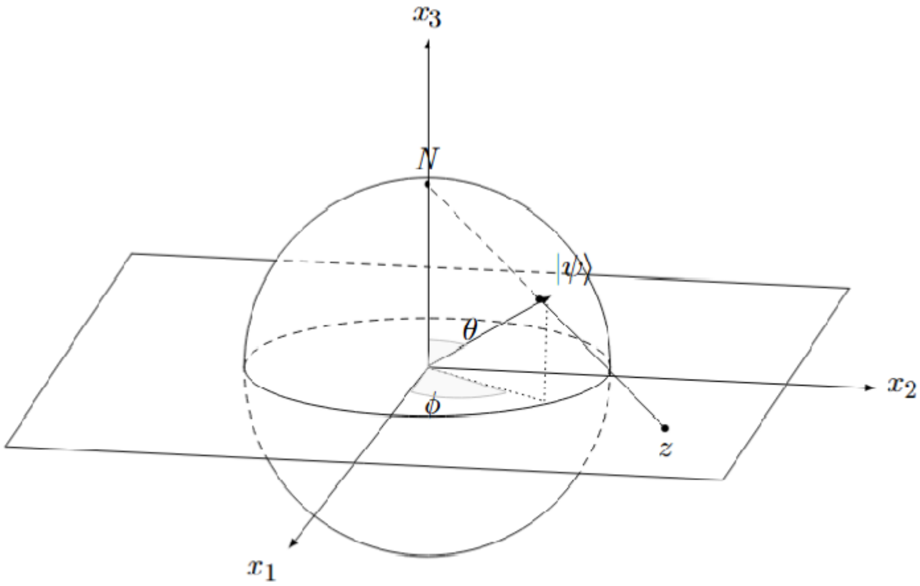

Figure 1.3: Bloch sphere

(qubit) with a similar mathematical expression as (1.1) (Nielsen and Chuang, 2010). Geometrically a single-qubit is formulated on the *Bloch sphere* (see Figure 1.3) which is the *Riemann sphere (or the extended complex plane)* $\mathbb{C}\mathrm{P}^1 \simeq C^\times := \mathbb{C} \cup \{\infty\}$ (Lee et al., 2002). From (1.1) up to the global phase factor, its pure quantum state is defined to be

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \qquad (1.10)$$

where $0 \leq \theta \leq \pi; 0 \leq \phi \leq 2\pi$. Thus, the Riemann sphere is thought to be the *internal space* of a single-qubit (or spin-$\frac{1}{2}$). A brief review of this is discussed in Chapter 2.

On the other hand, spin quantization can also be mathematically formulated in the realm of geometric quantization (GQ) formulation. It is a very well-known studied example of quantization on the non-cotangent bundle physical phase space that is of Riemann sphere S^2 which yields the spin-j systems (Woodhouse, 1997; Nair, 2016; Carosso, 2022). The family of symplectic 2-form of phase space S^2 is

$$\omega_{(k)} = \frac{ik\hbar dz \wedge d\bar{z}}{(1 + z\bar{z})^2}; \quad n \in \mathbb{Z}, \qquad (1.11)$$

where through the Bohr-Sommerfeld condition, it allows for the prequantization of the

S^2,

$$\frac{1}{2\pi\hbar}\int_{S^2}\omega = k \tag{1.12}$$

where k is an integer. Since,

$$\int_{S^2}\omega = 4\pi j^2,$$

then

$$j = \frac{\hbar k}{2};\; k \in \mathbb{Z}. \tag{1.13}$$

where the only symplectic manifolds that can be quantized correspond to classical systems whose angular momentum is an integer k multiple of $\frac{\hbar}{2}$. The quantization is completed by applying the holomorphic polarization $\mathbf{P}$ spanned by basis $\frac{\partial}{\partial \bar{z}^j}$ that yields the $(n+1)$-dimensional Hilbert space $\mathcal{H}_n$ of the square-integrable wavefunction on S^2. The wavefunction is in the form of an inhomogeneous polynomial with $\{1, z, \cdots, z^n\}$ as a basis for the Hilbert space. The generator of the rotations about z-axis on the S^2 is the Hamiltonian vector field of an observable J_3 on S^2. Hence the quantization of this observable gives an operator as the standard spin z-operator $\hat{S}_3$ with eigenvalues $\{-n, \cdots, n\}$ which $n = 2j$ where j is identified as spin quantum number.

Therefore, a similar physical prescription of spin quantization in the standard quantum mechanics view can be obtained geometrically in the GQ through the holomorphic polarization method. Such a method is to be studied in the CGQ to produce a new physical interpretation of spin quantization from the group-theoretical perspectives. More ideas on holomorphic polarization will be reviewed at length in Chapter 2.

1.3 Problem statement and objectives

As discussed, the quantization process on Riemann sphere S^2 in GQ involves *holomorphic polarization* method. Such a method reduces the quantum wavefunction dependence to half the number of phase space coordinates of position q or momentum p. The CGQ scheme proposed by Isham (1984) has an *in-built* polarization method that is not dissimilar to the GQ through the cotangent bundle structure T^*Q, where Q is a configuration space, of the phase space. In this case, the wavefunctions can be realized as wavefunctions on half of the phase space coordinate, position q or momentum p, *almost automatically* through Mackey's induced representation theory of the semi-direct product canonical group. The development of CGQ has shown that (as discussed in Chapter 2) it *only* works on the cotangent bundle phase space, unlike the GQ which works on both cotangent and non-cotangent bundle phase spaces, respectively. Therefore, the problem to be addressed is how to formulate the CGQ on a non-cotangent bundle physical phase space. The driving questions that could be asked to construct the premise of underlying formulations are as follows:

(i) Given a non-cotangent bundle physical phase space $\mathbb{C}\mathrm{P}^1$, how can one determine its appropriate canonical group to study its global kinematical symmetries?

(ii) If the canonical group is determined, can one possibly show the holomorphic polarization naturally arises through the unitary irreducible representation of such a canonical group?

(iii) If possible, can one get the different quantizations through inequivalent representations of the canonical group to produce spin quantizations?

In the current literature of CGQ formulations, it does not adequately address those formulated questions, for example by Isham (1984) himself and others like Bouketir and Zainuddin (1999); Bouketir (2000); Reyes-Lega and Benavides (2010); Silva and Jacobson (2021) *etc*.

Therefore the objectives are as follows:

(i) To determine the non-cotangent bundle physical phase space and compute its appropriate canonical group.

(ii) To construct the holomorphic wavefunction from the local holomorphic section of the associated vector bundle over phase space.

(iii) To obtain the holomorphic polarization by computing the unitary irreducible representations of the canonical group and generating its spin angular momentum operators.

(iv) To produce the matrix elements and characters of the representation operators and analyse their application in quantum information theory.

1.4 Scope of the research

The scope of research is limited to the new formulation of the CGQ method on $\mathbb{C}P^1$ as the whole compact phase space, rather than the configuration space, and the determination of its appropriate non-semidirect product canonical group and its unitary representations by not utilizing Mackey's induced representation theory. Secondly, the natural holomorphic polarization method is studied from the construction of holomorphic wavefunction on half of the coordinates on $\mathbb{C}P^1$ through the unitary representation to generate a set of angular momentum. Thirdly, the representation operators are determined as matrix elements in the form of Jacobi polynomials. Finally, the application of the result to describe a single-qubit pure state and its quantum logic gates geometrically in quantum information theory is discussed. Any other methods are not within the scope of the thesis and their exclusion is a limitation of this study.

1.5 Significance of the research

This study will be the first attempt to formulate the CGQ on a non-cotangent bundle phase space by treating $\mathbb{C}P^1$ as a whole compact phase space, instead of configuration space, along with its non-semidirect product canonical group for which the unitary irreducible representation not utilizing the Mackey's induced representation theory as proposed by Isham (1984). The focus of this study is the natural holomorphic polarization to yield spin quantization through the unitary representation and generate a set of spin angular momentum operators. The significance of this formulation is how the symmetry canonical group and its unitary representations can be a tool for describing single-qubit geometry and its quantum logic gates. Then, in general, one can use the unitary representation classifications to geometrically describe multiple qubit or qudit states as discussed in quantum information and quantum computation.

1.6 Thesis outlines

This thesis is outlined into seven chapters, and the discussions in each chapter are restricted and centralized to the problem statements and objectives formalized in the present chapter. Chapter 2 discusses the literature review of the CGQ formulations on the various configuration spaces in the past and emphasis is given on the case of the 2-sphere to show the research gap related to this study.

Chapter 3 discusses the mathematical tools and a new methodology proposed in the CGQ method to quantize non-cotangent bundle physical phase space. Then followed by Chapter 4 where the first part of the results, which is the classical part of the theory, emphasizes the construction of the non-cotangent bundle classical phase space. In particular, the determination of a compact manifold to be a non-cotangent bundle physical phase space and its kinematical global symmetries through the determination of the canonical groups.

Chapter 5 discusses the second part of the result, which is the quantum mechanical part of quantization formulation. Starting from the determination of holomorphic wavefunction through the holomorphic section bundle over phase space. Then, the computation of the notion of spin angular momentum operators and their algebraic structures and the discussion on the matrix elements and characters of the representation connected with the Jacobi polynomials.

Chapter 6 is the application of results in Chapter 5 in the field of quantum information theory. The discussion emphasized describing the notion of the qubit pure state in terms of the holomorphic wavefunction and its quantum logic gates, and also the three-level quantum systems (or qutrit) through the different values of representation classification. Lastly, Chapter 7 is the conclusion of this study and suggestions for future research.

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