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Article

Synchronization Control of Complex Spatio-Temporal Networks Based on Fractional-Order Hyperbolic PDEs with Delayed Coupling and Space-Varying Coefficients

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Abstract: This paper studies synchronization behaviors of two sorts of non-linear fractional-order complex spatio-temporal networks modeled by hyperbolic space-varying PDEs (FCSNHSPDEs), respectively, with time-invariant delays and time-varying delays, including one delayed coupling. One distributed controller with space-varying control gains is firstly designed. For time-invariant delayed cases, sufficient conditions for synchronization of FCSNHSPDEs are presented via LMIs, which have no relation to time delays. For time-varying delayed cases, synchronization conditions of FCSNHSPDEs are presented via spatial algebraic LMIs (SALMIs), which are related to time delay varying speeds. Finally, two examples show the validity of the control approaches.

Keywords: delayed coupling; space-varying; synchronization control; PDEs

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1. Introduction

There exists an array of important complex networks in real life, such as power networks [1], people group networks [2], influential spreader networks [3], transportation networks [4], and logistics networks [5]. Complex networks are an effective and significant instrument for perceiving the interconnections of elements. In view to their powerful function, complex networks have been applied to many fields, including virus spread [6], brain science [7], image processing [8], water quality assessment [9], network attack [10], feature extraction [11], and multi-agent systems [12–15].

As is well-known, synchronization is one of the most important dynamics behaviors [16–18]. Most of the literature has held the idea that the dynamics of nodes depends only on time. Actually, the dynamics of most processes depend not only on time but also on space [19–23], such as in the cases of flexible manipulators [24], flexible spacecraft [25], and reaction-diffusion systems [26]. Therefore, it is meaningful to study partial differential equation (PDE)-based complex spatio-temporal networks(CSNs), considering both time and space [27–29]. Kocarev et al. proposed synchronization methods of spatio-temporal chaos [30]. Xia and Scardovi studied synchronization analysis of linear CSNs [31]. Demetriou investigated control methods for synchronization of CSNs [32]. Kabalan et al. proposed synchronization of CSNs with in-domain coupling by boundary control [33]. Zheng et al. gave an control approach for synchronization of fractional-order CSNs with time delays [34]. Hu et al. proposed an adaptive approach for synchronization of intermittent CSNs using piecewise auxiliary functions [35]. Yang et al. proposed two boundary coupling ways of stochastic CSNs [36].

Yang et al. proposed exponential synchronization of fractional-order CSNs with hybrid delay-dependent impulses [37].

Most of the above references were modeled by parabolic PDEs, whereas few works studied the synchronization of CSNs based on hyperbolic space-varying PDEs (CSNHSPDEs). Li et al. studied synchronization of second-order CSNHSPDEs [38] and first-order CSNHSPDEs [39] by using boundary control. Lu proposed boundary control for local exact synchronization of quasi-linear CSNHSPDEs [40]. Ma and Yang studied synchronization control of CSNHSPDEs, respectively considering a single weight and multiple weights [41]. As a whole, these works studied synchronization of space-varying CSNHSPDEs, which fractional-order models have not considered.

Fractional-order systems are commonly found in a wide range of fields such as physics, electronics, biology, and engineering [42,43]. Yan et al. proposed boundary control of fractional-order parabolic multi-agent systems [44] as well as studying observer-based control [45]. Zhao et al. proposed an event-triggered boundary controller of fractional-order parabolic multi-agent systems [46]. Finite-time boundary control was studied for hyperbolic multi-agent systems [47]. However, the research of fractional-order CSNs based on hyperbolic space-varying PDEs (FCSNHSPDEs) with time-delayed couplings is significant and it remains challenging, not being solved yet.

The objective of this paper is to investigate a distributed controller for synchronization of a kind of FCSNHSPDEs with time-varying parameters and time delays. Firstly, a class of FCSNHSPDE models with time-invariant delays is given, and a distributed controller is studied to drive the following node to reach synchronization with the isolated node. Sufficient conditions are obtained for synchronization of FCSNHSPDEs in terms of spatial algebraic LMIs (SALMIs). A class of FCSNHSPDE models with time-varying delays is given. The same distributed controller is employed, and sufficient conditions are respectively obtained for synchronization of FCSNHSPDEs with time-varying delays. The key contributions of this paper are listed as:

(1) Non-linear fractional-order complex spatio-temporal networks are modeled by hyperbolic space-varying PDEs in this paper, and have potential applications for flexible manipulators, flexible strings, flexible articulated wings, and flexible appendages.
(2) One distributed controller with space-varying control gains is designed in this paper. It allows different nodes to own different gains.
(3) Synchronization conditions of FCSNHSPDEs are presented by spatial algebraic LMIs, which contain space-varying coefficients. By using spatial algebraic LMIs, time-invariant delays and multiple time-varying delays within FCSNHSPDEs have been addressed, respectively.

2. Problem Formulation

This paper firstly studies one fractional-order CSTN based on hyperbolic space-varying PDEs (FCSNHSPDEs) with time-invariant delays, and the i-th node has the behavior as

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha} z_i(\omega,t) = \Theta(\omega)\dfrac{\partial z_i(\omega,t)}{\partial \omega} + A(\omega)z_i(\omega,t) + A_d(\omega)z_i(\omega,t-\tau_1) + B(\omega)f(z_i(\omega,t)) \\ \qquad + B_d(\omega)f(z_i(\omega,t-\tau_2)) + c_1\sum_{j=1}^{N} g_{ij}(\omega)\Gamma(\omega)z_j(\omega,t) \\ \qquad + c_2\sum_{j=1}^{N} g_{ij}(\omega)\Gamma(\omega)z_j(\omega,t-\tau_3) + u_i(\omega,t), \\ z_i(L,t) = 0, \\ z_i(\omega,t) = z_i^0(\omega,t), (\omega,t) \in [0,L]\times[-\tau,0], \end{cases} \tag{1}$$

where $(\omega,t) \in [0,L]\times[0,\infty)$ are space and time, respectively. $z_i(\omega,t)$, $u_i(\omega,t) \in \mathbb{R}^n$. $\Theta(\omega) > 0$, $A(\omega)$, $A_d(\omega)$, $B(\omega)$, and $\Gamma(\omega) \in \mathbb{R}^{n\times n}$. $f(\cdot)$ is a non-linear function, $0 < L$ and

$0 < \alpha < 1$ are real scalars, time delays $0 \leqslant \tau_1, \tau_2, \tau_3 \leqslant \tau$, and constants $c_1 > 0$ and $c_2 > 0$ are the coupling strengths. $G(\omega) = (g_{ij}(\omega))_{N\times N}$ is such that $g_{ii}(\omega) = -\sum\limits_{j=1,j\neq i}^{N} g_{ij}(\omega)$.

The isolated node, $s(\omega,t) \in \mathbb{R}^n$, is assumed to be

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha} s(\omega,t) = \Theta(\omega)\dfrac{\partial s(\omega,t)}{\partial \omega} + A(\omega)s(\omega,t) + A_d(\omega)s(\omega,t-\tau_1) + B(\omega)f(s(\omega,t)) \\ \qquad\qquad + B_d(\omega)f(s(\omega,t-\tau_2)), \\ s(L,t) = 0, \\ s(\omega,t) = s^0(\omega,t), (\omega,t) \in [0,L]\times[-\tau,0]. \end{cases} \tag{2}$$

This study aims to explore a distributed controller driving FCSNHSPDE (1) synchronization to the isolated node (2) as

$$u_i(\omega,t) = d_i(\omega)(s(\omega,t) - z_i(\omega,t)), \tag{3}$$

in which $d_i(\omega)$ are space-varying control gains.

Definition 1. *FCSNHSPDE (1) reaches synchronization if*

$$\lim_{t\to\infty} ||z_i(\omega,t) - s(\omega,t)|| = 0, i \in \{1,2,\cdots,N\}. \tag{4}$$

Definition 2 ([48])**.** *For $\alpha \in (0,1)$, the Caputo partial derivative is defined as follows:*

$${}_{t_0}^{c}D_t^{\alpha} p(\omega,t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t} \frac{\partial p(\omega,\tau)}{\partial \tau}\frac{1}{(t-\tau)^{\alpha}}d\tau, \tag{5}$$

where $p(x,t)$: $\mathbb{R}\times[t_0,\infty)$ is a differentiable function with regard to t.

Assumption 1. *For any $\omega_1, \omega_2 \in \mathbb{R}$, there exists $0 < \mathcal{X} \in \mathbb{R}$ satisfying*

$$|f(\omega_1) - f(\omega_2)| \leq \mathcal{X}|\omega_1 - \omega_2|. \tag{6}$$

Lemma 1 ([49])**.** *For $\alpha \in (0,1)$, $z(\omega,t) : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ is differentiable, then the following inequation holds:*

$${}_{t_0}^{c}D_t^{\alpha}(z^T(\omega,t)z(\omega,t)) \leq 2z^T(\omega,t)\,{}_{t_0}^{c}D_t^{\alpha} z(\omega,t). \tag{7}$$

3. Synchronization of FCSNHSPDEs with Time-Invariant Delays

Let $e_i(\omega,t) \triangleq z_i(\omega,t) - s(\omega,t)$. The behavior of e_i is obtained as

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha} e_i(\omega,t) = (I_N \otimes \Theta(\omega))\dfrac{\partial e(\omega,t)}{\partial \omega} + (I_N \otimes A(\omega))e(\omega,t) + (I_N \otimes A_d(\omega))e(\omega,t-\tau_1) \\ \qquad + (I_N \otimes B(\omega))F(e(\omega,t)) + (I_N \otimes B_d(\omega))F(e(\omega,t-\tau_2)) \\ \qquad + c_1(G_1(\omega)\otimes\Gamma_1(\omega))e(\omega,t) + c_2(G_2(\omega)\otimes\Gamma_2(\omega))e(\omega,t-\tau_3) + u(\omega,t), \\ e(L,t) = 0, \\ e(\omega,t) = e^0(\omega,t), (\omega,t) \in [0,L]\times[-\tau,0], \end{cases} \tag{8}$$

where $e_i^0(\omega) \triangleq z_i^0(\omega) - s^0(\omega)$, $u \triangleq [u_1^T, u_2^T, \cdots, u_N^T]^T$, $e \triangleq [e_1^T, e_2^T, \cdots, e_N^T]^T$, $F(e_i) \triangleq f(z_i(\omega,t)) - f(s(\omega,t))$, and $F(e) \triangleq [F^T(e_1), F^T(e_2), \cdots, F^T(e_N)]^T$.

Theorem 1. *Under Assumption 1, FCSNHSPDE (1) achieves synchronization via the controller (2), if there exist $d_i(\omega) > 0$ such that the following SALMI holds:*

$$\Psi(\omega) \triangleq \begin{bmatrix} \Psi_{11}(\omega) & A_d(\omega) & 0.5c_2G_2(\omega) \otimes \Gamma_2(\omega) \\ * & -I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{aligned} \Psi_{11}(\omega) &\triangleq 0.5[I_N \otimes A(\omega) + c_1G_1(\omega) \otimes \Gamma_1(\omega) - D(\omega) \otimes I_n + *] \\ &\quad + 0.5\chi^2 I_N \otimes (B(\omega)B^T(\omega) + B_d(\omega)B_d^T(\omega)) + 3.5I_{Nn}, \\ D(\omega) &\triangleq diag\{d_1(\omega), d_2(\omega), \cdots, d_N(\omega)\}. \end{aligned}$$

Proof. Let the Lyapunov functional candidate be

$$\begin{aligned} V_1(t) =& {}_{t_0}^{c}D_t^{\alpha-1} V_a(t) + V_b(t), \\ V_a(t) =& 0.5\int_0^L e^T(\omega,t)e(\omega,t)d\omega, \\ V_b(t) =& \int_0^L \int_{t-\tau_1}^t e^T(\omega,\rho)e(\omega,\rho)d\rho d\omega \\ &+ \int_0^L \int_{t-\tau_2}^t e^T(\omega,\rho)e(\omega,\rho)d\rho d\omega \\ &+ \int_0^L \int_{t-\tau_3}^t e^T(\omega,\rho)e(\omega,\rho)d\rho d\omega. \end{aligned} \tag{10}$$

By using Lemma 1, one has

$$\begin{aligned} \dot{V}_1(t) \leqslant& \int_0^L e^T(\omega,t){}_{t_0}^{c}D_t^{\alpha} e(\omega,t)d\omega + 3\int_0^L e^T(\omega,t)e(\omega,t)d\omega \\ &- \int_0^L e^T(\omega,t-\tau_1)e(\omega,t-\tau_1)d\omega - \int_0^L e^T(\omega,t-\tau_2)e(\omega,t-\tau_2)d\omega \\ &- \int_0^L e^T(\omega,t-\tau_3)e(\omega,t-\tau_3)d\omega \\ =& \int_0^L e^T(\omega,t)\frac{\partial e(\omega,t)}{\partial \omega}d\omega \\ &+ \int_0^L e^T(\omega,t)(I_N \otimes A(\omega) + c_1G_1(\omega) \otimes \Gamma_1(\omega))e(\omega,t)d\omega \\ &+ \int_0^L e^T(\omega,t)(I_N \otimes A_d(\omega))e(\omega,t-\tau_1)d\omega \\ &+ \int_0^L e^T(\omega,t)(c_2G_2(\omega) \otimes \Gamma_2(\omega))e(\omega,t-\tau_3)d\omega + \int_0^L e^T(\omega,t)B(\omega)F(e(\omega,t))d\omega \\ &+ \int_0^L e^T(\omega,t)B_d(\omega)F(e(\omega,t-\tau_2))d\omega - \int_0^L e^T(\omega,t)(D(\omega) \otimes I_n)e(\omega,t)d\omega \\ &+ 3\int_0^L e^T(\omega,t)e(\omega,t)d\omega - \int_0^L e^T(\omega,t-\tau_1)e(\omega,t-\tau_1)d\omega \\ &- \int_0^L e^T(\omega,t-\tau_2)e(\omega,t-\tau_2)d\omega - \int_0^L e^T(\omega,t-\tau_3)e(\omega,t-\tau_3)d\omega. \end{aligned} \tag{11}$$

Using integration by parts,

$$\begin{aligned}
&\int_0^L e^T(\omega,t)\Theta(\omega)\frac{\partial e(\omega,t)}{\partial\omega}d\omega \\
=&e^T(\omega,t)\Theta(\omega)e(\omega,t)|_{\omega=0}^{\omega=L} \\
&-\int_0^L \frac{\partial e^T(\omega,t)}{\partial\omega}\Theta(\omega)e(\omega,t) \\
=&-e^T(0,t)\Theta(\omega)e(0,t) \\
&-\int_0^L e^T(\omega,t)\Theta(\omega)\frac{\partial e(\omega,t)}{\partial\omega}d\omega \\
\leq&-\int_0^L e^T(\omega,t)\Theta(\omega)\frac{\partial e(\omega,t)}{\partial\omega}d\omega,
\end{aligned} \tag{12}$$

which implies

$$\int_0^L e^T(\omega,t)\Theta(\omega)\frac{\partial e(\omega,t)}{\partial\omega}d\omega \leq 0. \tag{13}$$

By applying the triangle inequality, under Assumption 1, one has

$$\begin{aligned}
&\int_0^L e^T(\omega,t)B(\omega)F(e(\omega,t))d\omega + \int_0^L e^T(\omega,t)B_d(\omega)F(e(\omega,t-\tau_2))d\omega \\
\leq&0.5\chi^2\int_0^L e^T(\omega,t)(B(\omega)B^T(\omega)+B_d(\omega)B_d^T(\omega))e(\omega,t)d\omega \\
&+0.5\chi^{-2}\int_0^L (F^T(\omega,t)F(\omega,t)+F^T(\omega,t-\tau_2)F(\omega,t-\tau_2))d\omega \\
=&\int_0^L e^T(\omega,t)(0.5\chi^2 I_N\otimes(B(\omega)B^T(\omega)+B_d(\omega)B_d^T(\omega))+0.5I_{Nn})e(\omega,t)d\omega \\
&+0.5\int_0^L e^T(\omega,t-\tau_2)e(\omega,t-\tau_2)d\omega.
\end{aligned} \tag{14}$$

Substitution of (12)–(14) into (11) yields,

$$\dot{V}_1(t) \leqslant \int_0^L \bar{e}^T(\omega,t)\Psi\bar{e}(\omega,t)d\omega - 0.5\int_0^L e^T(\omega,t-\tau_2)e(\omega,t-\tau_2)d\omega, \tag{15}$$

where $\hat{e}(\omega,t) \triangleq [e^T(\omega,t), e^T(\omega,t-\tau_1), e^T(\omega,t-\tau_3)]^T$. Substituting (9) to (15), one has $\dot{V}(t) \leqslant -\lambda_{\min}(-\Psi)||\hat{e}(\cdot,t)|| \leqslant -\lambda_{\min}(-\Psi)||e(\cdot,t)|| < 0$, for all non-zero $e(\omega,t)$, implying synchronization of FCSNHSPDE (1). □

4. Synchronization of FCSNHSPDEs With Time-Varying Delays

This section studies time-varying delayed FCSNHSPDEs, such as

$$\begin{cases}
{}_{t_0}^{c}D_t^{\alpha}z_i(\omega,t) = \Theta(\omega)\dfrac{\partial z_i(\omega,t)}{\partial\omega} + A(\omega)z_i(\omega,t) + A_d(\omega)z_i(\omega,t-\tau_1(t)) + B(\omega)f(z_i(\omega,t)) \\
\qquad + B_d(\omega)f(z_i(\omega,t-\tau_2(t))) + c_1\sum\limits_{j=1}^{N} g_{ij}(\omega)\Gamma(\omega)z_j(\omega,t) \\
\qquad + c_2\sum\limits_{j=1}^{N} g_{ij}(\omega)\Gamma(\omega)z_j(\omega,t-\tau_3(t)) + u_i(\omega,t), \\
z_i(L,t) = 0, \\
z_i(\omega,t) = z_i^0(\omega,t), (\omega,t) \in [0,L]\times[-\tau,0],
\end{cases} \tag{16}$$

where $0 \leqslant \tau_1(t) \leqslant \tau$, $0 \leqslant \tau_2(t) \leqslant \tau$, $0 \leqslant \tau_1(t) \leqslant \tau$, $0 \leqslant \dot{\tau}_3(t) \leqslant \mu_1$, $0 \leqslant \dot{\tau}_2(t) \leqslant \mu_2$, and $0 \leqslant \dot{\tau}_3(t) \leqslant \mu_3$.

The isolated node is assumed to be

$$\begin{cases} {}^{c}_{t_0}D^{\alpha}_{t}s(\omega,t) = \Theta(\omega)\dfrac{\partial s(\omega,t)}{\partial\omega} + A(\omega)s(\omega,t) + A_d(\omega)s(\omega,t-\tau_1(t)) + B(\omega)f(s(\omega,t)) \\ \qquad\qquad + B_d(\omega)f(s(\omega,t-\tau_2(t))), \\ s(L,t) = 0, \\ s(\omega,t) = s^0(\omega,t), (\omega,t) \in [0,L]\times[-\tau,0]. \end{cases} \tag{17}$$

The error system between FCSNHSPDE (16) and (17) with time-varying delays can be obtained as

$$\begin{cases} {}^{c}_{t_0}D^{\alpha}_{t}e(\omega,t) = (I_N\otimes\Theta(\omega))\dfrac{\partial e(\omega,t)}{\partial\omega} + (I_N\otimes A(\omega))e(\omega,t) + (I_N\otimes A_d(\omega))e(\omega,t-\tau_1(t)) \\ \qquad\qquad + (I_N\otimes B(\omega))F(e(\omega,t)) + (I_N\otimes B_d(\omega))F(e(\omega,t-\tau_2(t))) \\ \qquad\qquad + c_1(G_1(\omega)\otimes\Gamma_1(\omega))e(\omega,t) + c_2(G_2(\omega)\otimes\Gamma_2(\omega))e(\omega,t-\tau_3(t)) + u(\omega,t), \\ e(L,t) = 0, \\ e(\omega,t) = e^0(\omega,t), (\omega,t)\in[0,L]\times[-\tau,0]. \end{cases} \tag{18}$$

Theorem 2. *Under Assumption 1, the FCSNHSPDE (17) achieves synchronization with the isolated node (18) via the controller (2), if there exist $d_i(\omega) > 0$ such that the following SALMI holds:*

$$\Xi \triangleq \begin{bmatrix} \Xi_{11} & A_d(\omega) & 0.5c_2G_2(\omega)\otimes\Gamma_2(\omega) \\ * & -I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{19}$$

where $D(\omega)$ is defined in (9) and

$$\begin{aligned} \Xi_{11} \triangleq\ & 0.5[I_N\otimes A(\omega) + c_1G_1(\omega)\otimes\Gamma_1(\omega) - D(\omega)\otimes I_n + *] \\ & + 0.5\chi^2 I_N\otimes(B(\omega)B^T(\omega) + B_d(\omega)B_d^T(\omega)) + 3.5I_{Nn}. \end{aligned}$$

Proof. Let the Lyapunov functional candidate be

$$\begin{aligned} V_2(t) =& {}^{c}_{t_0}D^{\alpha-1}_{t}V_a(t) + V_c(t), \\ V_a(t) =& 0.5\int_0^L e^T(\omega,t)e(\omega,t)d\omega, \\ V_c(t) =& \int_0^L\int_{t-\tau_1(t)}^t e^T(\omega,\rho)e(\omega,\rho)d\rho d\omega \\ & + \int_0^L\int_{t-\tau_2(t)}^t e^T(\omega,\rho)e(\omega,\rho)d\rho d\omega \\ & + \int_0^L\int_{t-\tau_3(t)}^t e^T(\omega,\rho)e(\omega,\rho)d\rho d\omega. \end{aligned} \tag{20}$$

By using Lemma 1, one has

$$
\begin{aligned}
\dot{V}_2(t) \leq & \int_0^L e^T(\omega,t)\,{}_{t_0}^{c}D_t^{\alpha} e(\omega,t)d\omega + 3\int_0^L e^T(\omega,t)e(\omega,t)d\omega \\
& - (1-\dot{\tau}_1(t))\int_0^L e^T(\omega,t-\tau_1(t))e(\omega,t-\tau_1(t))d\omega \\
& - (1-\dot{\tau}_2(t))\int_0^L e^T(\omega,t-\tau_2(t))e(\omega,t-\tau_2(t))d\omega \\
& - (1-\dot{\tau}_3(t))\int_0^L e^T(\omega,t-\tau_3(t))e(\omega,t-\tau_3(t))d\omega \\
= & \int_0^L e^T(\omega,t)\frac{\partial e(\omega,t)}{\partial \omega}d\omega \\
& + \int_0^L e^T(\omega,t)(I_N \otimes A(\omega) + c_1 G_1(\omega)\otimes \Gamma_1(\omega))e(\omega,t)d\omega \\
& + \int_0^L e^T(\omega,t)(I_N \otimes A_d(\omega))e(\omega,t-\tau_1(t))d\omega \\
& + \int_0^L e^T(\omega,t)(c_2 G_2(\omega)\otimes \Gamma_2(\omega))e(\omega,t-\tau_3(t))d\omega + \int_0^L e^T(\omega,t)B(\omega)F(e(\omega,t))d\omega \\
& + \int_0^L e^T(\omega,t)B_d(\omega)F(e(\omega,t-\tau_2(t)))d\omega - \int_0^L e^T(\omega,t)(D(\omega)\otimes I_n)e(\omega,t)d\omega \\
& + 3\int_0^L e^T(\omega,t)e(\omega,t)d\omega - (1-\dot{\tau}_1(t))\int_0^L e^T(\omega,t-\tau_1(t))e(\omega,t-\tau_1(t))d\omega \\
& - (1-\dot{\tau}_2(t))\int_0^L e^T(\omega,t-\tau_2(t))e(\omega,t-\tau_2(t))d\omega \\
& - (1-\dot{\tau}_3(t))\int_0^L e^T(\omega,t-\tau_3(t))e(\omega,t-\tau_3(t))d\omega,
\end{aligned}
\tag{21}
$$

where $D(\omega) \triangleq diag\{d_1(\omega), d_2(\omega), \cdots, d_N(\omega)\}$.

Under Assumption 1,

$$
\begin{aligned}
& \int_0^L e^T(\omega,t)B(\omega)F(e(\omega,t))d\omega + \int_0^L e^T(\omega,t)B_d(\omega)F(e(\omega,t-\tau_2(t)))d\omega \\
\leqslant & 0.5\chi^2(1-\mu_2)^{-1}\int_0^L e^T(\omega,t)(B(\omega)B^T(\omega) + B_d(\omega)B_d^T(\omega))e(\omega,t)d\omega \\
& + 0.5\chi^{-2}(1-\mu_2)\int_0^L (F^T(\omega,t)F(\omega,t) + F^T(\omega,t-\tau_2(t))F(\omega,t-\tau_2(t)))d\omega \\
= & \int_0^L e^T(\omega,t)(0.5\chi^2(1-\mu_2)^{-1} I_N \otimes (B(\omega)B^T(\omega) + B_d(\omega)B_d^T(\omega)) + 0.5 I_{Nn})e(\omega,t)d\omega \\
& + 0.5(1-\mu_2)\int_0^L e^T(\omega,t-\tau_2(t))e(\omega,t-\tau_2(t))d\omega.
\end{aligned}
\tag{22}
$$

According to the conditions of time-varying delays, one has

$$\begin{aligned} &-(1-\dot{\tau}_1(t))\int_0^L e^T(\omega,t-\tau_1(t))e(\omega,t-\tau_1(t))d\omega \\ &-(1-\dot{\tau}_2(t))\int_0^L e^T(\omega,t-\tau_2(t))e(\omega,t-\tau_2(t))d\omega \\ &-(1-\dot{\tau}_3(t))\int_0^L e^T(\omega,t-\tau_3(t))e(\omega,t-\tau_3(t))d\omega \\ \leqslant &-(1-\mu_1)\int_0^L e^T(\omega,t-\tau_1(t))e(\omega,t-\tau_1(t))d\omega \\ &-(1-\mu_2)\int_0^L e^T(\omega,t-\tau_2(t))e(\omega,t-\tau_2(t))d\omega \\ &-(1-\mu_3)\int_0^L e^T(\omega,t-\tau_3(t))e(\omega,t-\tau_3(t))d\omega. \end{aligned} \tag{23}$$

Substitution of (22) and (23) into (21) yields,

$$\dot{V}_2(t) \leqslant \int_0^L \tilde{e}^T(\omega,t)\Xi(\omega)\tilde{e}(\omega,t)d\omega - 0.5(1-\mu_2)\int_0^L e^T(\omega,t-\tau_2(t))e(\omega,t-\tau_2(t))d\omega, \tag{24}$$

where $\tilde{e}(\omega,t) \triangleq [e^T(\omega,t), e^T(\omega,t-\tau_1(t)), e^T(\omega,t-\tau_3(t))]^T$.

The rest of the proof is similar to that of Theorem 1, and so it is omitted. □

Remark 1. *There are many important works on synchronization of hyperbolic PDE-based CSTNs [38,39,50]; however, time delays are still not addressed, which has been considered in this paper.*

Remark 2. *This paper addresses synchronization of FCSNHSPDEs not only with multiple time-invariant delays but also with multiple time-varying delays, as well as considering delayed coupling.*

Remark 3. *PDEs' space-invariant parameters with based CSTNs have been studied for synchronization or consensus [50,51], while space-varying parameters models have not been considered. As is well-known, space-varying parameter models exist in processing [52–54]. This paper deals with space-varying parameter-based models.*

5. Numerical Examples

Example 1. *To demonstrate the effectiveness of Theorem 1, consider FCSNHSPDE (1) with random initial conditions and time-invariant delays as follows*

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha} z_{i1}(\omega,t) = \dfrac{\partial z_{i1}(\omega,t)}{\partial\omega} + 1.2z_{i1}(\omega,t) - 0.2z_{i2}(\omega,t) + 0.8z_{i1}(\omega,t-\tau_1) \\ \quad + 0.5z_{i2}(\omega,t-\tau_1) + f(z_{i1}(\omega,t)) - 0.2f(z_{i1}(\omega,t)) \\ \quad + f(z_{i1}(\omega,t-\tau_2)) - 0.2f(z_{i2}(\omega,t-\tau_2)) + 0.2\sum_{j=1}^{N} g_{ij}(\omega)z_{j1}(\omega,t) \\ \quad + 0.3\sum_{j=1}^{N} g_{ij}(\omega)z_{j1}(\omega,t-\tau_3) + u_{i1}(\omega,t), \\ {}_{t_0}^{c}D_t^{\alpha} z_{i2}(\omega,t) = \dfrac{\partial z_{i2}(\omega,t)}{\partial\omega} + 2.5sin(2\pi\omega)z_{i1}(\omega,t) - 1.8z_{i2}(\omega,t) \\ \quad + cos(\pi\omega)z_{i1}(\omega,t-\tau_1) + 1.6z_{i2}(\omega,t-\tau_1) + 0.2sin(2\pi\omega)f(z_{i1}(\omega,t)) \\ \quad - 2.5f(z_{i1}(\omega,t)) + 0.2sin(2\pi\omega)f(z_{i1}(\omega,t-\tau_2)) - 2.5f(z_{i2}(\omega,t-\tau_2)) \\ \quad + 0.2\sum_{j=1}^{N} g_{ij}(\omega)z_{j2}(\omega,t) + 0.3\sum_{j=1}^{N} g_{ij}(\omega)z_{j2}(\omega,t-\tau_3) + u_{i2}(\omega,t), \end{cases} \tag{25}$$

where

$$\begin{aligned}
&\Theta(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A(\omega) = \begin{bmatrix} 1.2 & -0.2 \\ 2.5sin(2\pi\omega) & -1.8 \end{bmatrix}, A_d(\omega) = \begin{bmatrix} 0.8 & 0.5 \\ cos(\pi\omega) & 1.6 \end{bmatrix}, \\
&B(\omega) = B_d(\omega) = \begin{bmatrix} 1 & -0.2 \\ 0.2sin(2\pi\omega) & -2.5 \end{bmatrix}, \Gamma_1(\omega) = \Gamma_2(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
&L = 1, \alpha = 0.95, c_1 = 0.2, c_2 = 0.3, t_0 = 0, \tau_1 = 3, \tau_2 = 2, \tau_3 = 4, f(\cdot) = tanh(\cdot), \\
&z_i = \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix}, G_1(\omega) = G_2(\omega) = \begin{bmatrix} 10 & -1 & -3 & -6 \\ -1 & 5 & -2 & -2 \\ -2 & -3 & 6 & -1 \\ -1 & -3 & -3 & 7 \end{bmatrix}.
\end{aligned} \quad (26)$$

This illustrates that FCSNHSPDE (1) cannot achieve synchronization without control in Figure 1. From (26), $\chi = 1$ is obtained. By Theorem 1, solve (9) by using Matlab, and the time-varying control gains are obtained as shown in Figure 2, where the parameter feasibility radius = 100 of feasp in the LMI toolkit. Figure 3 shows that FCSNHSPDE (1) reaches synchronization via the proposed controller (3) with the feedback gains shown in Figure 2, and the controller (3) is shown in Figure 4.

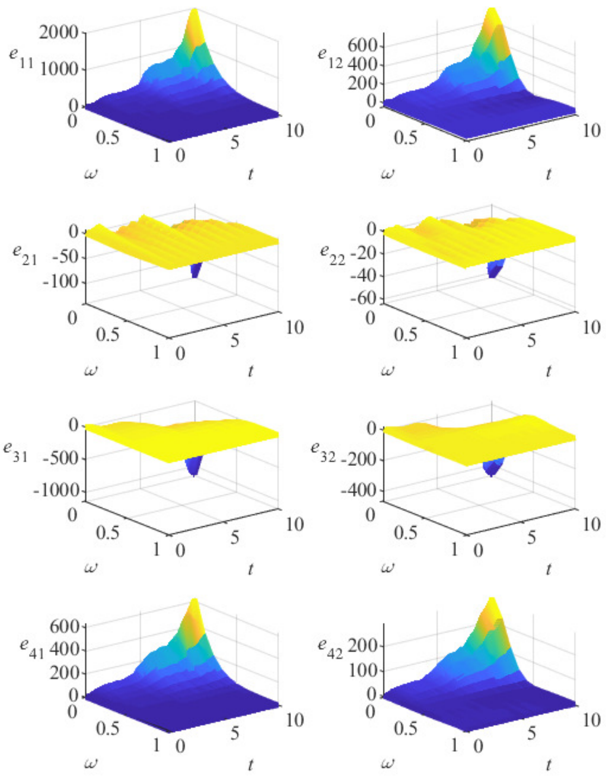

Figure 1. $e(\omega, t)$ of FCSNHSPDE (1) without control.

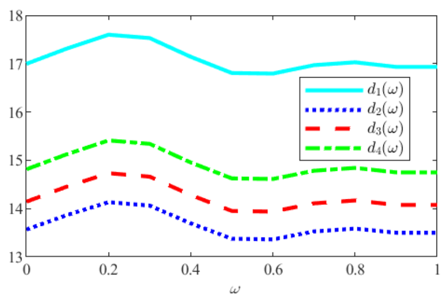

Figure 2. The space-varying control gains of FCSNHSPDE (1).

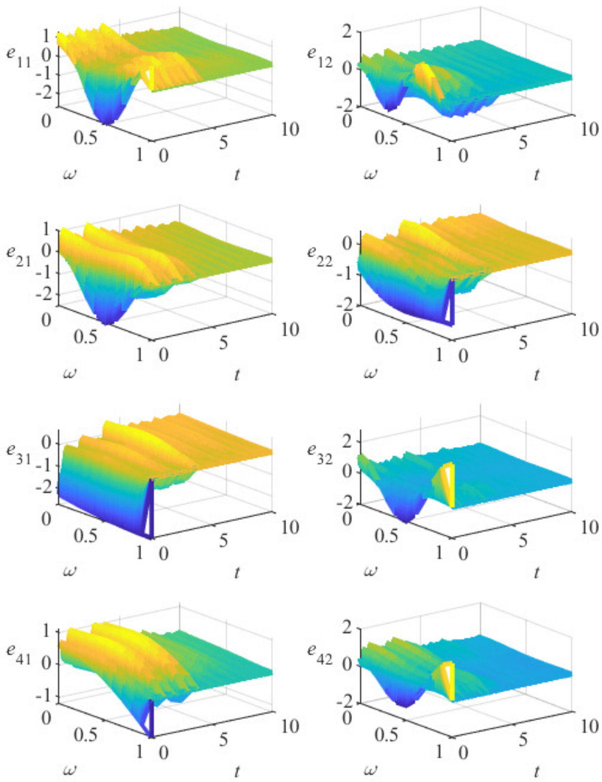

Figure 3. $e(\omega, t)$ of FCSNHSPDE (1) with control.

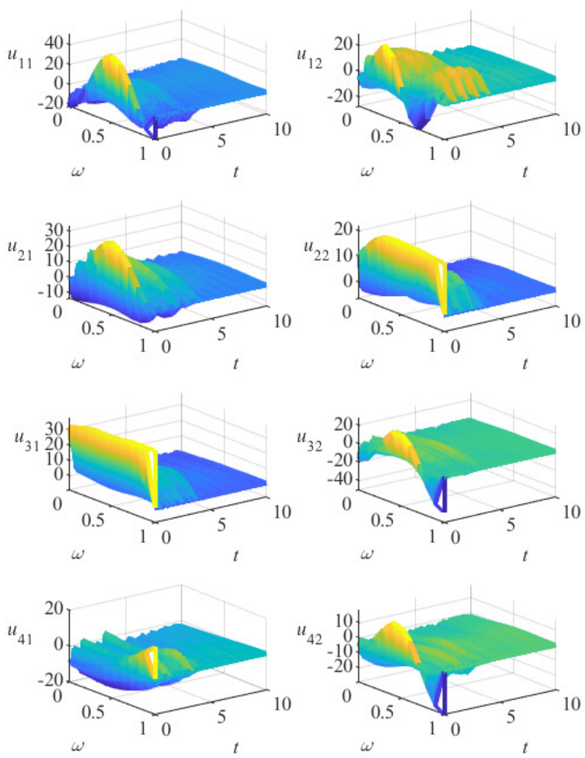

Figure 4. The control input of FCSNHSPDE (1).

Example 2. *To demonstrate the effectiveness of Theorem 2, consider FCSNHSPDE (16) with time-varying delays, and with the same coefficients to those of Example 1, except:*

$$\tau_1(t) = 1.2 + 0.2\sin(1.2\pi t), \tau_2 = 1.5 + 0.8\sin(0.2\pi t), \tau_3 = 1.8 + \sin(0.3\pi t). \tag{27}$$

Figure 5 shows that FCSNHSPDE (16) with time-varying delays cannot achieve synchronization without control. From (27), $\mu_1 = 0.24\pi, \mu_2 = 0.16\pi, \mu_3 = 0.3\pi$, and $\chi = 1$ are obtained. By Theorem 2, solve (20) and the time-varying control gains are obtained as shown in Figure 6, where the parameter feasibility radius=200 of feasp in the LMI toolkit. Figure 7 shows that FCSNHSPDE (16) with time-varying delays achieves synchronization via controller (3) with the control gains, while controller (3) is shown in Figure 8.

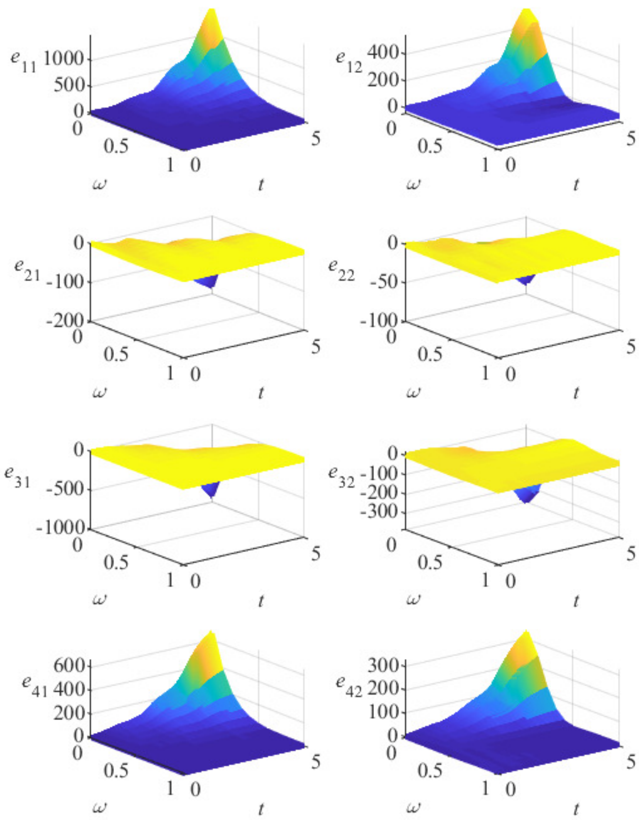

Figure 5. $e(\omega, t)$ of FCSNHSPDE (16) without control.

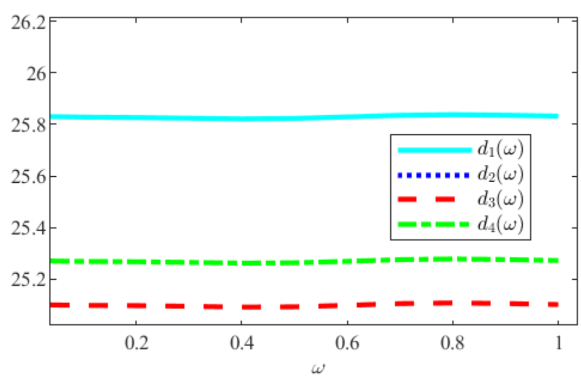

Figure 6. The space-varying control gains of FCSNHSPDE (16).

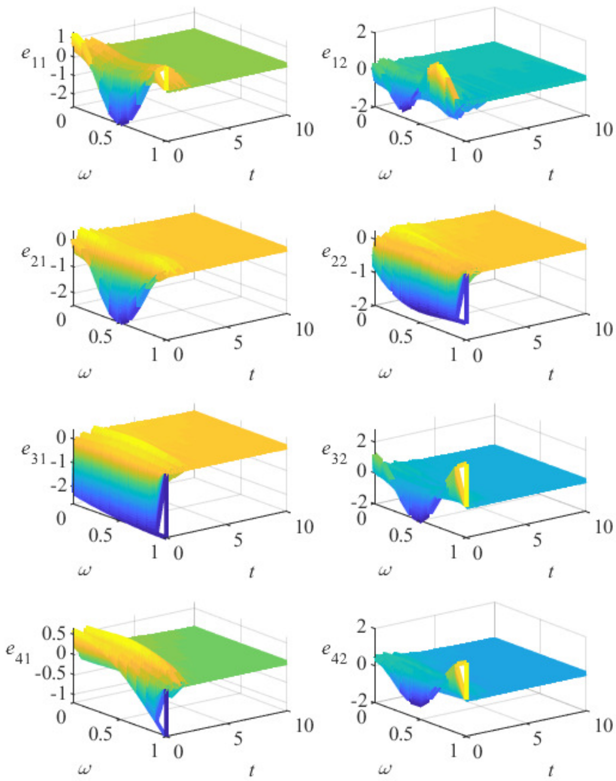

Figure 7. $e(\omega, t)$ of FCSNHSPDE (16) with control.

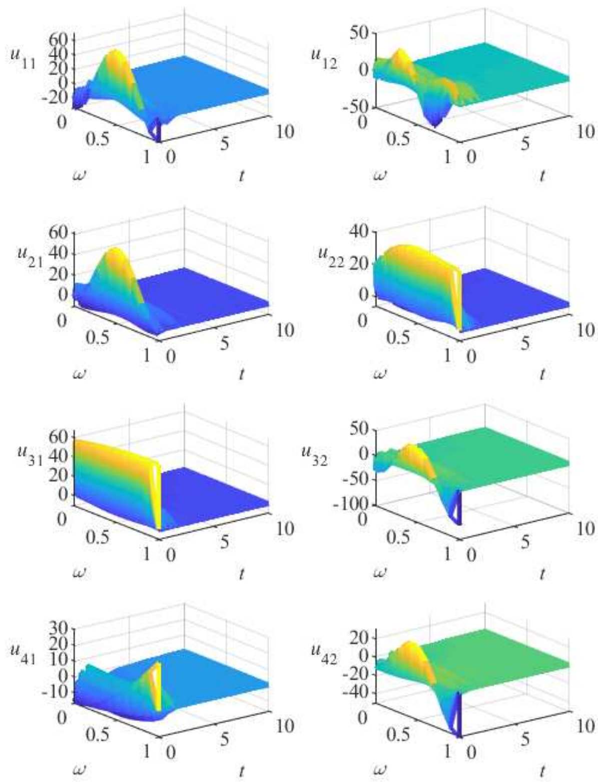

Figure 8. The control input of FCSNHSPDE (16).

6. Conclusions

This study addressed synchronization of two sorts of semi-linear space-varying FCSNHSPDEs, one with time-invariant delays, and the other with time-varying delays. To ensure FCSNHSPDEs achieve synchronization, a space-varying control gains-based control method was proposed. Sufficient conditions for synchronization of FCSNHSPDE with both time-invariant and time-varying delays were derived using SALMIs. The effectiveness of these methods was demonstrated through two examples. The proposed method has a potential application for flexible manipulators, flexible strings, flexible articulated wings, and flexible appendages, which will be considered in the future. The actuator is often prone to faults due to harsh environmental conditions, and so fault-tolerant control of FCSNHSPDEs will be studied in future.

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References

1. Zhou, D.; Hu, F.; Wang, S.; Chen, J. Power network robustness analysis based on electrical engineering and complex network theory. *Phys. A Stat. Mech. Its Appl.* **2021**, *564*, 125540. [CrossRef]
2. Lobsang, T.; Zhen, F.; Zhang, S.; Xi, G.; Yang, Y. Methodological framework for understanding urban people flow from a complex network perspective. *J. Urban Plan. Dev.* **2021**, *147*, 04021020. [CrossRef]
3. Liu, X.; Ye, S.; Fiumara, G.; De Meo, P. Influential spreaders identification in complex networks with TOPSIS and K-shell decomposition. *IEEE Trans. Comput. Soc. Syst.* **2023**, *10*, 347–361. [CrossRef]
4. Wen, T.; Gao, Q.; Chen, Y.; Cheong, K. Exploring the vulnerability of transportation networks by entropy: A case study of Asia–Europe maritime transportation network. *Reliab. Eng. Syst. Saf.* **2022**, *226*, 108578. [CrossRef]
5. Reddy, K.N.; Kumar, A.; Choudhary, A.; Cheng, T.E. Multi-period green reverse logistics network design: An improved Benders-decomposition-based heuristic approach. *Eur. J. Oper. Res.* **2022**, *303*, 735–752. [CrossRef]
6. Cheng, J.; Yin, P. Analysis of the complex network of the urban function under the lockdown of COVID-19: Evidence from Shenzhen in China. *Mathematics* **2022**, *10*, 2412. [CrossRef]
7. Rubinov, M.; Sporns, O. Complex network measures of brain connectivity: Uses and interpretations. *Neuroimage* **2010**, *52*, 1059–1069. [CrossRef]
8. Zhou, H.; Liu, Z.; Chu, D.; Li, W. Sampled-data synchronization of complex network based on periodic self-triggered intermittent control and its application to image encryption. *Neural Netw.* **2022**, *152*, 419–433. [CrossRef] [PubMed]
9. Sitzenfrei, R. Using complex network analysis for water quality assessment in large water distribution systems. *Water Res.* **2021**, *201*, 117359. [CrossRef]
10. Zhang, X.; Zhu, F.; Zhang, J.; Liu, T. Attack isolation and location for a complex network cyber-physical system via zonotope theory. *Neurocomputing* **2022**, *469*, 239–250. [CrossRef]
11. Barkoky, A.; Charkari, N.M. Complex Network-based features extraction in RGB-D human action recognition. *J. Vis. Commun. Image Represent.* **2022**, *82*, 103371. [CrossRef]
12. Yang, C.; Huang, T.; Zhang, A.; Qiu, J.; Cao, J.; Alsaadi, F.E. Output consensus of multi-agent systems based on PDEs with input constraint: A boundary control approach. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 370–377. [CrossRef]
13. Yan, X.; Yang, C.; Cao, J.; Korovin, I.; Gorbachev, S.; Gorbacheva, N. Boundary consensus control strategies for fractional-order multi-agent systems with reaction-diffusion terms. *Inf. Sci.* **2022**, *616*, 461–473. [CrossRef]
14. Yang, Y.; Qi, Q.; Hu, J.; Dai, J.; Yang, C. Adaptive fault-tolerant control for consensus of nonlinear fractional-order multi-agent systems with diffusion. *Fractal Fract.* **2023**, *7*, 760. [CrossRef]
15. Zheng, Y.; Yang, C.; Li, Z.; Zhang, A.; Qiu, J. Boundary containment control of multi-agent systems with time-invariant and time-varying delays. *J. Frankl. Inst.* **2024**, *361*, 106923. [CrossRef]
16. Wang, Z.; Jin, X.; Pan, L.; Feng, Y.; Cao, J. Quasi-synchronization of delayed stochastic multiplex networks via impulsive pinning control. *IEEE Trans. Syst. Man, Cybern. Syst.* **2022**, *52*, 5389–5397. [CrossRef]
17. Chai, L.; Liu, J.; Chen, G.; Zhao, X. Dynamics and synchronization of a complex-valued star network. *Sci. China Technol. Sci.* **2021**, *64*, 2729–2743. [CrossRef]

18. Zhang, J.; Zhu, J. Synchronization of high-dimensional Kuramoto models with nonidentical oscillators and interconnection digraphs. *IET Control Theory Appl.* **2022**, *16*, 244–255. [CrossRef]
19. Feng, Y.; Wang, Y.; Wang, J.W.; Li, H.X. Backstepping-based distributed abnormality localization for linear parabolic distributed parameter systems. *Automatica* **2022**, *135*, 109930. [CrossRef]
20. Wang, J.W.; Wang, J.M. Spatiotemporally asynchronous sampled-data control of a linear parabolic PDE on a hypercube. *Int. J. Control* **2022**, *95*, 3326–3335. [CrossRef]
21. Lin, C.; Cai, X. Stabilization of a class of nonlinear ODE/Wave PDE cascaded systems. *IEEE Access* **2022**, *10*, 35653–35664. [CrossRef]
22. Liu, Y.; Wang, J.W.; Wu, Z.; Ren, Z.; Xie, S. Robust H_∞ control for semilinear parabolic distributed parameter systems with external disturbances via mobile actuators and sensors. *IEEE Trans. Cybern.* **2022**, *53*, 4880–4893. [CrossRef] [PubMed]
23. Mathiyalagan, K.; Ragul, R.; Sangeetha, G.; Ma, Y.K. Exponential stability analysis of stochastic semi-linear systems with Lèvy noise. *IEEE Access* **2022**, *10*, 73871–73878.
24. Liu, Z.; Liu, J.; He, W. Partial differential equation boundary control of a flexible manipulator with input saturation. *Int. J. Syst. Sci.* **2017**, *48*, 53–62. [CrossRef]
25. Liu, Z.; Han, Z.; Zhao, Z.; He, W. Modeling and adaptive control for a spatial flexible spacecraft with unknown actuator failures. *Sci. China Inf. Sci.* **2021**, *64*, 152208. [CrossRef]
26. Cao, Y.; Kao, Y.; Park, J.H.; Bao, H. Global Mittag–Leffler stability of the delayed fractional-coupled reaction-diffusion system on networks without strong connectedness. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *33*, 6473–6483. [CrossRef]
27. Yang, C.; Cao, J.; Huang, T.; Zhang, J.; Qiu, J. Guaranteed cost boundary control for cluster synchronization of complex spatio-temporal dynamical networks with community structure. *Sci. China Inf. Sci.* **2018**, *61*, 052203. [CrossRef]
28. Xie, L.; Zhao, Y. Synchronization of some kind of PDE chaotic systems by invariant manifold method. *Int. J. Bifurc. Chaos* **2005**, *15*, 2303–2309. [CrossRef]
29. Yang, C.; Li, Z.; Chen, X.; Zhang, A.; Qiu, J. Boundary control for exponential synchronization of reaction-diffusion neural networks based on coupled PDE-ODEs. *IFAC-PapersOnLine* **2020**, *53*, 3415–3420. [CrossRef]
30. Kocarev, L.; Tasev, Z.; Parlitz, U. Synchronizing spatiotemporal chaos of partial differential equations. *Phys. Rev. Lett.* **1997**, *79*, 51. [CrossRef]
31. Xia, T.; Scardovi, L. Synchronization analysis of networks of linear parabolic partial differential equations. *IEEE Control Syst. Lett.* **2020**, *5*, 475–480. [CrossRef]
32. Demetriou, M.A. Synchronization and consensus controllers for a class of parabolic distributed parameter systems. *Syst. Control Lett.* **2013**, *62*, 70–76. [CrossRef]
33. Kabalan, A.; Ferrante, F.; Casadei, G.; Cristofaro, A.; Prieur, C. Leader-follower synchronization of a network of boundary-controlled parabolic equations with in-domain coupling. *IEEE Control Syst. Lett.* **2021**, *6*, 2006–2011. [CrossRef]
34. Zheng, B.; Hu, C.; Yu, J.; Jiang, H. Synchronization analysis for delayed spatio-temporal neural networks with fractional-order. *Neurocomputing* **2021**, *441*, 226–236. [CrossRef]
35. Hu, C.; He, H.; Jiang, H. Edge-based adaptive distributed method for synchronization of intermittently coupled spatiotemporal networks. *IEEE Trans. Autom. Control* **2021**, *67*, 2597–2604. [CrossRef]
36. Yang, C.; Yang, C.; Hu, C.; Qiu, J.; Cao, J. Two boundary coupling approaches for synchronization of stochastic reaction-diffusion neural networks based on semi-linear PIDEs. *J. Frankl. Inst.* **2022**, *359*, 10813–10830. [CrossRef]
37. Yang, S.; Jiang, H.; Hu, C.; Yu, J. Exponential synchronization of fractional-order reaction-diffusion coupled neural networks with hybrid delay-dependent impulses. *J. Frankl. Inst.* **2021**, *358*, 3167–3192. [CrossRef]
38. Li, T. From phenomena of synchronization to exact synchronization and approximate synchronization for hyperbolic systems. *Sci. China Math.* **2016**, *59*, 1–18. [CrossRef]
39. Li, T.; Lu, X. Exact boundary synchronization for a kind of first order hyperbolic system. *ESAIM Control. Optim. Calc. Var.* **2022**, *28*, 34. [CrossRef]
40. Lu, X. Local exact boundary synchronization for a kind of first order quasilinear hyperbolic systems. *Chin. Ann. Math. Ser. B* **2019**, *40*, 79–96. [CrossRef]
41. Ma, H.; Yang, C. Exponential synchronization of hyperbolic complex spatio-temporal networks with multi-weights. *Mathematics* **2022**, *10*, 2451. [CrossRef]
42. Wei, Y.; Chen, Y.; Zhao, X.; Cao, J. Analysis and synthesis of gradient algorithms based on fractional-order system theory. *IEEE Trans. Syst. Man, Cybern. Syst.* **2022**, *53*, 1895–1906. [CrossRef]
43. Ben Makhlouf, A.; Baleanu, D. Finite time stability of fractional order systems of neutral type. *Fractal Fract.* **2022**, *6*, 289. [CrossRef]
44. Yan, X.; Li, K.; Zhuang, J.; Yang, C.; Cao, J. Boundary control strategies for consensus of fractional-order multi-agent systems based on coupling PDE-ODEs. *IEEE Trans. Circuits Syst. II Express Briefs* **2023**, *71*, 2179–2183. [CrossRef]
45. Yan, X.; Li, K.; Yang, C.; Zhuang, J.; Cao, J. Consensus of fractional-order multi-agent systems via observer-based boundary control. *IEEE Trans. Netw. Sci. Eng.* **2024**, *11*, 3370–3382. [CrossRef]
46. Zhao, L.; Wu, H.; Cao, J. Event-triggered boundary consensus control for multi-agent systems of fractional reaction–diffusion PDEs. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *127*, 107538. [CrossRef]
47. Wang, X.; Huang, N. Finite-time consensus of multi-agent systems driven by hyperbolic partial differential equations via boundary control. *Appl. Math. Mech.* **2021**, *42*, 1799–1816. [CrossRef]

48. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Limited: Amsterdam, The Netherlands, 2006.
49. Lv, Y.; Hu, C.; Yu, J.; Jiang, H.; Huang, T. Edge-based fractional-order adaptive strategies for synchronization of fractional-order coupled networks with reaction–Diffusion terms. *IEEE Trans. Cybern.* **2018**, *50*, 1582–1594. [CrossRef]
50. Yang, C.; Yang, Y.; Yang, C.; Zhu, J.; Zhang, A.; Qiu, J. Adaptive control for synchronization of semi-linear complex spatio-temporal networks with time-invariant coupling delay and time-variant coupling delay. *Int. J. Adapt. Control Signal Process.* **2022**, *36*, 2640–2659. [CrossRef]
51. Dai, J.; Yang, C.; Yan, X.; Wang, J.; Zhu, K.; Yang, C. Leaderless consensus control of nonlinear PIDE-type multi-agent systems with time delays. *IEEE Access* **2022**, *10*, 21211–21218. [CrossRef]
52. Wang, J.W.; Wang, J.M. Spatiotemporal adaptive state feedback control of a linear parabolic partial differential equation. *Int. J. Robust Nonlinear Control* **2023**, *33*, 3850–3873. [CrossRef]
53. Ammari, O.; Giri, F.; Krstic, M.; Chaoui, F.; El Majdoub, K. Adaptive observer design for heat PDEs with discrete and distributed delays and parameter uncertainties. *IFAC-PapersOnLine* **2023**, *56*, 8952–8957. [CrossRef]
54. Wang, Z.P.; Wu, H.N.; Wang, X.H. Sampled-data fuzzy control with space-varying gains for nonlinear time-delay parabolic PDE systems. *Fuzzy Sets Syst.* **2020**, *392*, 170–194. [CrossRef]