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The chromaticity of s-bridge graphs and related graphs

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Abstract

The graph consisting of s paths joining two vertices is called an s-bridge graph. In this paper, we discuss the chromaticity of some families of s-bridge graphs, especially 4-bridge graphs, and some graphs related to s-bridge graphs.

1. The chromaticity of 4-bridge graphs

The graphs considered here are finite, undirected, simple and loopless. For a graph G, let $V(G)$ denote the vertex set of G and $E(G)$ the edge set of G. Let $P(G;\lambda)$, or simply $P(G)$ if there is no likelihood of confusion, denote the chromatic polynomial of G. In this paper, $y=\lambda-1$ and $x=-y$. Two graphs G and H are said to be chromatically equivalent if $P(G)=P(H)$. A graph G is said to be chromatically unique if $P(G)=P(H)$ implies H is isomorphic to G. Let $\mathscr{G}=\{G_0,\ldots,G_p\}$ and $\mathscr{K}=\{K_{i_1},\ldots,K_{i_p}\}$. Then all graphs obtained from $G_0,\ldots,G_p$ by overlapping in $K_{i_1},\ldots,K_{i_p}$ in different positions and different orders form a class of graphs. We denote it by $\{\mathscr{G},\mathscr{K}\}$. If the chromatic equivalence of H to $\{\mathscr{G},\mathscr{K}\}$ implies $H\in\{\mathscr{G},\mathscr{K}\}$, then $\{\mathscr{G},\mathscr{K}\}$ is said to be a complete class of chromatically equivalent graphs. For details and other symbols and definitions, readers can see [6].

A 2-connected graph G is called a generalized polygon tree if it can be decomposed into cycle class $\mathscr{G}=\{C_{i_1},\ldots,C_{i_r}\}$ and there exist an overlapping process: $H_1=C_{i_1}$, H_j is obtained from H_{j-1} and C_{i_j} by overlapping in path P_{i_j} where in each step of overlapping, the vertices on P_{i_j}, except end vertices, are with degree 2.

In [6], it was proved that a 2-connected graph is a generalized polygon tree if and only if it has no subgraphs homeomorphic to K_4. Obviously a generalized polygon

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tree is a planar graph. For a 2-connected planar graph G, we define $r(G)$ as the number of interior regions of G: $r(G)=|E(G)|-|V(G)|+1$. For a generalized polygon tree, we define intercourse number of G, $\sigma(G)$, as the number of nonadjacent vertex pairs, where there are at least three internally disjoint paths joining them. It was also proved in [6] that if G is a generalized polygon tree and $P(H)=P(G)$, then H is also a generalized polygon tree and $r(H)=r(G)$, $\sigma(H)=\sigma(G)$.

A graph consisting of s paths joining two vertices is called an s-bridge graph, which is denoted by $F(k_1,\ldots,k_s)$, where $k_1,\ldots,k_s$ are the lengths of s paths. Clearly an s-bridge graph is a generalized polygon tree.

It was proved by Chao and Whitehead [1] that the cycle C_n is chromatically unique. Later, Loerinc [3] proved that the generalized θ-graph θ_{abc} is chromatically unique. That is to say, 2-bridge graphs and 3-bridge graphs are all chromatically unique. In this paper, we consider the chromaticity of s-bridge graphs and some graphs related to s-bridge graphs. First we give the sufficient and necessary conditions for a 4-bridge graph to be chromatically unique.

Now let G be a 4-bridge graph $F(a,b,c,d)$, the lengths of the 4 paths joining two vertices u and v be a, b, c and d, where $a\geqslant b\geqslant c\geqslant d$ and $|V(G)|=n$. Then $|E(G)|=a+b+c+d=n+2$.

Theorem 1. *A 4-bridge graph G is not chromatically unique if and only if G satisfies one of the following conditions*:

(1) $d=1$,

(2) $d=2$ *and* $a=b+1=c+2$.

Proof. If $d=1$, G is a polygon tree obtained from C_{a+1}, C_{b+1} and C_{c+1} by overlapping on an edge. Let $\mathcal{G}=\{C_{a+1},C_{b+1},C_{c+1}\}$, $\mathcal{K}=\{K_2,K_2\}$. As shown in [6],$\{\mathcal{G},\mathcal{K}\}$ is a complete class of chromatically equivalent graphs and it is easy to see that $|\{\mathcal{G},\mathcal{K}\}|>1$ (see Fig. 1.) Hence G is not chromatically unique and all graphs which are chromatically equivalent to G belong to $\{\mathcal{G},\mathcal{K}\}$.

In the following, we always assume that $d>1$. Then G is a generalized polygon tree, the interior region number $r(G)=3$ and the intercourse number $\sigma(G)=1$. First, using the formula $P(G)=P(G+uv)+P(G.uv)$, we compute the chromatic

Fig. 1.

polynomial of G.

$$P(G)=\frac{1}{x^3(x-1)^3}P(C_{a+1})P(C_{b+1})P(C_{c+1})P(C_{d+1})$$

$$-\frac{1}{(x-1)^3}P(C_a)P(C_b)P(C_c)P(C_d)$$

$$=\frac{(-1)^{n-1}x}{(x-1)^2}[(1+x+x^2)-(x+1)(x^a+x^b+x^c+x^d)$$

$$+(x^{a+b}+x^{a+c}+x^{a+d}+x^{b+c}+x^{b+d}+x^{c+d})-x^{n+1}]$$

$$=\frac{(-1)^{n-1}x}{(x-1)^2}Q(G).$$

Suppose that there is a graph H such that $P(G)=P(H)$. By Lemma 4 in [6], we know that H is also a generalized polygon tree and the interior region number $r(H)=r(G)=3$, the intercourse number $\sigma(H)=\sigma(G)=1$, i.e. H is either a 4-bridge graph or a graph obtained from a generalized θ-graph and a cycle by overlapping on an edge.

Assume that H is a 4-bridge graph with a', b', c' and d' to be the lengths of its paths, where $a' \geqslant b' \geqslant c' \geqslant d'$. Comparing the coefficients of the terms with the lowest degrees in $Q(G)$ and $Q(H)$, we conclude that d' must be equal to d. By Lemma 4 in [6], the girth $g(H)=g(G)$, hence we have $c'=c$. It is easy to obtain that $b'=b$ and $a'=a$. Therefore, H is isomorphic to G.

Now suppose that H is obtained from a generalized θ-graph $\theta_{a'b'c'}$ and a cycle $C_{d'}$ by overlapping on an edge, where $a' \geqslant b \geqslant c' \geqslant 2$ and $d' \geqslant 3$, $a'+b'+c'+d'-3=|V(H)|=n$, i.e. $a'+b'+c'+d'=a+b+c+d+1$. Since the girth of G is more than 3, the girth of H is also more than 3, therefore $d' \geqslant 4$. We compute the chromatic polynomial of H.

$$P(H)=\frac{1}{x(x-1)}P(\theta_{a'b'c'})P(C_{d'})$$

$$=\frac{(-1)^{n-1}x}{(x-1)^2}[(1+x)-(x^{d'-1}+x^{d'}+x^{c'}+x^{b'}+x^{a'})$$

$$+(x^{c'+d'-1}+x^{b'+d'-1}+x^{a'+d'-1})+x^{a'+b'+c'-1}-x^{n+1}]$$

$$=\frac{(-1)^{n-1}x}{(x-1)^2}Q(H).$$

Now we solve the equation $Q(G)=Q(H)$. After canceling x^{n+1}, x and constant terms, we get

$$a+b+c+d=a'+b'+c'+d'-1,\ a \geqslant b \geqslant c \geqslant d \geqslant 2,\ a' \geqslant b' \geqslant c' \geqslant 2,\ d' \geqslant 4.$$

$$Q(G):\quad x^2-x^d-x^{d+1}-x^c-x^{c+1}-x^b-x^{b+1}-x^a-x^{a+1}$$
$$+x^{c+d}+x^{b+d}+x^{a+d}+x^{b+c}+x^{a+c}+x^{a+b}.$$
$$Q(H):\quad -x^{d'-1}-x^{d'}-x^{c'}-x^{b'}-x^{a'}+x^{c'+d'-1}$$
$$+x^{b'+d'-1}+x^{a'+d'-1}+x^{a'+b'+c'-1}$$

Since $d=\min\{d,d+1,c,c+1,b,b+1,a,a+1\}$ and $\min\{a'+d'-1,b'+d'-1,c'+d'-1,a'+b'+c'-1\}>2$, x^2 of $Q(G)$ can only be canceled with x^d of $Q(G)$, i.e. $d=2$.

$a+b+c+3=a'+b'+c'+d'$, $a\geqslant b\geqslant c\geqslant 2$, $a'\geqslant b'\geqslant c'\geqslant 2$, $d'\geqslant 4$.

$$Q(G):\quad -x^3-x^c-x^{c+1}-x^b-x^{b+1}-x^a-x^{a+1}$$
$$+x^{c+2}+x^{b+2}+x^{a+2}+x^{b+c}+x^{a+c}+x^{a+b}$$
$$Q(H):\quad -x^{d'-1}-x^{d'}-x^{c'}-x^{b'}-x^{a'}+x^{c'+d'-1}$$
$$+x^{b'+d'-1}+x^{a'+d'-1}+x^{a'+b'+c'-1}$$

Since $\min\{c+2,b+2,a+2,b+c,a+c,a+b\}>3$, $-x^3$ of $Q(G)$ can only be canceled with one term of $Q(H)$, i.e. $-x^{d'-1}$, $-x^{d'}$, $-x^{c'}$, $-x^{b'}$ or $-x^{a'}$. Because $d'\geqslant 4$, $-x^3$ cannot be canceled with $-x^{d'}$. If $-x^3$ is canceled with $-x^{d'-1}$, then $d'=4$; otherwise one of c', b' and a' must be equal to 3. Without loss of generality we may assume $d'=4$ or $c'=3$. Thus we consider the following two cases.

Case 1. $d'=4$.

In this case, we have

$$a+b+c=a'+b'+c'+1,\quad a\geqslant b\geqslant c\geqslant 2,\quad a'\geqslant b'\geqslant c'\geqslant 2.$$
$$Q(G):\quad -x^c-x^{c+1}-x^b-x^{b+1}-x^a-x^{a+1}$$
$$+x^{c+2}+x^{b+2}+x^{a+2}+x^{b+c}+x^{a+c}+x^{a+b}$$
$$Q(H):\quad -x^4-x^{c'}-x^{b'}-x^{a'}+x^{c'+3}+x^{b'+3}+x^{a'+3}+x^{a'+b'+c'-1}$$

By observing the terms with the highest power, we may conclude that $a+b=a'+b'+c'-1$ and $c=2$. Thus $c'=2$. We now have

$$a+b=a'+b'+1,\quad a\geqslant b\geqslant 2,\quad a'\geqslant b'\geqslant 2.$$
$$Q(G):\quad -x^3-x^b-x^{b+1}-x^a-x^{a+1}+x^4+2x^{b+2}+2x^{a+2}.$$
$$Q(H):\quad -x^4-x^{b'}-x^{a'}+x^5+x^{b'+3}+x^{a'+3}.$$

Then we have $a=a'+1=b'+1=b+1$.

$$Q(G):\quad -x^3-2x^{b+1}+x^4+x^{b+2}.$$
$$Q(H):\quad -x^4-x^b+x^5.$$

The solution is $b=3$. Therefore $F(4,3,2,2)$ is chromatic equivalent with the graph obtained from $F(3,3,2)$ and C_4 by overlapping on an edge.

Case 2. $c'=3$.

In this case we have

$$a+b+c=a'+b'+d', \quad a \geqslant b \geqslant c \geqslant 2, \quad a' \geqslant b' \geqslant 2, \quad d' \geqslant 5.$$

$$Q(G): \quad -x^c-x^{c+1}-x^b-x^{b+1}-x^a-x^{a+1}$$
$$+x^{c+2}+x^{b+2}+x^{a+2}+x^{b+c}+x^{a+c}+x^{a+b},$$

$$Q(H): \quad -x^{d'-1}-x^{d'}-x^{b'}-x^{a'}+x^{d'+2}+x^{b'+d'-1}+x^{a'+d'-1}+x^{a'+b'+2}.$$

By comparing the least power terms, we can see that $c=d'-1$ or $c=b'$. If $c=d'-1$, then the girth of G is $c+2>d'$ which is a contradiction. So $c=b'$. We now have $a+b=a'+d'$, $a \geqslant b \geqslant c$, $a' \geqslant c \geqslant 2$, $d' \geqslant 5$.

$$Q(G): \quad -x^{c+1}-x^b-x^{b+1}-x^a-x^{a+1}$$
$$+x^{c+2}+x^{b+2}+x^{a+2}+x^{b+c}+x^{a+c}+x^{a+b}.$$

$$Q(H): \quad -x^{d'-1}-x^{d'}-x^{a'}+x^{d'+2}+x^{c+d'-1}+x^{a'+d'-1}+x^{a'+c+2}.$$

We now compare the terms of highest power. Since $c+d'-1 \leqslant a'+d'-1=a+b-1$, we have $a+b=a'+c+2=a'+d'$. Thus $d'=c+2$. So we get

$$a+b=a'+c+2, \quad a \geqslant b \geqslant c, \quad a' \geqslant c \geqslant 3.$$

$$Q(G): \quad -x^b-x^{b+1}-x^a-x^{a+1}+x^{c+2}+x^{b+2}+x^{a+2}+x^{b+c}+x^{a+c}.$$

$$Q(H): \quad -x^{c+2}-x^{a'}+x^{c+4}+x^{2c+1}+x^{a'+c+1}.$$

In order that x^{c+2} can be canceled, at least two of a, b, $a+1$ and $b+1$ must be equal to $c+2$. If $a=b=c+2$, then $a'=c+2$. There is no solution. If $a=b=c+1$, then $a'=c$. Again, no solution. If $a=b+1=a+2$, then $a'=a+1$ and we have the solution. Thus $F(c+2,c+1,c,2)$ is chromatic equivalent with the graph obtained from $F(c+1,c,3)$ and C_{c+2} by overlapping on an edge.

Now we have solved the equation $Q(G)=Q(H)$ and completed the proof. From the above argument we can see that when $a=b+1=c+2$ and $d=2$, the graphs obtained from $\theta_{3,c,c+1}$ and C_{c+2} by overlapping on an edge are the only graphs chromatically equivalent to $F(c+2,c+1,c,2)$. □

2. Some complete classes of chromatically equivalent graphs

Now we consider the chromaticity of some s-bridge graphs and graphs related to s-bridge graphs. First we consider the class $\{\{\theta_{abc}, C_d\}, \{K_2\}\}$, i.e. the set of those graphs that are obtained from a θ-graph and a cycle by overlapping on an edge.

Theorem 2. *$\{\{\theta_{abc}, C_d\}, \{K_2\}\}$, where $a \geqslant b \geqslant c \geqslant 2$ and $d \geqslant 3$, is not a complete class of chromatically equivalent graphs if and only if $c=3$ and $d=a+1=b+2$.*

Proof. Let $G \in \{\{\theta_{abc}, C_d\}, \{K_2\}\}$. Then G is a generalized polygon tree, $r(G)=3$ and $\sigma(G)=1$. By Lemma 4 in [7], if there is a graph H such that $P(G)=P(H)$, then H is either a 4-bridge graph or a graph obtained from a generalized θ-graph and a cycle by overlapping on an edge. If H is a 4-bridge graph, and the lengths of whose 4 paths are a', b', c' and d', then by Theorem 1, $P(G)=P(H)$ if and only if $c=3$, $d=a+1=b+2$, $d'=2$, $a'=b'+1=c'+2$ and $b=c'$. So we only need to prove that if $H \in \{\{\theta_{a'b'c'}, C_{d'}\}, \{K_2\}\}$, where $a' \geqslant b' \geqslant c'$, and $P(G)=P(H)$, then $a'=a$, $b'=b$, $c'=c$ and $d'=d$. Let $a+b+c+d=a'+b'+c'+d'=n$. If $d'=d$, then by the chromatic uniqueness of the generalized θ-graph, it follows $a'=a$, $b'=b$ and $c'=c$. So in the following we prove that if $d \neq d'$, then there is no solution for $P(G)=P(H)$.

As done in Section 1, let $P(G)=[(-1)^{n-1}x/(x-1)^2]Q(G)$ and $P(H)=[(-1)^{n-1}x/(x-1)^2]Q(H)$. We try to solve the equation $Q(G)=Q(H)$. Canceling some terms, we get

$$a+b+c+d=a'+b'+c'+d', \quad a \geqslant b \geqslant c \geqslant 2, \quad a' \geqslant b' \geqslant c' \geqslant 2, \quad d \neq d'.$$

$$Q(G): \quad -x^{d-1}-x^d-x^c-x^b-x^a+x^{c+d-1}+x^{b+d-1}+x^{a+d-1}+x^{a+b+c-1}.$$

$$Q(H): \quad -x^{d'-1}-x^{d'}-x^{c'}-x^{b'}-x^{a'}+x^{c'+d'-1}+x^{b'+d'-1}+x^{a'+d'-1}+x^{a'+b'+c'-1}$$

Without loss of generality, we can assume $d>d'$. Then the girth of G must be $b+c$ satisfying $d>b+c$ and the girth of H is either d' or $b'+c'$ if $b'+c'<d'$. In both cases, comparing the terms with lowest degrees of $Q(G)$ and $Q(H)$ yields $c'=c$ and $b'=b$.

$$Q(G): \quad -x^{d-1}-x^d-x^a+x^{c+d-1}+x^{a+b+c-1}.$$

$$Q(H): \quad -x^{d'-1}-x^{d'}-x^{a'}+x^{c'+d'-1}+x^{a'+b'+c'-1}.$$

The solutions are (1) $a'=a$ and $d'=d$; (2) $a'=d-1$, $a=d'-1$ and $c=c'=1$, where the first one contradicts the assumption that $d>d'$ and the second one contradicts the condition that $c \geqslant 2$. Thus we have finished the proof. □

Since $|\{\{\theta_{2,2,2}, C_d\}, \{K_2\}\}|=1$, the graph obtained from $\theta_{2,2,2}$ and a cycle by overlapping on an edge is chromatically unique.

Theorem 3. *Let $\mathscr{G}=\{G_0, \dots, G_k\}$ and $\mathscr{K}=k\{K_2\}$, where $G_0, \dots, G_k$ are all nonseparable generalized polygon trees. Then, if $\{\mathscr{G}, \mathscr{K}\}$ is a complete class of chromatically equivalent graphs, so is $\{\mathscr{G} \cup s\{K_3\}, \mathscr{K} \cup s\{K_2\}\}$ for any positive integer s.*

Proof. We first prove that if a graph H is a nonseparable generalized polygon tree with $r(H)>1$, then $g(H)>3$. Now H can be obtained from some cycles by overlapping on paths. If one cycle has length 3, it must overlap on edges with other subgraphs, i.e. H is separable. A contradiction.

Suppose that G is chromatically equivalent to $\{\mathscr{G}\cup s\{K_3\},\mathscr{K}\cup s\{K_2\}\}$. Then G is a generalized polygon tree. Let $G\in\{\mathscr{G}',\mathscr{K}'\}$. Then all graphs in $\mathscr{G}'$ must be generalized polygon trees. Let $\mathscr{G}'=\{H_0,\ldots,H_t\}$ and $\mathscr{K}'=t\{K_2\}$, where $H_0,\ldots,H_t$ are all nonseparable. If $r(H_i)>1$, then $g(H_i)>3$. Therefore $\mathscr{G}'$ must contain at least s triangles. So $\{\mathscr{G}'\backslash s\{K_3\},\ \mathscr{K}'\backslash s\{K_2\}\}$ is chromatically equivalent to $\{\mathscr{G},\mathscr{K}\}$. Since $\{\mathscr{G},\mathscr{K}\}$ is a complete class of chromatically equivalent graphs, $\{\mathscr{G}'\backslash s\{K_3\},\mathscr{K}'\backslash s\{K_2\}\}=\{\mathscr{G},\mathscr{K}\}$, i.e. $\{\mathscr{G}',\mathscr{K}'\}=\{\mathscr{G}\cup s\{K_3\},\mathscr{K}\cup s\{K_2\}\}$. Therefore $\{\mathscr{G}\cup s\{K_3\},\mathscr{K}\cup s\{K_2\}\}$ is a complete class of chromatically equivalent graphs. □

Theorem 4. *Let G be an s-bridge graph, where the lengths of the s paths are all m. Then G is chromatically unique.*

Proof. The vertex number of G is $(n-1)s+2$, $g(G)=2n$, the number of cycles in G with length $2n$ is $\binom{s}{2}$, and G is a generalized polygon tree with $r(G)=s-1$ and $\sigma(G)=1$.

Now if there is a graph H such that $P(H)=P(G)$, then $|V(H)|=(n-1)s+2$, $g(H)=2n$ and H is also a generalized polygon tree with $\sigma(H)=1$. So H is obtained from a t-bridge graph H' and $s-t$ cycles by overlapping on edges where $t\leqslant s$. Since $g(H)=2n$, if one path of t-bridge graph H' has length $n'<n$, the others must have lengths at least $2n-n'$. So $|V(H')|\geqslant(n-1)t+2$ and the number of cycles in H' with length $2n$ is at most $\binom{t}{2}$. It is easy to see that the number of cycles in H with length $2n$ is at most $\binom{t}{2}+(n-1)(s-t)/(2n-2)\leqslant\binom{s}{2}$ and the equality holds if and only if $t=s$ and H is isomorphic to G. Hence G is chromatically unique. □

As a special case, $K_{2,s}$ is chromatically unique, which was proved in [4] using other methods.

Theorem 5. *Let G be an s-bridge graph, the lengths of whose s paths satisfy $j_1\geqslant\cdots\geqslant j_s\geqslant s-1$. Then G is chromatically unique.*

Proof. Suppose that there is a graph H such that $P(H)=P(G)$. Then H is either an s-bridge graph or a graph obtained from a t-bridge graph and $s-t$ cycles by overlapping on edges, where $t<s$.

If H is an s-bridge graph, we solve equation $P(H)=P(G)$ and it is easy to conclude that H is isomorphic to G.

Suppose that H is obtained from a t-bridge graph and $s-t$ cycles by overlapping on edges, where the lengths of t paths in t-bridge graph are $k_1\geqslant\cdots\geqslant k_t$, and the lengths of $s-t$ cycles are $l_1\geqslant\cdots\geqslant l_{s-t}$. First we give the chromatic polynomials of G and H.

$$P(G)=\frac{y}{(y+1)^{s-1}}\prod(y^{j_i}+(-1)^{j_i+1})+\frac{y^s}{(y+1)^{s-1}}\prod(y^{j_i-1}+(-1)^{j_i})$$

$$=\frac{y}{(y+1)^{s-1}}\left(\prod(y^{j_i}+(-1)^{j_i+1})+y^{s-1}\prod(y^{j_i-1}+(-1)^{j_i})\right)$$

$$=\frac{y}{(y+1)^{s-1}}Q(G),$$

$$P(H)=\frac{y}{(y+1)^{s-1}}\left(\prod(y^{k_i}+(-1)^{k_i+1})+y^{t-1}\prod(y^{k_i-1}+(-1)^{k_i})\right)$$

$$\prod(y^{l_i-1}+(-1)^{l_i})$$

$$=\frac{y}{(y+1)^{s-1}}Q(H).$$

We solve the equation $Q(G)=Q(H)$. Since $|V(G)|=|V(H)|$, we have $\sum j_i-s+2=\sum k_i+\sum l_i-2s+t+2$. It is easy to see that the term with the lowest power in

$$Q(G)-(-1)^{\sum j_i+s}$$

must be either

$$(-1)^{\sum j_i}y^{s-1} \text{ or } (-1)^{\sum_{i\neq s}j_i-s+1}y^{j_s},$$

which cannot be canceled with each other. The corresponding term in

$$Q(H)-(-1)^{\sum k_i+\sum l_i+t}$$

must be one of the three terms:

$$(-1)^{\sum_{i\neq t}k_i+\sum l_i-t+1}y^{k_t},\ (-1)^{\sum k_i+\sum l_i}y^{t-1} \quad\text{and}\quad (-1)^{\sum_{i\neq s-t}l_i+\sum k_i-t}y^{l_{s-t}-1},$$

which cannot be canceled either, so

$$\min\{t-1,k_t,l_{s-t}-1\}=\min\{s-1,j_s\}.$$

Since $t-1<s-1\leqslant j_s$, the equality cannot hold, i.e. there is no solution for $P(G)=P(H)$. □

Theorem 6. *Let M be an s-bridge graph, the lengths of whose s paths are all n. Then $\{\{M,C_p\},\{K_2\}\}$ is a complete class of chromatically equivalent graphs.*

Proof. First we give the chromatic polynomial of $G\in\{\{M,C_p\},\{K_2\}\}$ as follows:

$$P(G)=\frac{y}{(y+1)^s}((y^n+(-1)^{n+1})^s+y^{s-1}(y^{n-1}+(-1)^n)^s)(y^{p-1}+(-1)^p)$$

$$=\frac{y}{(y+1)^s}Q(G).$$

The term of $Q(G)-(-1)^{ns+s+p}$ with the lowest power must be one of the three terms: $(-1)^{(s-1)(n+1)+p}sy^n$, $(-1)^{sn+p}y^{s-1}$ and $(-1)^{sn+s}y^{p-1}$ which cannot be canceled with each other.

Suppose that there is a graph H such that $P(H)=P(G)$. Then H is either an $(s+1)$-bridge graph or a graph obtained from a t-bridge graph ($t\leqslant s$) and cycles by overlapping on edges.

Case 1. H is an $(s+1)$-bridge graph. If $g(G)=2n\leqslant p$, then because G has $\binom{s}{2}$ or $\binom{s}{2}+1$ cycles with length $2n$, it is easy to conclude that there is no solution for $P(G)=P(H)$. So $p<2n$. Let the lengths of $s+1$ paths of H be $j_1\geqslant\cdots\geqslant j_{s+1}$. Then $j_s+j_{s+1}=p$ and

$$P(H)=\frac{y}{(y+1)^s}\left(\prod(y^{j_i}+(-1)^{j_i+1})+y^s\prod(y^{j_i-1}+(-1)^{j_i})\right)$$

$$=\frac{y}{(y+1)^s}Q(H).$$

The term in

$$Q(H)-(-1)^{\sum j_i+s+1}$$

with the lowest power is either

$$(-1)^{\sum_{i\neq s+1}j_i+s}y^{j_{s+1}} \quad \text{or} \quad (-1)^{\sum j_i}y^s,$$

which cannot be canceled with each other. So

$$\min\{j_{s+1},s\}=\min\{s-1,n,p-1\}.$$

We can only get $j_{s+1}=s-1$ and $s=n$. However, there is no solution for $Q(G)=Q(H)$.

Case 2. H is obtained from a t-bridge graph and the cycles $C_{l_1},\ldots,C_{l_{s-t+1}}$ by overlapping on edges, where $t<s$, the lengths of the t paths of the t-bridge graph are $j_i\geqslant\cdots\geqslant j_t$ and $l_1\geqslant\cdots\geqslant l_{s-t+1}$.

If the equality $l_k=p$ holds for an integer k, we can know that $H\in\{\{M,C_p\},\{K_2\}\}$. If $2n\leqslant p$, since $g(G)=2n$ and G has $\binom{s}{2}$ or $\binom{s}{2}+1$ cycles with length $2n$, it is easy to conclude that $H\in\{\{M,C_p\},\{K_2\}\}$. Now $g(G)=p<2n$, $j_t+j_{t-1}=p<l_{s-t+1}$ and

$$P(H)=\frac{y}{(y+1)^s}\left(\prod(y^{j_i}+(-1)^{j_i+1})+y^{t-1}\prod(y^{j_i-1}+(-1)^{j_i})\right)$$
$$\prod(y^{l_i-1}+(-1)^{l_i})$$
$$=\frac{y}{(y+1)^s}Q(H).$$

Then

$$\min\{p-1,n,s-1\}=\min\{l_{s-t+1}-1,t-1,j_t\}.$$

Since $t<s$, $j_t<n$ and $j_t<p-1$, there is no solution for $Q(G)=Q(H)$.

Since $|\{\{K_{2,s}, C_p\}, \{K_2\}\}| = 1$ the graph obtained from $K_{2,s}$ and C_p by overlapping on an edge is chromatically unique.

At last, we give the following theorem without proof, which is similar to Theorems 5 and 6.

Theorem 7. *Let M be an s-bridge graph with s paths whose lengths are* $j_1 \geqslant \cdots \geqslant j_s > s$. *Then* $\{\{M, C_p\}, \{K_2\}\}$ *is a complete class of chromatically equivalent graphs.*

Now if M is an s-bridge graph, then $|\{\{M, C_p\}, \{K_2\}\}| = ?$ Let $\lceil a \rceil$ be the minimum integer equal to or more than a. The following result is obvious: Suppose that the s-bridge graph M has t paths with different lengths $j_1 > \cdots > j_t$. Then

$$|\{\{M, C_p\}, \{K_2\}\}| = \sum_{i=1}^{t} \lceil j_i/2 \rceil.$$

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