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Minimization of Curvature with Cubic Spline Interpolation for Optimal Racing Line

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ABSTRACT

In motor sports, racing line optimization is important since different race line can result in different speed and time taken to complete a race track. In competitive motor sport events such as NASCAR cup, Formula 1 and WRC, the lap time of top racers can differ in mere milliseconds. Hence, it is important to formulate a racing strategy to achieve the best possible lap time. Optimizing the lap time requires one to minimize the curvature around the corner, and this in turn can be formulated as a variational problem. This study proposes cubic spline interpolation as a strategy to find the curvature and time taken for a race car to complete a corner at the race track, bounded by physical limits of the race car and the radius of turn of the race track. The strategy, which allows one to explore different points on the Cartesian plane, is illustrated through an example on a simple corner.

Keywords: Optimal Racing Line, Minimum Curvature, Cubic Spline Interpolation

INTRODUCTION

In competitive motor sports, the difference between victory and defeat often comes down to milliseconds. One of the most crucial factors in achieving optimal lap times is the selection of the ideal racing line - the path a vehicle takes through a race track. The optimization of racing lines can be solved by means of mathematical principles in the calculus of variations. In particular, the optimal racing line can be formulated as a variational problem, where the objective is to minimize the time taken to complete a given track while adhering to the physical constraints such as track boundaries, track condition, vehicle dynamics and traction control.

In their work on generating optimal velocity profiles for vehicles with acceleration techniques, Velenis and Tsiotras (2008) formulated the minimum-time cornering problem as a constrained optimization problem. Their work demonstrated that the optimal route can be derived as a variational problem, considering both kinematic and dynamic constraints of the vehicle. Apart from dynamics of the car, Braghin et al. (2008) also considered the geometry of the track on the problem of trajectory planning that was formulated as a constrained optimization problem.

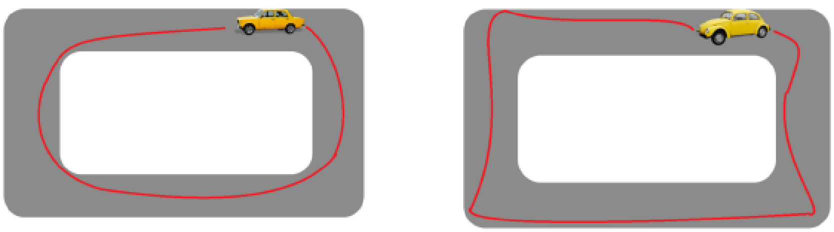
According to Xiong (2010), several factors affect the racing line at corners such as turn in (starting) point, braking point, apex and the direction and position of the next corner. A right strategy would be one that minimizes total time and maximizes overall speed, of which the latter requires the driver to maximize the radius of curvature around corners. Combining both artificial intelligence (AI) method and Euler spiral method, Xiong introduced an integrated method that is more applicable in generating optimal racing lines for different track shapes, subject to pre-analysis of the tracks.

In the context of Formula One race car, Perantoni and Limebeer (2013) solved the optimal control problem for minimizing lap time via direct transcription and nonlinear programming. Through optimization of driven line, parameters of car set-up according to track condition and driver controls, they managed to reduce the solution time for a full lap. The study on minimum lap time had also been conducted by Bianco et al. (2018) who did simulations of direct method via nonlinear programming problem and indirect method based on Pontryagin's principle on the trajectory-free full dynamic models. The results shows that both methods displayed similar behaviour on the overall simulation. Other notable works on minimizing lap time include car maneuvering (Casanova, 2000) and ideal driver control (Kelly and Sharp, 2009).

More recently, Heilmeier et al. (2019) generated minimum curvature path for autonomous race care via a quadratic optimization problem while improving its accuracy. It is clear that minimizing curvature and optimizing lap time comes hand-in-hand. In this study, we formulate the problem of minimizing the curvature as a variational problem which can be solved via the Euler-Lagrange equations analytically. Cubic spline interpolation is introduced as a strategy to find the curvature and time taken for a race car to complete a corner at the race track, bounded by the physical limits of a race car and the radius of turn of the race track. The strategy allows one to explore different points on the xy-plane such that the optimal race line can be obtained.

PRELIMINARIES

Cars need to slow down when it is entering a corner, and accelerate when it is exiting the corner. For all race tracks, a racing car can follow an infinitely many numbers of possible racing lines. However, not all racing line that follows the same track or circuit can result in the shortest lap time.



(a) (b)

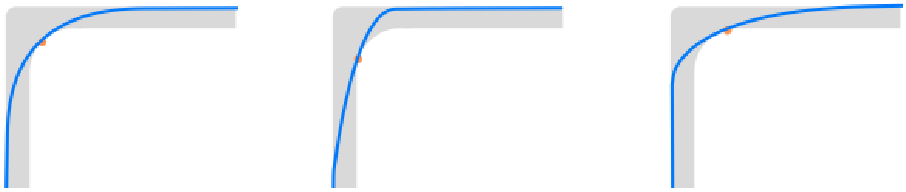
Figure 1: Comparison of racing lines

Figure 1 shows two different racing lines on the same track. Racing line (a) seems to be a better line as the driver is making as little turns as possible with smoother curves at the corners. Meanwhile, racing line (b) is not optimal due to many unnecessary turns which will affect the overall speed. It is at these corners that differentiates the quality of one's racing technique as the curvature around the corners determines the lap time.

One of the contributing factors in determining the curvature is the apex. Every racer will seek to touch a point called the apex, depending on the type of turn. In general, the apex, is a point in the inner edge of the race track corner with following characteristics:

(a) It is the point at which the car is closest and reached minimum speed in the inner edge of a corner.
(b) At the apex, the car is closest to the inner edge of the corner.

There are three types of apexes and the choice of apex depends on the type of car, condition of track surface, and shape of the corner. Figure 2 shows three types of apexes with the orange point as the apex.



(a) Geometric apex (b) Early apex (c) Late apex

Figure 2: Types of apexes

This study only considers geometrical apex of which it maintains the radius of curvature and it is suitable for low to average powered cars. Among its advantages are it has a balanced entry and exit speed and it smooths out curves, thus reducing the possibility of oversteering and understeering.

Another important factor in optimization of racing line is the traction circle which is defined as a vehicle's capability to handle deceleration with cornering when entering a corner and acceleration with cornering when exiting a corner (Mitchell et al., 2004). In Figure 3, the direction of the arrow indicates the direction a driver experience normal force from the car as it moves, and the length of the arrow indicates the magnitude of the force. For every car, there exists a maximum amount of grip which its tyres can provide to stabilize the motion of the car.

When a car fails to attain the minimum turning angle in order to turn, the vehicle is identified as understeer. On the other hand, if the turning angle is too large, the vehicle is losing control due to oversteer. The former is due to excessive steering input and braking input at a same time, while the latter is due to excessive steering input and throttle input at a given time. Figure 4 shows the traction circles when a car is losing control due to oversteering and understeering. The former is due to excessive steering input and braking input at a same time, while the latter is due to excessive steering input and throttle input at a given time.

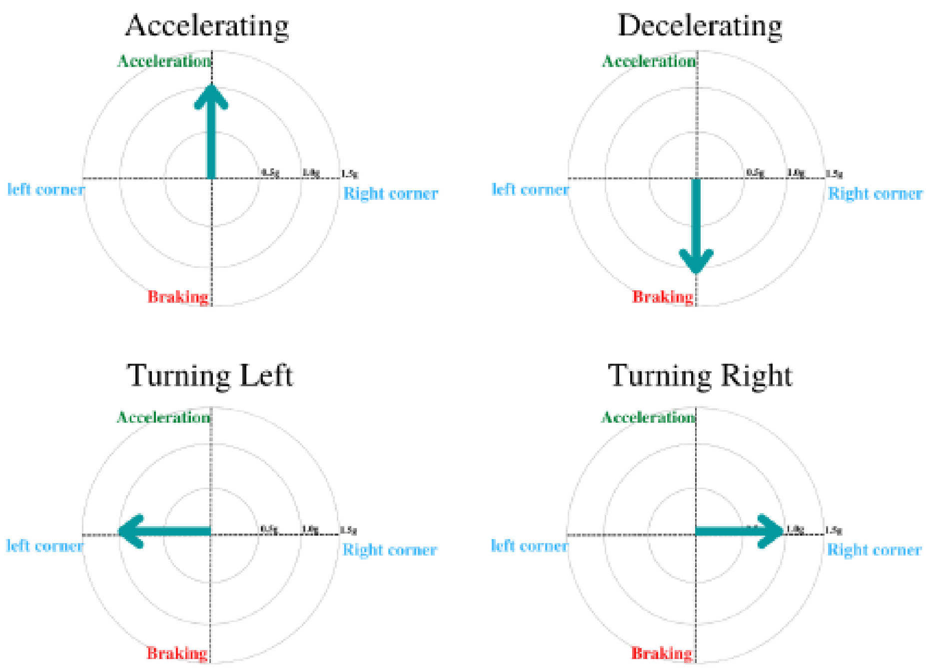


Figure 3: Traction circles when a car is accelerating, decelerating, turning left and right

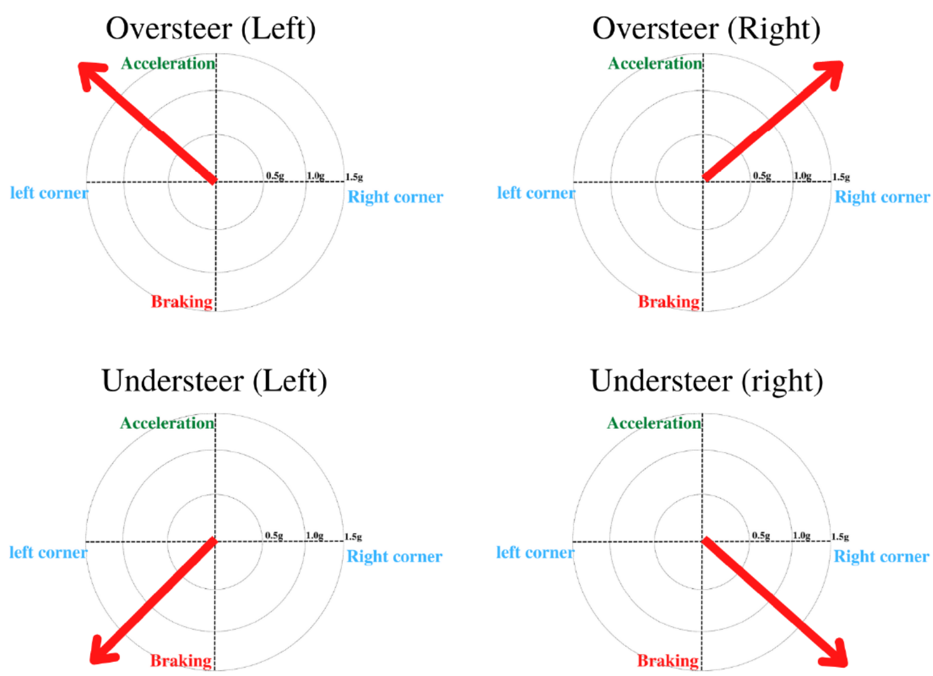


Figure 4: Traction circles due to oversteering and understeering

PROBLEM FORMULATION

Let μ be the friction coefficient of the racing track surface which is an asphalted road. Dry asphalt road with a friction coefficient of about 0.7 to 0.8 shows very low wheel slip ratio (Zhao at al., 2017). Therefore, this road condition is able to provide enough traction so that grip racing technique is faster than drift racing technique.

Let

(a) $F_n = mg$ be the normal force acting on the car, where m is the mass of the car and g is the gravitational acceleration;
(b) F_d is the downward force due to the mass of the car;
(c) r be the radius of curvature;
(d) $F_c = \frac{mv^2}{r}$ be the centripetal force, where v is the velocity of the car;
(e) $F_v = \mu F_n$ be the tangential force;
(f) F_r be the resultant force due to the centripetal and tangential forces.

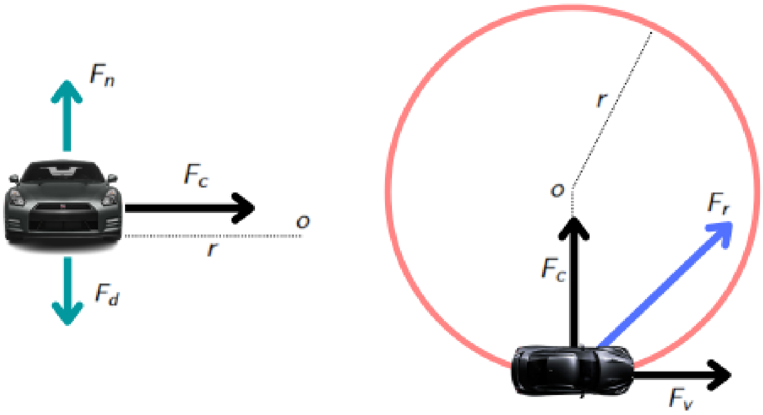


Figure 5: Forces acting on a car while being driven in a circle with radius r

For a car to maintain no slip condition, it must satisfy

$$F_c \leq F_v. \tag{1}$$

Otherwise, the car will slip out of the circle. Substituting the formulas of F_c and F_v results in

$$v \leq \sqrt{9.8\mu r}, \tag{2}$$

where the gravitational acceleration g is 9.8 ms^{-2}. It is clear that v^2 is proportional to r. Since curvature and radius of curvature is related in a way that $\kappa = \frac{1}{r}$, we have $v^2 \propto \frac{1}{\kappa}$.

It is clear from inequality (2) that the maximum speed a car can safely navigate a corner is constrained by the friction and radius of curvature. As the curve's radius changes, so does the critical speed threshold, of which exceeding this limit will result in the tyres losing grip on the track, potentially causing the car to slide or spin out. Tighter corners (smaller radius) require lower speeds to maintain traction, while gentler curves (larger radius) allow for higher speeds while keeping the tyres firmly planted.

A race car can maintain a higher cornering speed without slipping when it follows a path with a larger radius of turning. Hence, a path with minimum curvature is preferred. However, the curvature κ is restricted by the built of the car which has minimum turning radius $r_{\min}$:

$$r \geq r_{\min} \ \text{ or } \ \kappa \leq \kappa_{\max}, \tag{3}$$

where $\kappa_{\max}$ is the maximum curvature of the car.

Suppose $y = y(x)$ where x and y are the coordinates on the Cartesian plane. The aim is to obtain the curvature of the function at any point in the given domain. Let the position vector be

$$\boldsymbol{r}(x) = x\boldsymbol{i} + y(x)\boldsymbol{j}. \tag{4}$$

By means of vector calculus, the curvature κ for $\boldsymbol{r}(x)$ is given by

$$\kappa(x) = \frac{|y''(x)|}{\left(1 + \left(y'(x)\right)^2\right)^{\frac{3}{2}}}. \tag{5}$$

Denote by $v_{\max}$ the maximum speed limit, $a_{\min} < 0$ the maximum deceleration and $a_{\max} > 0$ the maximum acceleration.

In order for an optimal race line to exist, all parts of the race track must be wide enough and do not have a high curvature to the point that a race car cannot clear it without having its racing line intersecting with the border of racetrack. Hence for a solution to exist, the racing line has to satisfy:

$$r_{\min} \leq r_{\text{turn}}, \tag{6}$$

where r_{turn} denotes the maximum turning radius allowed by the corner.

Let

$$\kappa(x) = \int_a^b f\left(x, y(x), y'(x), y''(x), y'''(x)\right) dx, \tag{7}$$

where

$$f\left(x, y(x), y'(x), y''(x), y'''(x)\right) = \frac{d}{dx}\left[\frac{|y''(x)|}{\left(1 + \left(y'(x)\right)^2\right)^{\frac{3}{2}}}\right]. \tag{8}$$

The variational problem for minimizing curvature along a track to form an optimal racing line around a corner is given by

$$\begin{aligned}
\text{minimize} \quad & \{\kappa(x): y(x) \in C^2([b,c]), \kappa(b) = 0, \kappa(c) = 0\}, \\
\text{subject to} \quad & v(x) \leq \sqrt{9.8\mu r} \leq v_{\max}, \\
& r_{\min} \leq r(x) \leq r_{\text{turn}}, \\
& a_{\min} \leq a(x) \leq a_{\max}.
\end{aligned} \tag{9}$$

Here b and c are the starting point and exit point of which the corner, respectively, and $a(x)$ the acceleration of the car at position x.

In this study, we consider the race takes place on a dry asphalt road with friction coefficient of 0.8 with the car Toyota MR2 which has a maximum speed of $220\ \text{kmh}^{-1}$, maximum acceleration of $3.9683\ \text{ms}^{-2}$, maximum deceleration of $-10.9128\ \text{ms}^{-1}$, minimum turning radius of 4.6 m and down force of 5000 N. To simplify calculation, we minimize the square of the curvature (5) instead:

$$\kappa^2(x) = \frac{\left(y''(x)\right)^2}{\left(1 + \left(y'(x)\right)^2\right)^3}. \tag{10}$$

Therefore, problem (9) is now expressed as

$$\begin{aligned}
\text{minimize} \quad & \{\kappa^2(x): y(x) \in C^2([b,c]), \kappa(b) = 0, \kappa(c) = 0\}, \\
\text{subject to} \quad & v(x) \leq \sqrt{7.84r} \leq v_{\max}, \\
& 4.6 \leq r(x) \leq r_{\text{turn}}, \\
& -10.9128 \leq a(x) \leq 3.9683.
\end{aligned} \tag{11}$$

This problem can be solved by means of Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right) = 0. \tag{12}$$

Note that in (8), f does not depend on x and hence we can use Beltrami's identity to find the function f that represents the equation of the racing line:

$$f - \left[y'\frac{\partial f}{\partial y'}\right] - \left[y'\left(\frac{\partial f}{\partial y''}\right)' - y''\left(\frac{\partial f}{\partial y''}\right)\right] - \left[y'\left(\frac{\partial f}{\partial y''}\right)'' + y''\left(\frac{\partial f}{\partial y'''}\right)' - y'''\left(\frac{\partial f}{\partial y'''}\right)\right] = c. \tag{13}$$

In reality, race tracks are built with multiple turns and straights. Straights are sections of a race track in which a race car can be driven in a straight line. The optimal race line in a straight will simply be a straight line that connects the starting point to the exit point. On the other hand, a corner is a section of a racing track which is not a straight. To find an optimal racing line around a corner, we have to first find the starting point (entry), the apex, and the exit point.

The first step is to decompose a path with multiple corners into segments consisting of one corner and analyze each segment independently. One of such examples is given in Figure 6, where the starting point, apex and exit point of each corner (T3, T4 and T5) is found by fitting a circle into each corner with its radius as large as possible, or simply r_{turn}. The starting point is labeled in green, the apex in orange, and the exit point in red.

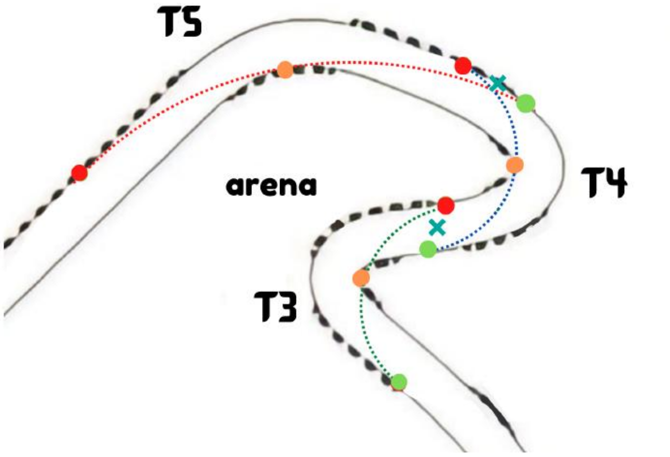


Figure 6: Analysis of T3, T4 and T5 of Silverstone racing track by fitting circles into each corner with largest possible radius around the respective apex

Since a racing line must be continuous, the exit point of previous segment must be the same as the starting point of the next segment. We can take the midpoint between the exit point of the previous corner and the starting point of the next corner, as illustrated by the cyan cross in Figure 6. The midpoint will replace both these points and the new starting and exit points are shown in Figure 7. Each segment is then interpolated by using cubic spline interpolation method to ensure C^2 continuity, which in turn will be used to approximate the curvature and time taken to complete each corner.

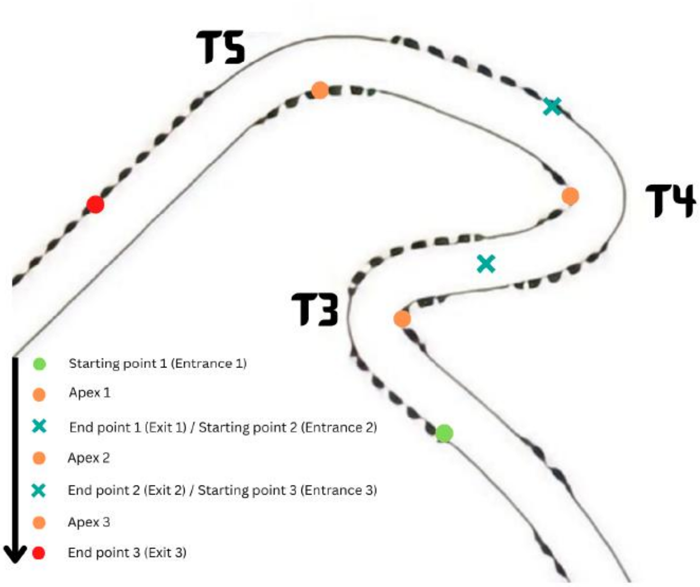


Figure 7: T3, T4 and T5 of Silverstone racing track with adjusted starting and end (exit) points

CUBIC SPLINE INTERPOLATION FOR APPROXIMATION OF RACING LINE

Instead of solving equation (13) directly, we use cubic spline interpolation (Wolberg, 1988) to approximate the racing line around the corner. Cubic spline interpolation, a powerful mathematical tool that ensures smoothness of curves with continuity of up to the second order derivatives, is a piecewise polynomial function consisting of $n-1$ cubic polynomials that takes the form:

$$y(x) = \begin{cases} y_1(x), & \text{if } x_1 \le x \le x_2, \\ y_2(x), & \text{if } x_2 \le x \le x_3, \\ \vdots & \vdots \\ y_{n-1}(x), & \text{if } x_{n-1} \le x \le x_n. \end{cases} \tag{14}$$

where

$$y_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \qquad i = 1, 2, \dots, n - 1. \tag{15}$$

with $b = x_1 < x_2 < \cdots < x_{n-1} < x_n = c$. Here a_i, b_i, c_i and d_i are constants determined by means of C^2 continuity of y_i at any given data point. Note that this problem requires $y"(x_1) = y"(x_n) = 0$ since the car is about to exit the corner and continue in a straight line. Hence $c_1 = c_n = 0$.

Denote by $h_i = \Delta x_i = x_{i+1} - x_i$, we obtain

$$\begin{aligned} a_i &= y_i(x_i), \\ a_{i+1} &= a_i + b_i h_i + c_i h_i^2 + d_i h_i^3, \\ b_{i+1} &= b_i + 2c_i h_i + 3d_i h_i^2, \\ c_{i+1} &= c_i + 3d_i h. \end{aligned}$$

The values of a_i, b_i, c_i and d_i are then determined using the following formulas:

$$a_i = y_i(x_i), \tag{16}$$

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1}), \tag{17}$$

$$b_i = \frac{a_{i+1} - a_i}{h_i} - \frac{c_{i+1} + 2c_i}{3}h_i, \tag{18}$$

$$d_i = \frac{c_{i+1} - c_i}{3h_i}. \tag{19}$$

To obtain the time taken for a race car to complete a turn, one needs to first obtain the arc length taken by the car using the formula

$$s = \int_a^b \sqrt{1 + \left(y'(x)\right)^2}\, dx. \tag{20}$$

The value of the arc length is then used to evaluate the velocity of the car at starting point (u), velocity at apex (v_a) and velocity at exit point (v) where

$$v^2 = u^2 + 2as. \tag{21}$$

Lastly, the time taken can be easily evaluated using the formula

$$v = u + at. \tag{22}$$

EXAMPLE OF A SIMPLE CORNER FOR APPROXIMATION OF CURVATURE AND TIME TAKE OF A RACING LINE

Figure 8 shows a parabola shaped corner with labeled starting point, apex and exit point. The outer edge of this corner is plotted using black bold curve, the inner edge of the corner is plotted using black dotted curve, and the center path is plotted using blue dashed curve.

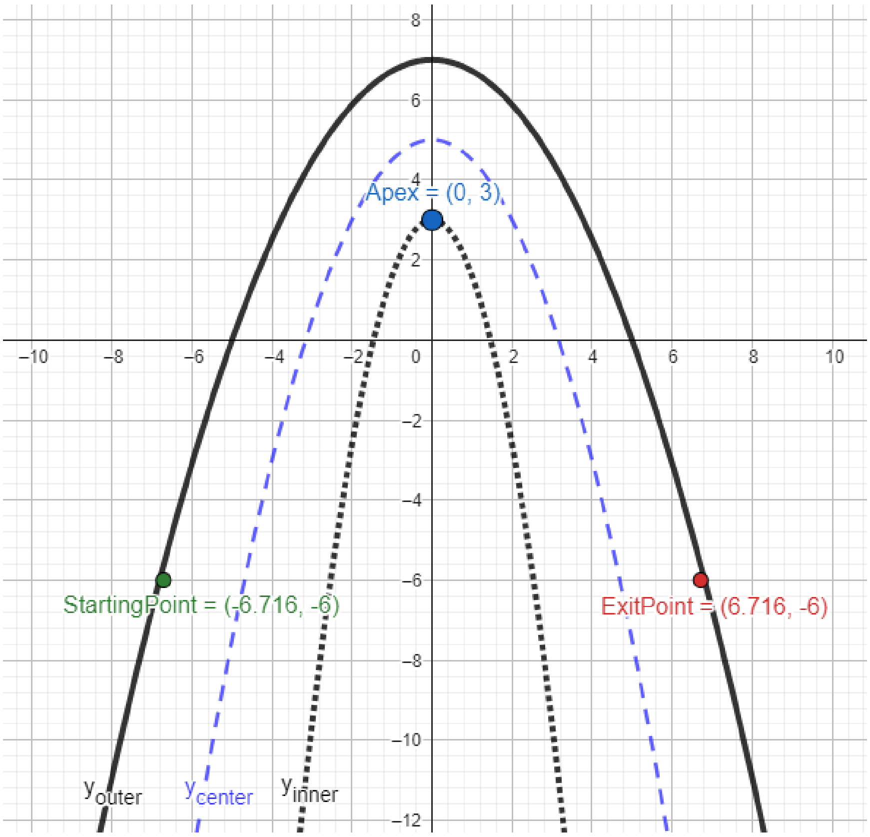


Figure 8: Parabola shaped turn with starting point, apex and exit point

The equations used to generate the corners are:

$$y = -0.28x^2 + 7, \quad \text{(outer edge)}$$
$$y = -1.4x^2 + 3, \quad \text{(inner edge)}$$
$$y = -0.5x^2 + 5. \quad \text{(center path)}$$

Consider the following points taken from the race track in Figure 8.

Table 1: Points taken from the race track in Figure 8

n	1	2	3	4	5	6	7
x	-6.716	-4.489	-2.283	0	2.283	4.489	6.716
$y(x)$	-6	0	2.393	3	2.393	0	-6

Implementing cubic spline interpolation (14) – (15) and finding a_i using (16) gives

$$a_1 = -6, \; a_2 = 0, \; a_3 = 2.393, \; a_4 = 3, \; a_5 = 2.393, \; a_6 = 0, \; a_7 = -6.$$

We obtain the values of $\boldsymbol{c_i}$ using (17) and the fact that $c_1 = c_7 = 0$:

$$c_2 = -0.5154,\ \ c_3 = -0.1176,\ \ c_4 = -0.1158,\ \ c_5 = -0.1179,\ \ c_6 = -0.5141.$$

Solving for b_i and d_i by means of (18) and (19) gives

$$b_1 = 3.0768,\ \ b_2 = 1.9292,\ \ b_3 = 0.5330,\ \ b_4 = 0.0001,\ \ b_5 = -0.5333,\ \ b_6 = -1.9309,$$
$$d_1 = -0.0770,\ \ d_2 = 0.06,\ \ d_3 = 0.0002,\ \ d_4 = -0.0003,\ \ d_5 = -0.0598,\ \ d_6 = 0.0769.$$

Hence, the function $y(x)$ is

$$y(x) = \begin{cases} -6 + 3.0768(x+6.716) - 0.077(x+6.716)^3, & -6.716 \le x \le -4.489, \\ 1.9292(x+4.489) - 0.5154(x+4.489)^2 + 0.06(x+4.489)^3, & -4.489 \le x \le -2.283, \\ 2.393 + 0.533(x+2.283) - 0.1176(x+2.283)^2 + 0.0002(x+2.283)^3, & -2.283 \le x \le 0, \\ 3 - 0.0001x - 0.1158x^2 - 0.0003x^3, & 0 \le x \le 2.283, \\ 2.393 - 0.5333(x-2.283) - 0.1179(x-2.283)^2 - 0.0598(x-2.283)^3, & 2.283 \le x \le 4.489, \\ -1.9309(x-4.489) - 0.5141(x-4.489)^2 + 0.0769(x-4.489)^3, & 4.489 \le x \le 6.716. \end{cases} \tag{23}$$

Figure 9 illustrates the racing line obtained from cubic spline interpolation based on the points taken from Table 1. The resulting racing line (y_1 – blue, y_2 – orange, y_3 – yellow, y_4 – purple, y_5 – green, y_6 – light blue) is the approximation obtained from cubic spline interpolation method. It started from near the outer edge, touched the apex in the inner edge and exit the corner while aiming for the outer edge. The circle on the figure is a circle with radius 4.6m which touches the apex. The equation of the circle is

$$x^2 + (y+1.6)^2 = 4.6^2.$$

Note that more points are taken around the apex and the points taken are never inside the circle. The reason is to avoid taking racing line which intersects with the circle, which is the physical limit of car turning radius.

The main objective of an optimal racing line is to minimize the curvature, hence maintaining maximum possible speed. Recall the formula of a function's curvature (5). It is possible to find the curvature of this racing line for any point in its domain as all functions generated by cubic spline interpolation is C^2 continuous. This implies that the curvature of the racing line will be continuous. Figure 9 also shows its curvature (brown) plotted on the same graph. Based on the graph, we can observe that the curvature is very low as the racing car enters, touches the apex and exits the corner.

The next step is to evaluate the time taken for the race car to drive past the corner. Since the radius of curvature is directly proportional to maximum allowed velocity of a car before it spins out of control, we need to first determine the minimum radius of curvature of this corner. This can be done by first determining the limiting factor of radius of curvature at the apex.

The inner edge of the corner is given by $y(x) = -1.4x^2 + 3$ and therefore the maximum curvature at the apex (when $x = 0$) is

$$\kappa(0) = \frac{|-2.8|}{(1)^{3/2}} = 2.8.$$

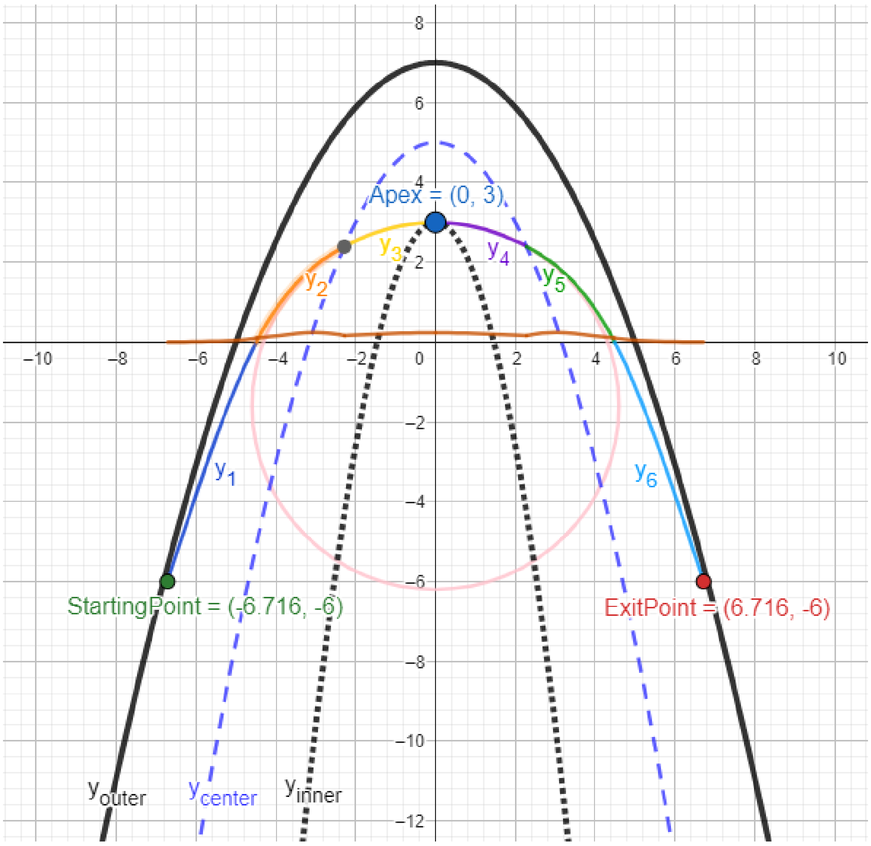


Figure 9: Race line obtained from cubic spline interpolation

Taking 4.6 m as the minimum radius of curvature, we deduce the velocity at the apex v_a by using the first constraint in (11):

$$v_a = \sqrt{7.84(4.6)} = 6.0053 \text{ ms}^{-1} = 21.6191 \text{ kmh}^{-1}.$$

We can calculate the distance traveled by the car by using the arc length formula

$$s_{y_i} = \int_{x_i}^{x_{i+1}} \sqrt{1 + \left(y'(x)\right)^2}\, dx, \qquad i = 1, 2, \dots, n-1.$$

Therefore, the arc length of each path s_{y_i} for $[x_i, x_{i+1}]$ are as follows:

$$s_{y_1} = 6.4075, \; s_{y_2} = 3.3096, \; s_{y_3} = 2.3894, \; s_{y_4} = 2.3866, \; s_{y_5} = 3.3095, \; s_{y_6} = 6.4064.$$

We now calculate the final velocity at each path using the formula

$$v_i^2 = v_{i-1}^2 + 2a_{\min}\, s_{y_i}, \qquad i = 4, 3, 2,$$

where we consider maximum deceleration $a_{\min}$ before reaching the apex. Since $v_4 = v_a = 6.0053 \text{ ms}^{-1}$, we first compute v_3:

$$v_4^2 = v_3^2 + 2a_{\min}\, s_{y_3} \;\Rightarrow\; v_3 = 9.3922 \text{ ms}^{-1} = 33.8119 \text{ kmh}^{-1}.$$

Subsequently,

$$v_2 = 12.6668 \text{ ms}^{-1} = 45.6005 \text{ kmh}^{-1},$$
$$v_1 = 17.329 \text{ ms}^{-1} = 62.3844 \text{ kmh}^{-1},$$

where v_1 is the velocity of the car at point $(-6.716, 6)$.

After leaving the apex, the car starts to accelerate at maximum acceleration of $a_{\max} = 3.9683 \text{ ms}^{-2}$ and hence the car has a velocity of

$$v_5 = 7.4165 \text{ ms}^{-1} = 26.6994 \text{ kmh}^{-1},$$
$$v_6 = 9.0150 \text{ ms}^{-1} = 32.454 \text{ kmh}^{-1},$$
$$v_7 = 11.4941 \text{ ms}^{-1} = 41.3788 \text{ kmh}^{-1},$$

where v_7 is the velocity of the car at point $(6.716, 6)$.

Finally, we evaluate the time taken to complete the corner using the formula

$$v_i = v_{i-1} + at_i, \qquad i = 1, 2, \dots, 7.$$

However, since we assumed that the car achieves maximum deceleration and maximum acceleration before and after the apex, respectively, we have the following:

$$v_4 = v_1 + a_{\min}(t_1 + t_2 + t_3),$$
$$v_7 = v_4 + a_{\max}(t_4 + t_5 + t_6).$$

Subsequently,

$$t_1 + t_2 + t_3 = 1.0377 \text{ s},$$
$$t_4 + t_5 + t_6 = 1.3832 \text{ s},$$

which means the total time take to clear the corner is 2.4209 s.

CONCLUSION

This study introduced the optimization of racing line as a variational problem and implemented cubic spline interpolation as a practical approach for minimizing curvature around the corner, subject to the vehicle dynamics and track limitations. The approach provides racing teams with a systematic tool for strategy development, allowing for simulations of different points around corners for optimal racing lines to achieve minimum lap time.

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