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Exponentiated gamma Burr-type X distribution: model, theory, and applications

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Abstract

Several extended Burr-type X distributions have been formed in the past decade. These distributions are widely used in modeling lifetime data as their hazard functions can fit various shapes, such as bathtub, decreasing, and increasing. However, certain extended Burr-type X distributions may not adequately fit the unimodal hazard function. Thus, this paper proposes a new extended distribution with greater flexibility to solve this deficiency: exponentiated gamma Burr-type X distribution. We provide the expressions for the probability density and cumulative distribution functions of the proposed distribution, along with its statistical properties, such as limit behavior, quantile function, moment function, moment-generating function, Renyi entropy, and order statistics. To estimate the model parameters, we employ the maximum likelihood estimation method, and we assess its performance through a simulation study with different sample sizes and parameter values. Finally, to demonstrate the application of this new distribution, we apply it to a real dataset concerning the failure times of aircraft windshields. The results indicate that the new distribution provides a superior fit compared to its submodels and the extended Burr-type X distributions. Moreover, it proves to be highly competitive and can serve as an alternative to certain nonnested models. In summary, the new distribution is highly flexible, capable of modeling a variety of hazard-function shapes, including decreasing, increasing, bathtub, and unimodal patterns.

Keywords: Burr-type X; Gamma generalized; Exponentiated; Unimodal

1 Introduction

In survival analysis, statistical distributions are widely used to describe lifetime data. However, while there are numerous distributions available for modeling lifetime data, some datasets, particularly those with unimodal and bathtub-shaped hazard functions, may not be well represented by existing models. Such shapes are commonly observed in survival data analysis [25]. As a result, there has been increasing interest in developing more flexible distributions by introducing additional parameters to the baseline distribution. Subsequently, several new distributions have been introduced, such as Kumaraswamy exponentiated Burr XII [4], beta Burr-type-V [11], Weibull Burr XII [3], type-I half-logistic modified Weibull [12], and Marshall–Olkin generalized Burr XII [2] distributions. With the additional parameters added, all these new distributions have greater flexibility compared to their baseline distribution. Thus, they can be used as an alternative distribution for their submodels.

The Burr-type X distribution, a well-known model in survival analysis, is one of twelve new distributions introduced by Burr in 1942 using a differential equation approach. According to Surles and Padgett [26], its initial form is a one-parameter distribution and has been extended to a two-parameter distribution by adding a scale parameter, known as two-parameter Burr-type X (BX). In recent years, several extended BX distributions have been proposed, such as beta Burr-type X [19], gamma Burr-type X [16], Weibull Burr-type X [15], exponentiated generalized Burr-type X [17], beta Kumaraswamy Burr-type X [18], exponentiated Burr-type X [5], exponentiated Weibull Burr-type X [20] and exponentiated beta Burr-type X [21]. These distributions are created by incorporating additional parameters into the BX distribution. Hence, with the additional parameters added, these distributions have greater flexibility than the BX distribution and can accommodate a broader range of hazard functions.

Khaleel et al. [16] introduced a three-parameter extended BX distribution known as the gamma Burr-type X (GBX) distribution. It is formed using gamma-G [23] with the BX distribution as the baseline distribution, where an additional parameter is added to the BX distribution. The probability density function (pdf) and cumulative distribution function (cdf) of GBX are given as

$$g(x,\alpha,\lambda,\theta) = \frac{2\theta\lambda^2 x e^{-(\lambda x)^2}}{\Gamma(\alpha)}\left(1-e^{-(\lambda x)^2}\right)^{\theta-1}\left[-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\alpha-1}, x,\alpha,\lambda,\theta > 0$$

and

$$G(x,\alpha,\lambda,\theta) = 1 - \frac{\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)}, x,\alpha,\lambda,\theta > 0, \tag{1}$$

respectively, where $\Gamma(\alpha)$ is the gamma function, and $\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]$ is the lower incomplete gamma function. It is noted that this distribution can only model increasing, decreasing, and bathtub-shaped hazard functions but not the unimodal-shaped hazard function. Thus, we propose a new distribution, namely exponentiated gamma Burr-type X (EGBX) distribution. It is a four-parameter distribution where an additional parameter is added to the GBX distribution using the exponentiated type of distributions [13]. Hence, with the additional parameter, we expect the new distribution to be more flexible and solve the inadequacy of GBX distribution.

This paper is outlined as follows. In Sect. 2, we derive the cdf, pdf, and hazard function of the EGBX distribution. Sections 3 and 4 explain the derivation of the statistical properties and the likelihood function, respectively. In addition, the limit behavior of both cdf and pdf of the EGBX distribution when x approaches infinity and zero are also discussed in Sect. 3. Section 5 shows the performance of the EGBX distribution via simulation studies. The application of the EGBX distribution with two real datasets is then illustrated in Sect. 6. Finally, we conclude the outcomes of the study in Sect. 7.

2 Exponentiated gamma Burr-type X distribution

In this study, we introduce a new four-parameter distribution called the exponentiated gamma Burr-type X distribution. This distribution is constructed by applying the exponentiated distribution approach [13] to the GBX distribution as the baseline, with an additional parameter included in the GBX distribution. The EGBX distribution cdf can be obtained by taking the γ exponent on equation (1),

$$G(x,\alpha,\gamma,\lambda,\theta) = \left[1 - \frac{\gamma\left[\alpha, -\theta ln\left(1 - e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)}\right]^{\gamma}, x,\alpha,\gamma,\lambda,\theta > 0 \tag{2}$$

and its pdf is derived by differentiating equation (2),

$$\begin{aligned} g(x,\alpha,\gamma,\lambda,\theta) = {} & \frac{2\gamma\theta\lambda^2 x e^{-(\lambda x)^2}}{\Gamma(\alpha)}\left(1 - e^{-(\lambda x)^2}\right)^{\theta-1}\left[-\theta ln\left(1 - e^{-(\lambda x)^2}\right)\right]^{\alpha-1} \\ & \times \left[1 - \frac{\gamma\left[\alpha, -\theta ln\left(1 - e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)}\right]^{\gamma-1}, x,\alpha,\gamma,\lambda,\theta > 0, \end{aligned} \tag{3}$$

where $\Gamma(\alpha)$ and $\gamma\left[\alpha, -\theta ln\left(1 - e^{-(\lambda x)^2}\right)\right]$ are the gamma function and lower incomplete gamma function, respectively. The hazard function of the EGBX distribution is written as

$$\begin{aligned} h(x,\alpha,\gamma,\lambda,\theta) = {} & \frac{2\gamma\theta\lambda^2 x e^{-(\lambda x)^2}}{\Gamma(\alpha)}\left(1 - e^{-(\lambda x)^2}\right)^{\theta-1}\left[-\theta ln\left(1 - e^{-(\lambda x)^2}\right)\right]^{\alpha-1} \\ & \times \frac{1}{\left(\left[1 - \frac{\gamma\left[\alpha,-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)}\right]^{1-\gamma} + \frac{\gamma\left[\alpha,-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)} - 1\right)}, \end{aligned}$$

which is derived by taking the ratio of its pdf to its survival function. The pdf and hazard function of the EGBX distribution with different parameter values are displayed in Figs. 1 and 2, respectively. Figure 2 shows that the EGBX distribution's hazard function can be determined in different shapes, including decreasing, increasing, unimodal, and bathtub. Additionally, the EGBX distribution is highly flexible and can be simplified to several well-known distributions by setting certain parameters to 1. These distributions are known as

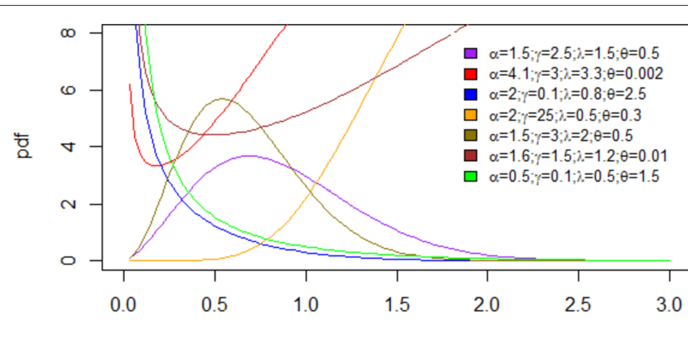

Figure 1 Probability Density Functions for the EGBX Distribution

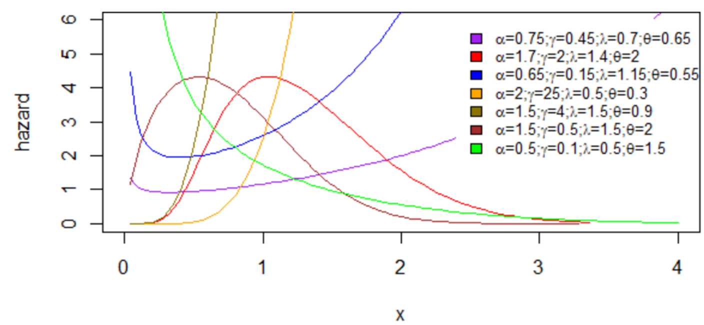

Figure 2 Hazard Functions for the EGBX Distribution

Table 1 Submodels for the EGBX Distribution

Distribution	Parameter values			
	α	γ	λ	θ
Gamma Burr-Type X	α	1	λ	θ
Burr-Type X	1	1	λ	θ
Rayleigh	1	1	λ	1

its submodels, as presented in Table 1. For example, the EGBX distribution approaches the GBX distribution when $\gamma = 1$, and reduces to the BX distribution when $\alpha = \gamma = 1$. It is proved that the EGBX distribution can cover the characteristics of its submodels, and it has superior flexibility to its submodels.

3 Statistical properties

This section outlines several key statistical properties of the EGBX distribution, including its limit behavior, linear form, quantile function, moment function (mf), moment-generating function (mgf), Renyi entropy, and order statistics.

Referring to equations (2) and (3), when x approaches zero, we have

$$\lim_{x\to 0}\left(1-e^{-(\lambda x)^2}\right)=0$$

and

$$\lim_{x\to 0} ln\left(1-e^{-(\lambda x)^2}\right)=-\infty.$$

Subsequently, the lower incomplete gamma function in equations (2) and (3), $\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]$, is reduced to the gamma function, $\Gamma(\alpha)$. Hence, as x approaches zero this gives

$$\lim_{x\to 0} g(x,\alpha,\gamma,\lambda,\theta)=\lim_{x\to 0} G(x,\alpha,\gamma,\lambda,\theta)=0.$$

In contrast, as x approaches infinity, we obtain

$$\lim_{x\to +\infty}\left(1-e^{-(\lambda x)^2}\right)=1$$

and

$$\lim_{x\to+\infty} ln\left(1-e^{-(\lambda x)^2}\right)=0.$$

Also, the lower incomplete gamma function in equations (2) and (3), $\gamma\left[\alpha,-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]$, become zero. Thus, when x approaches infinity, the limits of the cdf and pdf of the EGBX distribution are given as

$$\lim_{x\to+\infty} G\left(x,\alpha,\gamma,\lambda,\theta\right)=1$$

and

$$\lim_{x\to+\infty} g\left(x,\alpha,\gamma,\lambda,\theta\right)=0,$$

respectively. Additionally, $1-e^{-(\lambda x)^2}$ increases from zero to 1 when x increases. Subsequently, $-\theta ln\left(1-e^{-(\lambda x)^2}\right)$ reduces from infinity to zero. Hence, the cdf of the EGBX distribution is a nondecreasing function. In conclusion, both the cdf and pdf of the EGBX distribution satisfy the characteristics of a probability distribution. By implementing the binomial expansion and power series, the pdf of the EGBX distribution in equation (3) can be expressed in a linear form, such as

$$\begin{aligned} g\left(x,\alpha,\gamma,\lambda,\theta\right) = 2\gamma\theta^{\alpha}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} V_{ijk}x^{2k+1}\left(\gamma\left[\alpha,-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]\right)^{j} \\ \times\left[ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\alpha-1}, \end{aligned} \tag{4}$$

where

$$V_{ijk}=\frac{(-1)^{i+j+k+\alpha-1}\left(i+1\right)^{k}\lambda^{2(k+1)}}{k!\left[\Gamma\left(\alpha\right)\right]^{j+1}}\binom{\theta-1}{i}\binom{\gamma-1}{j}.$$

The linear form above is useful for deriving the statistical properties of the EGBX distribution.

The quantile function of the EGBX distribution can be found by inverting its cdf, that is

$$Q\left(u\right)=\frac{1}{\lambda}\left[-ln\left(1-e^{-\frac{v}{\theta}}\right)\right]^{\frac{1}{2}}, \tag{5}$$

where $u=g(\alpha,\gamma,\lambda,\theta)$ and $v=\gamma^{-1}\left[\alpha,\ \left(1-u^{\frac{1}{\gamma}}\right)\Gamma(\alpha)\right]$, which is the inverse function of the lower incomplete gamma function. The quantile function in equation (5) is then used for generating the EGBX random variable by setting $U\sim U(0,\ 1)$.

Also, the rth moment of the EGBX distribution is defined as

$$E\left(X^r\right)=\int_0^{\infty} x^r g\left(x,\alpha,\gamma,\lambda,\theta\right)dx.$$

Using equation (4), we obtain

$$E\left(X^r\right) = 2\gamma\theta^\alpha \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} V_{ijk} \times \int_0^\infty \left[x^{2k+r+1}\left(\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]\right)^j \left[ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\alpha-1}\right] dx.$$

The rth moment of the EGBX distribution is helpful in exploring the characteristics of the distribution. It can be used to obtain the mean, median, coefficient of variation, kurtosis, and skewness of the EGBX distribution.

As its name suggests, the moment-generating function (mgf) can be used to produce the rth moment of the distribution. The mgf of the EGBX distribution is given as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} g(x, \alpha, \gamma, \lambda, \theta)\, dx. \tag{6}$$

By substituting equation (4) into equation (6), we obtain

$$E\left(X^r\right) = 2\gamma\theta^\alpha \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} V_{ijk} \int_{-\infty}^{\infty} \left[e^{tx}x^{2k+1}\left(\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]\right)^j \times \left[ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\alpha-1}\right] dx$$

and this can be rewritten as

$$E\left(x^r\right) = 2\gamma\theta^\alpha \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} V_{ijk} \int_{-\infty}^{\infty} \left[\frac{t^l x^{2k+l+1}}{l!}\left(\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]\right)^j \times \left[ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\alpha-1}\right] dx$$

by using the power series of an exponential function.

Suppose $x_1, x_2, \ldots, x_n$ is the random sample of the EGBX distribution. Then, the ith-order statistic of the EGBX distribution is defined as

$$f_{x(i)}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)\,[F(x)]^{i-1}\,[1-F(x)]^{n-i}, \tag{7}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the EGBX distribution, respectively. With the binomial expansion and incorporating the beta function, equation (7) can be rewritten as

$$f_{x(i)}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{i+j-1}. \tag{8}$$

By inserting equations (2) and (3) into (8), we have

$$f_{x(i)}(x, \alpha, \gamma, \lambda, \theta) = \frac{2\gamma\theta\lambda^2 x e^{-(\lambda x)^2}}{\Gamma(\alpha)\,\mathrm{B}(i, n-i+1)} \left(1-e^{-(\lambda x)^2}\right)^{\theta-1} \left[-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\alpha-1}$$

$$\times \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[1 - \frac{\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)}\right]^{\gamma(i+j)-1}$$

and this then can be simplified as

$$\times \sum_{j=0}^{n-i}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{\lambda^{2(m+1)} x^{2m+1} V_{ijkl}\left(\gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]\right)^l}{\left[\Gamma(\alpha)\right]^{l+1}},$$

where

$$V_{ijklm} = \frac{(-1)^{j+k+l+m}}{m!}(k+1)^m \binom{n-i}{j}\binom{\theta-1}{k}\binom{\gamma(i+j)-1}{l}.$$

In addition, the φ-order Renyi entropy of the EGBX distribution is

$$I_\varphi(x) = \frac{1}{1-\psi} log\left(\int_{-\infty}^{\infty}\left[g(x)\right]^\varphi dx\right). \tag{9}$$

By referring to equations (3) and (9), we use the binomial expansion and obtain

$$\begin{aligned}\left[g(x)\right]^\varphi &= \left(2\gamma\theta\lambda^2 x\right)^\varphi\left[-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\varphi(\alpha-1)} \\ &\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\varphi(\theta-1)}{i}\binom{\varphi(\gamma-1)}{j}\frac{(-1)^{i+j} w^j}{\left[\Gamma(\alpha)\right]^{j+1}} e^{-(i+\varphi)(\lambda x)^2},\end{aligned} \tag{10}$$

where $w = \gamma\left[\alpha, -\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]$. Subsequently, we may rewrite equation (10) as

$$\begin{aligned}\left[g(x)\right]^\varphi &= (2\gamma)^\varphi \theta^{\alpha\varphi}\left[-ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\varphi(\alpha-1)} \\ &\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_{ijk} w^j \lambda^{2(\varphi+k)} x^{\varphi+2k}}{\left[\Gamma(\alpha)\right]^{j+1}},\end{aligned} \tag{11}$$

by expanding the exponential term using its power series, where

$$B_{ijk} = \binom{\varphi(\theta-1)}{i}\binom{\varphi(\gamma-1)}{j}\frac{(-1)^{i+j+k}}{k!}(i+\varphi)^k.$$

Finally, the Renyi entropy of the EGBX distribution is derived as

$$\begin{aligned}I_\varphi(x) &= \frac{(2\gamma)^\varphi\theta^{\alpha\varphi}}{1-\psi} \\ &\times log\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty}\frac{B_{ijk} w^j \lambda^{2(\varphi+k)} x^{\varphi+2k}\left[-ln\left(1-e^{-(\lambda x)^2}\right)\right]^{\varphi(\alpha-1)}}{\left[\Gamma(\alpha)\right]^{j+1}} dx\right)\right),\end{aligned}$$

by substituting equation (11) into equation (9).

4 Parameters estimation

We use the maximum likelihood estimation approach to estimate the unknown parameters by maximizing the likelihood or log-likelihood functions. Here, the log-likelihood function of the EGBX distribution is

$$\begin{aligned} l(x,\alpha,\gamma,\lambda,\theta) = \sum_{i=0}^{n}\left[ln(2\gamma\theta) + 2ln(\lambda) + ln(x_i) - (\lambda x_i)^2 - ln[\Gamma(\alpha)]\right] \\ + (\theta-1)\sum_{i=0}^{n} ln\left(1-e^{-(\lambda x)^2}\right) + (\alpha-1)\sum_{i=0}^{n} ln\left[-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right] \\ + (\gamma-1)\sum_{i=0}^{n} ln\left[1-\frac{\gamma\left[\alpha,-\theta ln\left(1-e^{-(\lambda x)^2}\right)\right]}{\Gamma(\alpha)}\right]. \end{aligned} \tag{12}$$

To maximize equation (12), we obtain all the first-order partial derivatives for parameters α, γ, λ, and θ as below

$$\frac{\partial l}{\partial\alpha} = -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} ln(-w_i) + (\gamma-1)\sum_{i=1}^{n}\frac{\gamma(\alpha,-w_i)\Gamma'(\alpha) - \gamma'(\alpha,-w_i)\Gamma(\alpha)}{\Gamma(\alpha)\left[\Gamma(\alpha)-\gamma(\alpha,-w_i)\right]},$$

$$\frac{\partial l}{\partial\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} ln\left[1-\frac{\gamma(\alpha,-w_i)}{\Gamma(\alpha)}\right],$$

$$\begin{aligned} \frac{\partial l}{\partial\lambda} = \frac{2n}{\lambda} + (\theta-1)\sum_{i=1}^{n}\frac{2\lambda x_i^2}{e^{(\lambda x_i)^2}-1} + (\gamma-1)\sum_{i=1}^{n}\frac{2\lambda\theta x_i^2 e^{-(\lambda x_i)^2+\theta}(-w_i)^{\alpha-1}}{\Gamma(\alpha)-\gamma(\alpha,-w_i)} \\ + (\alpha-1)\sum_{i=1}^{n}\frac{2\lambda\theta x_i^2}{\left(e^{(\lambda x_i)^2}-1\right) ln\left(1-e^{-(\lambda x_i)^2}\right)} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \frac{n(\alpha-1)}{\theta} + \sum_{i=1}^{n} ln\left(1-e^{-(\lambda x_i)^2}\right) \\ + (1-\gamma)\sum_{i=1}^{n}\frac{\theta^{\alpha-1}e^{\theta}\left(1-e^{-(\lambda x_i)^2}\right)\left[-ln\left(1-e^{-(\lambda x_i)^2}\right)\right]^{\alpha}}{\Gamma(\alpha)-\gamma(\alpha,-w_i)}, \end{aligned}$$

where $w_i = \theta ln\left(1-e^{-(\lambda x_i)^2}\right)$, $\Gamma'(\alpha) = \int_0^{\infty} t^{\alpha-1} ln(t) e^{-t} dt$, and $\gamma'(\alpha,-w_i) = \int_0^{-w_i} t^{\alpha-1} \times ln(t) e^{-t} dt$. The numerical approach BFGS is implemented to obtain the maximum likelihood estimates of the EGBX distribution parameters.

5 Simulation

In this section, we assess the performance of the EGBX distribution through simulation studies with various sample sizes (n = 50, 150, 300, 400, 500) and different sets of parameter values ($(\alpha,\gamma,\lambda,\theta) = \{(2, 25, 0.5, 0.3), (0.75, 0.45, 0.7, 0.65), (1.5, 0.5, 1.5, 2)\}$). Different sample sizes are chosen to investigate the performance of the EGBX distribution in cases with small to large sample sizes. Meanwhile, the three sets of parameter values

Table 2 Average, RMSE, and Bias of the EGBX distribution for Different Sets of Parameter Values

	$\alpha = 2$			$\gamma = 25$			$\lambda = 0.5$			$\theta = 0.3$		
	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias
Set 1												
$n = 50$	2.0376	0.4918	0.0376	25.0024	0.0112	0.0024	0.5182	0.1097	0.0182	0.3212	0.1064	0.0212
$n = 150$	2.0027	0.2280	0.0027	25.0004	0.0046	0.0004	0.5067	0.0563	0.0067	0.3053	0.0503	0.0053
$n = 300$	1.9997	0.1602	–0.0003	25.0002	0.0031	0.0002	0.5042	0.0368	0.0042	0.3025	0.0366	0.0025
$n = 400$	2.0006	0.0751	0.0006	25.0001	0.0014	0.0001	0.5014	0.0303	0.0014	0.3019	0.0218	0.0019
$n = 500$	1.9968	0.0882	–0.0032	25.0000	0.0016	0.0000	0.5024	0.0230	0.0024	0.3008	0.0218	0.0008
	$\alpha = 0.75$			$\gamma = 0.45$			$\lambda = 0.7$			$\theta = 0.65$		
	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias
Set 2												
$n = 50$	0.9895	0.6974	0.2395	0.6741	0.3276	0.2241	0.7800	0.3886	0.0800	0.5243	0.2461	–0.1257
$n = 150$	0.8582	0.4595	0.1082	0.5486	0.1734	0.0986	0.7463	0.2644	0.0463	0.5796	0.1789	–0.0704
$n = 300$	0.8082	0.3374	0.0582	0.5033	0.1101	0.0533	0.7311	0.1962	0.0311	0.6081	0.1461	–0.0419
$n = 400$	0.7822	0.2722	0.0322	0.4840	0.0759	0.0340	0.7259	0.1616	0.0259	0.6176	0.1169	–0.0324
$n = 500$	0.7907	0.2572	0.0407	0.4807	0.0670	0.0307	0.7145	0.1426	0.0145	0.6219	0.1060	–0.0281
	$\alpha = 1.5$			$\gamma = 0.5$			$\lambda = 1.5$			$\theta = 2.0$		
	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias
Set 3												
$n = 50$	1.8066	0.6155	0.3066	0.6940	0.4770	0.1940	1.4176	0.2846	–0.0824	1.9500	0.5095	–0.0500
$n = 150$	1.6479	0.3979	0.1479	0.5910	0.2637	0.0910	1.4527	0.2005	–0.0473	1.9384	0.3510	–0.0616
$n = 300$	1.5923	0.3008	0.0923	0.5518	0.1689	0.0518	1.4701	0.1522	–0.0299	1.9593	0.2925	–0.0407
$n = 400$	1.5787	0.2850	0.0787	0.5391	0.1507	0.0391	1.4721	0.1870	–0.0279	1.9789	0.2860	–0.0211
$n = 500$	1.5729	0.2533	0.0729	0.5380	0.1386	0.0380	1.4736	0.1270	–0.0264	1.9632	0.2425	–0.0368

cover increasing, bathtub, and unimodal hazard functions, respectively. We use the quantile function of the EGBX distribution in equation (5) to generate the random variable and then we fit the data using the EGBX distribution. For each case considered in this study, the simulation is conducted 2000 times, with the average value (Ave), root mean square error (RMSE), and bias of all estimations recorded and shown in Table 2.

Referring to Table 2, for all three sets of parameter values we consider, we observe that all the average values are close to their true values, while the differences between the average value and true value become smaller as the sample size increases. On the other hand, the bias is closer to zero, while the RMSE decreases when the sample size increases from 50 to 500. Thus, all the parameter estimators are asymptotically unbiased. This shows that the maximum likelihood estimation approach performed well for all the selected sample sizes. Therefore, we can conclude that maximum likelihood is suitable for estimating the EGBX distribution's parameters and performs better for a larger sample size.

6 Numerical examples

To illustrate the application of the EGBX distribution, a real dataset: the failure time of 84 aircraft windshields [27] is used in this section. We investigate its performance along with its submodels, some extended Burr-type X distributions, and several nonnested models. The submodels and extended Burr-type X distributions are Weibull Burr-type X (WBX), beta Burr-type X (BBX), exponentiated generalized Burr-type X (EGEBX), GBX, and BX distributions. Meanwhile, the nonnested models have exponentiated Burr-type XII Poisson (EBXIIP) [10], exponentiated Weibull Burr-type XII (EWBXII) [1], beta Burr-type XII (BBXII) [22], and generalized Marshall–Olkin extended Burr-XII (GMOBXII) [14] distributions. These nonnested models are chosen because they are an extension of the

Table 3 Chi-Square Goodness of Fit Test of All Competing Models for Failure Time of 84 Aircraft Windshields

Model	Test statistics	p-value
EGBX	4.7260	0.4502
WBX	3.1590	0.6755
BBX	3.8452	0.5719
EGEBX	5.5518	0.3523
GBX	3.8420	0.6981
BX	3.4803	0.8373
Nonnested Model		
GMOBXII	3.1485	0.6771
EBXIIP	2.9773	0.5616
EWBXII	5.5987	0.2312
BBXII	7.2030	0.1255

Burr-type XII distribution, another distribution introduced by Burr [9]. Additionally, these nonnested models have four or five parameters, similarly to the EGBX distribution. We use the maximum likelihood approach to estimate the parameters of all competing models. Next, we fit the competing models to the Kaplan–Meier survival function to guess the initial value of each parameter.

The performance of all competing models is assessed through the chi-square goodness of fit test, Akaike (AIC) [6], Bayesian (BIC) [24], correlated Akaike (CAIC), and Hannan–Quinn (HQIC) information criteria, as presented in Tables 3 and 4. The chi-square goodness of fit test examines whether the data follows a specific distribution. Meanwhile, the information criteria assess the model fit, where a smaller value of these criteria implies a superior fit. Lastly, the corresponding survival functions are also plotted in Fig. 3.

As shown in Table 3, all the p-values are greater than 0.05. Hence, we can conclude that all the competing distributions can be used to model the first dataset. At the same time, from Table 4, we can see that the EGBX distribution is the best-fitted distribution among its submodels and the extended Burr-type X distributions because it has the smallest values for all criteria except for BIC. However, the difference between its BIC value and the smallest BIC value is small, indicating no significant difference in their performance. Furthermore, the EGBX distribution is a strong contender among nonnested models, demonstrating the second smallest values for AIC, BIC, CAIC, and HQIC. Therefore, we can conclude that EGBX provides a good fit for the dataset and is a formidable alternative to nonnested models.

7 Conclusion

This study introduced a new four-parameter distribution known as the exponentiated gamma Burr-type X distribution that is an extension of the gamma Burr-type X distribution to solve the deficiency of the GBX distribution. It has high flexibility and can accommodate hazard functions in numerous shapes, including decreasing, increasing, bathtub, and unimodal, which is not covered by the GBX distribution. We explore its cdf and pdf together with various statistical properties, including limit behavior, linear form, quantile function, mf, mgf, order statistics, and Renyi entropy. The quantile function is then used to simulate the EGBX random variable. We utilize the maximum likelihood estimation approach to estimate the parameters of the EGBX distribution. Next, we perform a simulation study using various parameter values and sample sizes to evaluate the performance of the EGBX distribution. The results indicate that the maximum likelihood estimation

Table 4 MLEs, log-likelihood, AIC, BIC, CAIC, and HQIC of All Competing Models for Failure Time of 84 Aircraft Windshields

Model	MLEs	Negative log-likelihood	AIC	BIC	CAIC	HQIC
EGBX	$\hat{\alpha} = 0.1657$ $\hat{\gamma} = 3.5116$ $\hat{\lambda} = 1.0737$ $\hat{\theta} = 0.0301$	127.130	262.259	272.030	262.759	266.189
WBX	$\hat{\alpha} = 113.3565$ $\hat{\beta} = 0.1087$ $\hat{\lambda} = 0.0499$ $\hat{\theta} = 11.1418$	129.343	266.687	276.410	267.193	270.596
BBX	$\hat{\alpha} = 11.7643$ $\hat{\beta} = 0.4201$ $\hat{\lambda} = 0.5738$ $\hat{\theta} = 0.0824$	130.067	268.134	277.857	268.641	272.043
EGEBX	$\hat{\alpha} = 11.3297$ $\hat{\beta} = 0.3559$ $\hat{\lambda} = 0.1931$ $\hat{\theta} = 2.7592$	127.234	262.467	272.190	262.973	266.376
GBX	$\hat{\gamma} = 0.4184$ $\hat{\lambda} = 0.5750$ $\hat{\theta} = 0.9456$	130.060	266.120	273.412	266.420	269.051
BX	$\hat{\lambda} = 0.3801$ $\hat{\theta} = 1.1988$	130.470	264.939	269.801	265.087	266.894
GMOBXII	$\hat{\alpha} = 146.5491$ $\hat{\beta} = 1.9643$ $\hat{\lambda} = 1.6207$ $\hat{\theta} = 4.3481$	127.854	263.709	273.432	264.215	267.617
EBXIIP	$\hat{\alpha} = 9.1600$ $\hat{\beta} = 2.7440$ $\hat{\gamma} = 0.5574$ $\hat{\lambda} = 9.7147$ $\hat{\theta} = 4.3524$	129.718	269.437	281.591	270.206	274.322
EWBXII	$\hat{\alpha} = 6.7793$ $\hat{\beta} = 3.1281$ $\hat{\gamma} = 0.7518$ $\hat{\lambda} = 0.0255$ $\hat{\theta} = 0.1497$	124.348	258.695	270.849	259.464	263.581
BBXII	$\hat{\alpha} = 6.6405$ $\hat{\beta} = 3.4900$ $\hat{\gamma} = 4.0034$ $\hat{\lambda} = 0.3570$ $\hat{\theta} = 4.9392$	126.784	263.568	275.723	264.338	268.454

method is effective for estimating the parameters of the EGBX distribution. Finally, we applied the EGBX distribution to a real dataset to demonstrate its performance. The result revealed that the suggested distribution provides a better fit than its submodels and some extended Burr-type X distributions. Additionally, it is a strong competitor to other competing models in this study, particularly the four nonnested models. The proposed distribution can model various types of survival data across multiple fields, including engineering and medicine. However, this study does not explore scenarios involving censored

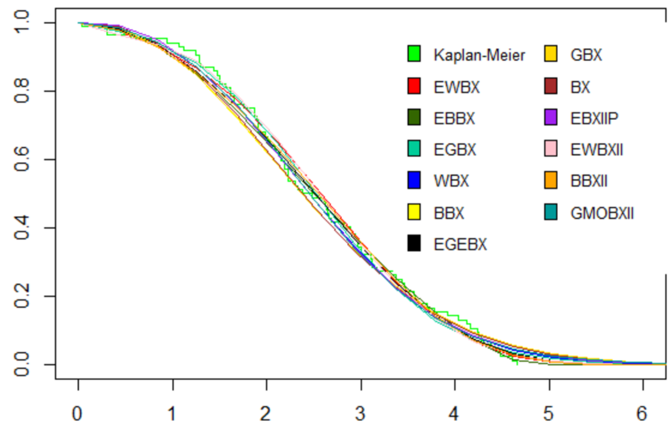

Figure 3 Estimated Survival Function for All Competing Models for Failure Time of 84 Aircraft Windshields

observations and covariates, which are crucial in survival analysis [7, 8]. Hence, future research should consider the existence of censoring and covariates.

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Author contributions
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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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References

1. Abouelmagd, T.H.M., Hamed, M.S., Afify, A.Z.: The extended Burr XII distribution with variable shapes for the hazard rate. Pak. J. Stat. Oper. Res. **13**(3), 687–698 (2017)
2. Afify, A., Abdellatif, A.: The extended Burr XII distribution: properties and applications. J. Nonlinear Sci. Appl. **13**, 133–146 (2020)
3. Afify, A.Z., Cordeiro, G.M., Ortega, E.M., Yousof, H.M., Butt, N.S.: The four-parameter Burr XII distribution: properties, regression model, and applications. Commun. Stat., Theory Methods **47**(11), 2605–2624 (2018)
4. Afify, A.Z., Mead, M.E.: On five-parameter Burr XII distribution: properties and applications. S. Afr. Stat. J. **51**(1), 67–80 (2017)
5. Ahmed, M.T., Khaleel, M.A., Oguntunde, P.E., Abdal-Hammed, M.K.: A new version of the exponentiated Burr X distribution. J. Phys. Conf. Ser. **1818**(1), 012116 (2021). IOP Publishing
6. Akaike, H.: A new look at the statistical model identification. IEEE Trans. Autom. Control **19**(6), 716–723 (1974)
7. Al-Hakeem, H.A., Arasan, J., Mustafa, M.S.B., Lim, F.P.: Parameter estimation for the generalized exponential distribution in the presence of interval censored data and covariate. Int. J. Nonlinear Anal. Appl. **14**(1), 739–751 (2023)

8. Al-Hakeem, H.A., Arasan, J., Mustafa, M.S.B., Lim, F.P.: Generalized exponential distribution with interval-censored data and time dependent covariate. Commun. Stat., Simul. Comput. **52**(12), 6149–6159 (2023)
9. Burr, I.W.: Cumulative frequency functions. Ann. Math. Stat. **13**(2), 215–232 (1942)
10. Da Silva, R.V., Gomes-Silva, F., Ramos, M.W.A., Cordeiro, G.M.: The exponentiated Burr XII Poisson distribution with application to lifetime data. Int. J. Stat. Probab. **4**(4), 112 (2015)
11. Dikko, H.G., Aliyu, Y., Alfa, S.: The beta-Burr type v distribution: its properties and application to real life data. Int. J. Adv. Stat. Probab. **5**(2), 83–86 (2017)
12. Elbatal, I., Zayad, M., Rasekhi, M., Afify, A., Iqbal, Z.: A new extended Weibull model for lifetime data. J. Appl. Probab. Stat. **14**(1), 57–73 (2019)
13. Gupta, R.C., Gupta, P.L., Gupta, R.D.: Modeling failure time data by Lehman alternatives. Commun. Stat., Theory Methods **27**(4), 887–904 (1998)
14. Handique, L., Chakraborty, S.: A new four-parameter extension of Burr-XII distribution: its properties and applications. Jpn. J. Stat. Data Sci. **1**(2), 271–296 (2018)
15. Ibrahim, N.A., Khaleel, M.A., Merovci, F., Kilicman, A., Shitan, M.: Weibull Burr X distribution properties and application. Pak. J. Stat. **33**(5), 315–336 (2017)
16. Khaleel, M.A., Ibrahim, N.A., Shitan, M., Merovci, F.: Some properties of gamma Burr type X distribution with application. AIP Conf. Proc. **1739**(1), 020087 (2016). AIP Publishing LLC
17. Khaleel, M.A., Ibrahim, N.A., Shitan, M., Merovci, F.: New extension of Burr type X distribution properties with application. J. King Saud Univ., Sci. **30**(4), 450–457 (2018)
18. Madaki, U.Y., Bakar, M.R.A., Handique, L.: Beta Kumaraswamy Burr type X distribution and its properties. Prepr. - Am. Chem. Soc., Div. Pet. Chem. **1**, 1–34 (2018)
19. Merovci, F., Khaleel, M.A., Ibrahim, N.A., Shitan, M.: The beta Burr type X distribution properties with application. SpringerPlus **5**(1), 1–18 (2016)
20. Oh, Y.L., Lim, F.P., Chen, C.Y., Ling, W.S.Y., Loh, Y.F.: Exponentiated Weibull Burr type X distribution's properties and its applications. Electron. J. Appl. Stat. Anal. **15**(3), 553–573 (2022)
21. Oh, Y.L., Lim, F.P., Chen, C.Y., Ling, W.S.Y., Loh, Y.F.: A new exponentiated beta Burr type X distribution: model, theory, and applications. Sains Malays. **52**(1), 281–294 (2023)
22. Paranaíba, P.F., Ortega, E.M., Cordeiro, G.M., Pescim, R.R.: The beta Burr XII distribution with application to lifetime data. Comput. Stat. Data Anal. **55**(2), 1118–1136 (2011)
23. Ristić, M.M., Balakrishnan, N.: The gamma-exponentiated exponential distribution. J. Stat. Comput. Simul. **82**(8), 1191–1206 (2012)
24. Schwarz, G.: Estimating the dimension of a model. Ann. Stat. **6**(2), 461–464 (1978)
25. Silva, G.O., Ortega, E.M., Cordeiro, G.M.: The beta modified Weibull distribution. Lifetime Data Anal. **16**(3), 409–430 (2010)
26. Surles, J.G., Padgett, W.J.: Inference for reliability and stress-strength for a scaled Burr type X distribution. Lifetime Data Anal. **7**(2), 187–200 (2001)
27. Tahir, M.H., Cordeiro, G.M., Mansoor, M., Zubair, M.: The Weibull-Lomax distribution: properties and applications. Hacet. J. Math. Stat. **44**(2), 455–474 (2015)

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