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Efficient solutions to time-fractional telegraph equations with Chebyshev neural networks

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Abstract

This study aims to employ artificial neural networks (ANNs) as a novel method for solving time fractional telegraph equations (TFTEs), which are typically addressed using the Caputo fractional derivative in scientific investigations. By integrating Chebyshev polynomials as a substitute for the traditional hidden layer, computational performance is enhanced, and the range of input patterns is broadened. A feed-forward neural network (NN) model, optimized using the adaptive moment estimation (Adam) technique, is utilized to refine network parameters and minimize errors. Additionally, the Taylor series is applied to the activation function, which removes any limitation on taking fractional derivatives during the minimization process. Several benchmark problems are selected to evaluate the proposed method, and their numerical solutions are obtained. The results demonstrate the method's effectiveness and accuracy, as evidenced by the close agreement between the numerical solutions and analytical solutions.

1. Introduction

The applications of fractional calculus (FC) offer a valuable framework for addressing a wide array of problems in various scientific and technical disciplines. Recently, there has been a growing interest in FC among scientists and academics. Research indicates that fractional models often provide higher accuracy and efficiency compared to traditional models. Moreover, fractional derivative operators are essential for explaining physical processes. Various integral and derivative operators have been introduced [1]. In recent years, fractional equations have been utilized in fields such as mechanics, physics, chemistry, biology, medicine, economics, and signal processing to model a range of real-world phenomena. These phenomena include, but are not limited to, image processing, control systems, viscoelasticity and damping, and diffusion and wave propagation. Researchers from diverse scientific fields, including mathematics, physics, biology, chemistry, and engineering, have extensively explored the use of FC to model significant events within these domains [2–8].

Oliver Heaviside introduced the telegraph equation (TE). This linear second-order hyperbolic partial differential equation (PDE) models the current and voltage dynamics in an electrical transmission line, considering both distance and time. As noted by [9], this model explains the reflection of electromagnetic waves by the wire and the formation of wave patterns along it. The theory covers all frequency ranges addressed by transmission lines, including high-frequency communications and direct current. While initially developed for telegraph wires, the concept extends to conductors functioning across various frequency ranges, such as radio waves, low frequencies, audio frequencies, and direct current pulses. Wire radio antennas, which are electrical analogs of single-conductor transmission lines, also fall under this theory [10, 11]. The TE is applicable to many phenomena in electrical engineering, chemistry, biology, and the physical sciences.

The TE has been solved numerically using various methods over time. These include the spline radial basis function method [12], the Chebyshev Tau method [13], the Legendre multiwavelet Galerkin method [14], and the homotopy perturbation method [15]. Other methods employed are the Chebyshev spectral collocation method [16], the differential quadrature method [17], the Haar wavelet method [18], the Bessel functions method [19], and the dual reciprocity boundary integral equation method [20].

The numerical solution of time fractional telegraph (TFT) equations has been extensively studied by various researchers. [21] employed hybrid Legendre functions to approximate solutions to TFT equations [22]. suggested a spectral meshless radial point interpolation approach for solving two-dimensional fractional telegraph equations [23]. demonstrated the use of the Chebyshev Tau method for numerically solving the two-sided fractional-order telegraph equation [24]. developed a computational Tau technique based on Legendre polynomials for solving TFT equations . For space and time fractional telegraph equations, [25] used a novel projected differential transform technique . The separation of variables method was employed by Chen to find analytical solutions of TFT equations with different boundary conditions [26]. [27] investigated the Sinc-Legendre collocation approach for solving TFT equations . Fourier and Laplace transforms have been used in multiple studies to find analytical solutions of TFT equations [28–30]. [31] presented an effective Legendre wavelets approach for numerically solving TFT equations . Additionally, several researchers utilized semi-analytical approaches to solve TFTEs [32–35].

Recently, researchers have been investigating the use of NN methods to solve a variety of differential equations (DEs), including both partial and ordinary DEs [36, 37]. This success has led to further exploration into using NNs for solving differential equations of fractional orders [38–40]. Here, we provide a summary of the recent research in this area. In their pioneering work, [41] introduced a novel NN approach for addressing differential equations, representing a significant advancement in the field. Subsequent studies have identified several innovative features of the NN method [42].

- The NN-derived solution is analytical, enabling calculations at any point within the domain.
- Coordinate transformations are unnecessary in the solution process.
- The method demands relatively few parameters, and its complexity does not escalate rapidly with an increase in sample points.

ANNs have been advanced to address a wide array of mathematical challenges, including PDEs [43], inverse problems [44, 45], DEs [46–49], and fractional PDEs [50–52]. The recent progress underscores the effectiveness of NN techniques. Consequently, we leverage these innovative attributes to pioneer new methodologies tailored for TFTEs.

This study makes the following specific contributions:

- This work proposes an ANN, called the Chebyshev neural network (CHNN), that utilizes Chebyshev polynomials to solve TFTEs. In the CHNN model, Chebyshev polynomials augment input patterns by substituting for the hidden layer, thereby eliminating the need for it. This approach leads to a reduction in network parameters and an enhancement in computational efficiency.
- This method stands apart by solving TFTEs with an unlimited selection of activation functions, achieved by applying the Taylor series to the Gaussian function. Because of its adaptability, it is more valuable and effective in handling TFTEs.
- This research utilized a forward-propagating NN that was trained using the Adam optimizer, a method known for its efficiency in converging during training.

The structure of the paper is as follows: section 2 provides key definitions, including some fractional derivatives and the Chebyshev polynomial. Section 3 details the approach to addressing TFTEs and delves into the comprehensive design of the CHNN. Section 4 analyzes several examples of TFTEs by evaluating their absolute errors. Lastly, section 5 summarizes the findings from our analysis.

2. Preliminary concepts and notations

This section presents definitions of fractional derivatives and Chebyshev polynomials.

Definition 2.1. [53] defines the following Riemann-Liouville derivative of α order:

$$ {}^{R}D^{\alpha}\nu(\zeta) = \begin{cases} \dfrac{1}{\Gamma(k-\alpha)}\dfrac{d^k}{d\zeta^k}\displaystyle\int_a^{\zeta}\dfrac{\nu(s)}{(\zeta-s)^{\alpha+1-k}}ds, & for\ k-1<\alpha<k, \\ \dfrac{d^k}{d\zeta^k}\nu(\zeta), & for\ \alpha = k, \end{cases} $$

Given a natural number k.

Definition 2.2. The following is the definition of the α-order Caputo derivative given by [53]:

$$ {}^{C}D^{\alpha}\nu(\zeta) = \begin{cases} \dfrac{1}{\Gamma(k-\alpha)}\displaystyle\int_a^{\zeta}\dfrac{\nu^{(k)}(s)}{(\zeta-s)^{\alpha+1-k}}ds, & for\ k-1<\alpha<k, \\ \dfrac{d^k}{d\zeta^k}\nu(\zeta), & for\ \alpha = k, \end{cases} $$

Given a natural number k.

2.1. Chebyshev polynomial

The Chebyshev polynomial of the first kind, denoted as $CH_n(\tau)$, is a polynomial of degree n in the variable τ, defined by the equation [54]:

$$ CH_n(\tau) = \cos(n\varphi) \quad \text{where} \quad \tau = \cos(\varphi). \tag{2.1} $$

When τ varies within the interval $[-1, 1]$, the corresponding variable φ varies within $[0, \pi]$. These intervals are traversed in opposite directions, with $\tau = -1$ corresponding to $\varphi = \pi$ and $\tau = 1$ corresponding to $\varphi = 0$.

It is well known that $\cos(n\varphi)$ is a polynomial of degree n in $\cos(\varphi)$. The basic formulas for these polynomials are familiar:

$$ \cos(0\varphi) = 1, \quad \cos(1\varphi) = \cos(\varphi), \quad \cos(2\varphi) = 2\cos^2(\varphi) - 1, $$
$$ \cos(3\varphi) = 4\cos^3(\varphi) - 3\cos(\varphi), \quad \cos(4\varphi) = 8\cos^4(\varphi) - 8\cos^2(\varphi) + 1, \ldots $$

From equation (2.1), the first few Chebyshev polynomials can be derived directly:

$$ CH_0(\tau) = 1, \quad CH_1(\tau) = \tau, \quad CH_2(\tau) = 2\tau^2 - 1, \quad CH_3(\tau) = 4\tau^3 - 3\tau, \quad CH_4(\tau) = 8\tau^4 - 8\tau^2 + 1, \ldots $$

The fundamental recurrence relation for Chebyshev polynomials is given by:

$$ CH_n(\tau) = 2\tau CH_{n-1}(\tau) - CH_{n-2}(\tau), \quad n = 2, 3, \ldots $$

with the initial conditions:

$$ CH_0(\tau) = 1, \quad CH_1(\tau) = \tau. $$

3. Chebyshev neural techniques

We give a thorough explanation of our technique in this section. This study's neural network design is shown in figure 1.

A non-homogeneous TFTE can be expressed in general terms as follows :

$$ \frac{\partial^{\alpha}\varphi(x, t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha-1}\varphi(x, t)}{\partial t^{\alpha-1}} - \frac{\partial^2\varphi(x, t)}{\partial x^2} + \varphi(x, t) = f(x, t), \quad 1 < \alpha \leqslant 2. \tag{3.1} $$

with the following initial and boundary conditions:

$$ \begin{aligned} \varphi(x, 0) = \phi_0(x), \quad \varphi(0, t) = \psi_0(t), \\ \varphi(x, 1) = \phi_1(x), \quad \varphi(1, t) = \psi_1(t). \end{aligned} \tag{3.2} $$

The trial solution, parameterized by flexible parameters Θ, is represented by the expression $\hat{\varphi}(\eta, \Theta)$. Thus, the following formulation of equation (3.1) can be used:

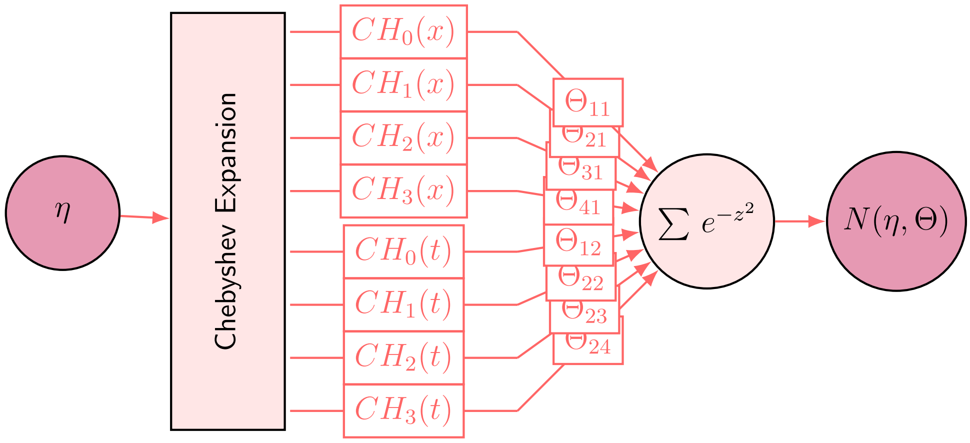


Figure 1. Chebyshev neural network architecture.

$$\frac{\partial^\alpha \hat{\varphi}(\eta, \Theta)}{\partial t^\alpha} + \frac{\partial^{\alpha-1}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha-1}} - \frac{\partial^2 \hat{\varphi}(\eta, \Theta)}{\partial x^2} + \hat{\varphi}(\eta, \Theta) = f(x, t), \ 1 < \alpha \leqslant 2. \tag{3.3}$$

The trial solution, denoted as $\hat{\varphi}(\eta, \Theta)$, can be represented in the following manner [41]:

$$\hat{\varphi}(\eta, \Theta) = \zeta(x, t) + x(1 - x)t(1 - t)N(\eta, \Theta). \tag{3.4}$$

The function $\zeta(x, t)$ solution satisfies the boundary conditions and is given by:

$$\begin{aligned}\zeta(x, t) = (1 - x)\psi_0(t) + x\psi_1(t) \\ + (1 - t)[\phi_0(x) - \{(1 - x)\phi_0(0) + x\phi_0(1)\}] \\ + t[\phi_1(x) - \{(1 - x)\phi_1(0) + x\phi_1(1)\}].\end{aligned} \tag{3.5}$$

The first expression, $\zeta(x, t)$, is unaffected by adjustable parameters and fulfils boundary requirements. The second formula includes the variable $N(\eta, \Theta)$, which represents the outcome of the CHNN. The parameters of the CHNN are denoted by Θ and can be adjusted flexibly. The Taylor series is used to expand the activation function e^{-z^2}. The initial four terms of the resulting series are being considered: $N(\eta, \Theta) = e^{-z^2} = 1 - z^2 + \frac{z^4}{2} - \frac{z^6}{6}$.

z represents a summation in which each term in the expansion of the input data is multiplied by its corresponding weight, expressed as:

$$z = \sum_{i=1}^{2}\left(\sum_{j=1}^{M} \Theta_{ji} CH_{j-1}(\eta_i)\right), \tag{3.6}$$

The input vector is fed into the CHNN and is labeled as $\eta_i = (x, t)^T$. Here, the expanded input data is represented by $CH_{j-1}(\eta)$ and the vector of weights associated with the neural network is Θ_{ji}, where j ranges from 1 to M. Chebyshev polynomials are used to generate the improved pattern as follows:

$$[CH_0(x_{1l}), CH_1(x_{2l}), CH_2(x_{3l}), CH_3(x_{4l}),\ldots;CH_0(t_{1l}), CH_1(t_{2l}), CH_2(t_{3l}), CH_3(t_{4l}),\ldots;\ldots], \ 1 \leqslant l \leqslant r.$$

$N(\eta, \Theta)$ has a α-th derivative that takes η into account. It is as follows:

$$\begin{aligned}\frac{\partial^\alpha N(\eta, \Theta)}{\partial \eta_i^\alpha} &= \frac{\partial^\alpha}{\partial \eta_i^\alpha}\left[-\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^2 + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^4 \right. \\ &\quad \left. - \frac{1}{6}\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i)))^6\right], \\ &= -\frac{\partial^\alpha}{\partial \eta_i^\alpha}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^2\right) + \frac{1}{2}\frac{\partial^\alpha}{\partial \eta_i^\alpha}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^4\right) \\ &\quad - \frac{1}{6}\frac{\partial^\alpha}{\partial \eta_i^\alpha}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^6\right)\Bigg].\end{aligned}$$

The first and second derivatives of $N(\eta, \Theta)$ with consideration for η is given by:

$$\begin{aligned}\frac{\partial N(\eta,\Theta)}{\partial \eta_i} &= \frac{\partial}{\partial \eta_i}\left[-\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^2 + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^4 \right.\\ &\quad \left. - \frac{1}{6}\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i)))^6\right],\\ &= -\frac{\partial}{\partial \eta_i}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^2\right) + \frac{1}{2}\frac{\partial}{\partial \eta_i}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^4\right)\\ &\quad - \frac{1}{6}\frac{\partial}{\partial \eta_i}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^6\right)\Bigg],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 N(\eta,\Theta)}{\partial \eta_i^2} &= \frac{\partial^2}{\partial \eta_i^2}\left[-\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^2 + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^4 \right.\\ &\quad \left. - \frac{1}{6}\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i)))^6\right],\\ &= -\frac{\partial^2}{\partial \eta_i^2}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^2\right) + \frac{1}{2}\frac{\partial^2}{\partial \eta_i^2}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^4\right)\\ &\quad - \frac{1}{6}\frac{\partial^2}{\partial \eta_i^2}\left(\sum_{i=1}^{2}\sum_{j=1}^{M}(\Theta_{ji}CH_{j-1}(\eta_i))^6\right)\Bigg].\end{aligned}$$

The fractional partial derivative of $\hat{\varphi}(\eta, \Theta)$ with regard to x and t in (3.4) is represented as follows:

$$\begin{aligned}\frac{\partial^\alpha \hat{\varphi}(\eta,\Theta)}{\partial t^\alpha} &= \frac{\partial^\alpha \zeta(x,t)}{\partial t^\alpha} + x(1-x)t(1-t)\frac{\partial^\alpha}{\partial t^\alpha}N(\eta,\Theta)\\ &\quad + x(1-x)\left[\frac{\Gamma(1+1)}{\Gamma(1-\alpha+1)}t^{1-\alpha} - \frac{\Gamma(2+1)}{\Gamma(2-\alpha+1)}t^{2-\alpha}\right]N(\eta,\Theta),\\ \frac{\partial \hat{\varphi}(\eta,\Theta)}{\partial x} &= \frac{\partial \zeta(x,t)}{\partial x} + x(1-x)t(1-t)\frac{\partial}{\partial x}N(\eta,\Theta) + (1-2x)t(1-t)N(\eta,\Theta),\\ \frac{\partial^2 \hat{\varphi}(\eta,\Theta)}{\partial x^2} &= \frac{\partial^2 \zeta(x,t)}{\partial x^2} + x(1-x)t(1-t)\frac{\partial^2}{\partial x^2}N(\eta,\Theta) + 2(1-2x)t(1-t)\frac{\partial}{\partial x}N(\eta,\Theta)\\ &\quad - 2t(1-t)N(\eta,\Theta).\end{aligned}$$

The error function relating to the problems of TFTEs represented by equation (3.3) is expressed as:

$$E(\eta,\Theta) = \frac{1}{2}\sum_{l=1}^{r}\left[\frac{\partial^\alpha \hat{\varphi}(\eta,\Theta)}{\partial t^\alpha} + \frac{\partial^{\alpha-1}\hat{\varphi}(\eta,\Theta)}{\partial t^{\alpha-1}} - \frac{\partial^2 \hat{\varphi}(\eta,\Theta)}{\partial x^2} + \hat{\varphi}(\eta,\Theta) - f(x_l,t_l)\right]^2. \tag{3.7}$$

Then, weights are adjusted by using the following formula:

$$\begin{aligned}&\text{biased first moment estimate: } m^n = \beta_1 m^{n-1} + (1-\beta_1)\frac{\partial E}{\partial \Theta^n},\\ &\text{biased second raw moment estimate: } v^n = \beta_2 v^{n-1} + (1-\beta_2)\left(\frac{\partial E}{\partial \Theta^n}\right)^2,\\ &\text{bias}-\text{corrected first moment estimate: } \hat{m}^n = \frac{m^n}{1-\beta_1},\\ &\text{bias}-\text{corrected second raw moment estimate: } \hat{v}^n = \frac{v^n}{1-\beta_2},\\ &\text{Updated the parameters: } \Theta^{n+1} = \Theta^n - \frac{\xi}{\sqrt{\hat{v}^n + \epsilon}}\hat{m}^n.\end{aligned} \tag{3.8}$$

Where ξ is the learning rate. The parameters β_1 and β_2, both within the range (0, 1), represent the decay rates for moment estimates. The variables m and v serve as biased estimates for the gradient's first and second moments, respectively, and are initially set to zero. Additionally, ϵ is a small value, typically set to 1×10^{-8}, to prevent division by zero.

Table 1. Comparison of analytic and CHNN results for different α values for Example 4.1.

(x,t)	Analytic $\alpha = 1.95$	CHNN $\alpha = 1.45$	CHNN $\alpha = 1.7$	CHNN $\alpha = 1.95$
(0.1, 0.1)	0.1105	0.1113	0.1116	0.1119
(0.2, 0.2)	0.2447	0.2423	0.2443	0.2454
(0.3, 0.3)	0.4084	0.3972	0.4011	0.4020
(0.4, 0.4)	0.6127	0.5926	0.5955	0.5930
(0.5, 0.5)	0.8777	0.8575	0.8520	0.8416
(0.6, 0.6)	1.2386	1.2323	1.2095	1.1866
(0.7, 0.7)	1.7542	1.7713	1.7287	1.6914
(0.8, 0.8)	2.5179	2.5596	2.5081	2.4613
(0.9, 0.9)	3.6737	3.7300	3.6877	3.6462
(1, 1)	5.4366	5.4365	5.4366	5.4366

The derivative of $E(\eta, \Theta)$ with respect to parameters is as follows:

$$\frac{\partial E(\eta, \Theta)}{\partial \Theta_{ji}} = \frac{\partial}{\partial \Theta_{ji}^{n}}\left(\frac{1}{2}\sum_{l=1}^{r}\left[\frac{\partial^{\alpha}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha}} + \frac{\partial^{\alpha-1}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha-1}} - \frac{\partial^{2}\hat{\varphi}(\eta, \Theta)}{\partial x^{2}} + \hat{\varphi}(\eta, \Theta) - f(x_l, t_l)\right]^{2}\right),$$
$$= \sum_{l=1}^{r}\left[\frac{\partial^{\alpha}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha}} + \frac{\partial^{\alpha-1}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha-1}} - \frac{\partial^{2}\hat{\varphi}(\eta, \Theta)}{\partial x^{2}} + \hat{\varphi}(\eta, \Theta) - f(x_l, t_l)\right]$$
$$\times\left(\frac{\partial}{\partial \Theta_{ji}^{n}}\left(\frac{\partial^{\alpha}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha}}\right) + \frac{\partial}{\partial \Theta_{ji}^{n}}\left(\frac{\partial^{\alpha-1}\hat{\varphi}(\eta, \Theta)}{\partial t^{\alpha-1}}\right) - \frac{\partial}{\partial \Theta_{ji}^{n}}\left(\frac{\partial^{2}\hat{\varphi}(\eta, \Theta)}{\partial x^{2}}\right) + \frac{\partial}{\partial \Theta_{ji}^{n}}(\hat{\varphi}(\eta, \Theta))\right).$$

The algorithm provided illustrates the implementation of the CHNN technique for solving TFTEs.

Algorithm 1. CHNN algorithm for solving TFTEs

Require: Solution domain $[0, 1] \times [0, 1], l, M, k$
1: **for** $j = 1$ **to** r **do**
2: Generate mesh points (x_r, t_r) within $[0, 1] \times [0, 1]$
3: **end for**
4: **for** $l = 1$ **to** M **do**
5: Initialize randomly weighted vectors Θ_l
6: **end for**
7: Construct CHNN functional block using input data
8: **for** epoch $= 1$ **to** k **do**
9: Compute z using equation (3.6), $N(\eta, \Theta)$ using z
10: Compute $\hat{\varphi}(\eta, \Theta)$ using Eq.(3.4)
11: Calculate error function $E(\eta, \Theta)$
12: Adjust weight vectors Θ using equation (3.8)
13: **end for**
14: Calculate $\hat{\varphi}(\eta, \Theta)$.

4. Computational results

The following section presents several instances to illustrate the efficacy of the CHNN in solving TFTEs.

Example 4.1. consider the following TFTEs [55]:

$$\begin{cases} \dfrac{\partial^{\alpha}\varphi(x, t)}{\partial t^{\alpha}} + \dfrac{\partial^{\alpha-1}\varphi(x, t)}{\partial t^{\alpha-1}} - \dfrac{\partial^{2}\varphi(x, t)}{\partial x^{2}} + \varphi(x, t) = f(x, t), \\ \varphi(x, 0) = 0,\ \varphi(x, 1) = 2e^{x},\ 0 \leqslant x \leqslant 1, \\ \varphi(0, t) = t + t^{3+\alpha},\ \varphi(1, t) = (t + t^{3+\alpha})e,\ 0 < t \leqslant 1. \end{cases}$$

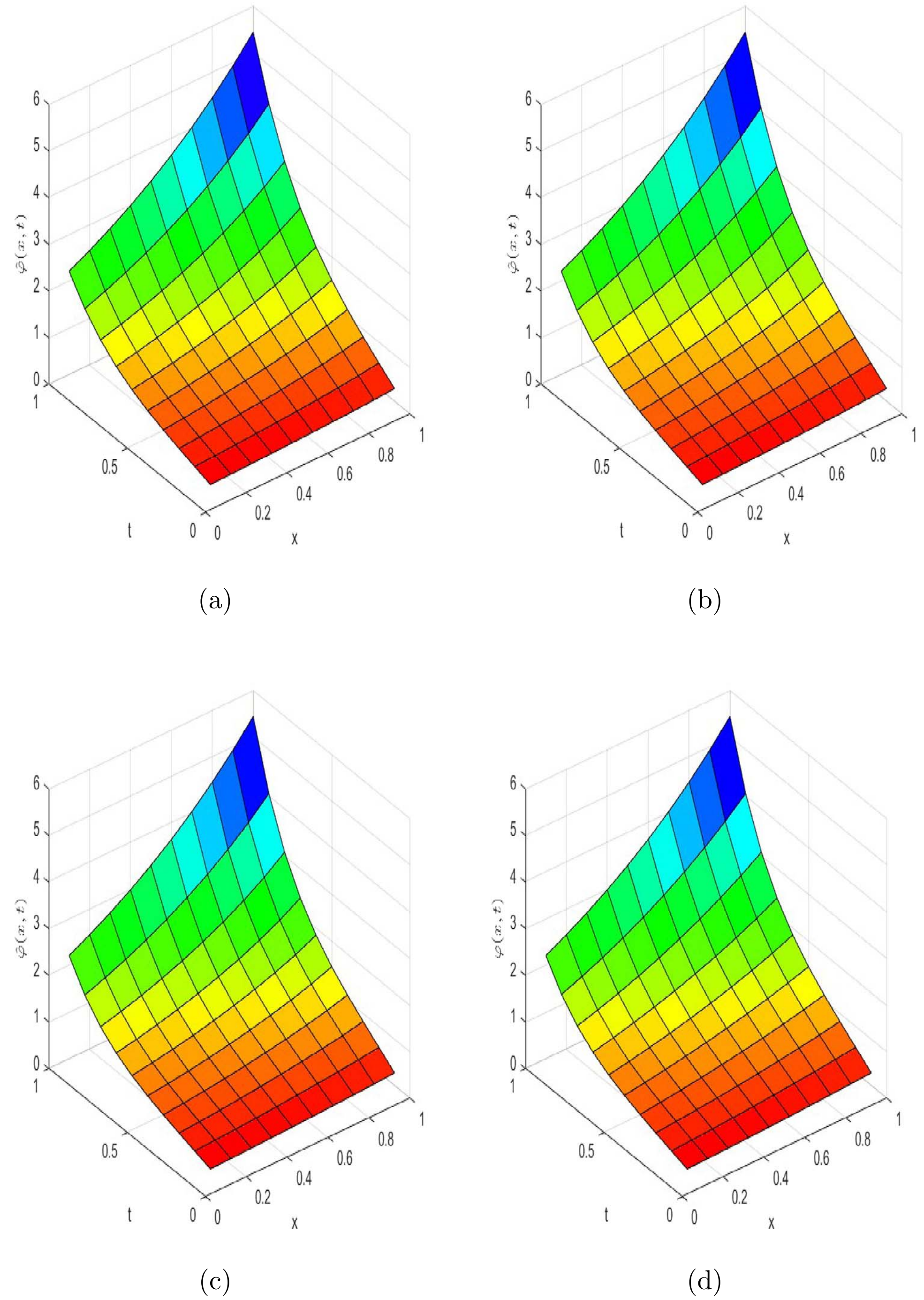


Figure 2. The CHNN solution is illustrated in (a), (b) and (c) with $\alpha = 1.3, 1.8, 1.95$, respectively and analytic solution is illustrated in (d) with $\alpha = 1.95$ for Example 4.1.

While $\varphi(x,\ t) = (t + t^{3+\alpha})e^x$ and $f(x,\ t) = (\dfrac{\Gamma(4+\alpha)}{\Gamma(4)}t^3 + \dfrac{\Gamma(4+\alpha)}{\Gamma(5)}t^4 + \dfrac{\Gamma(2)}{\Gamma(3-\alpha)}t^{2-\alpha})e^x$.The suggested method involves utilizing a neural network with eight neurons to analyze input data. The input data is processed using mesh points placed in a 10×10 grid that spans the region $[0, 1] \times [0, 1]$. The neural network is then trained for a total of 1000 epochs. The values of β_1 and β_2 are fixed at 0.95 and 0.99, respectively. The learning rate, denoted by ξ, is assigned a value of 0.0001, while ϵ is set to 1×10^{-8}. The outcomes of this training for CHNN when $\alpha = 1.45$, 1.7, and 1.95 are contrasted with the analytical solution for $\alpha = 1.95$, as presented in table 1. Figure 2 shows visual representations of the closeness between the solutions given by the CHNN with $\alpha = 1.3$, 1.8, and 1.95 and the analytic solution for $\alpha = 1.95$. These statistics validate the exceptional precision of the CHNN model in providing a very effective and close-to-perfect solution. Finally, figure 3 shows a visual comparison between the CHNN solution and the solution obtained through the method in [55] for $\alpha = 1.1$, 1.3, 1.5, 1.7, and 1.9.

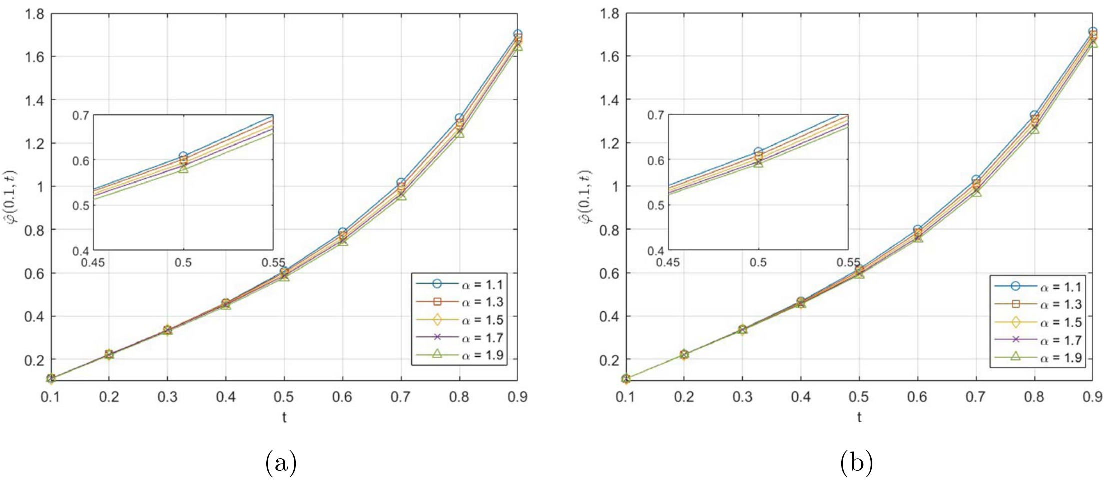


Figure 3. Comparison of CHNN solution in (a) and the method in [55] in (b) when $x = 0.1, \alpha = 1.1, 1.3, 1.5, 1.7, 1.9$ for Example 4.1.

Table 2. Comparison of analytic and CHNN results for different α values for Example 4.2.

(x,t)	Analytic solution	CHNN $\alpha = 1.1$	CHNN $\alpha = 1.7$	CHNN $\alpha = 1.9$
(0.1, 0.1)	0.1100	0.1178	0.1148	0.1137
(0.2, 0.2)	0.2400	0.2646	0.2537	0.2503
(0.3, 0.3)	0.3900	0.4317	0.4135	0.4077
(0.4, 0.4)	0.5600	0.6125	0.5946	0.5869
(0.5, 0.5)	0.7500	0.8027	0.7966	0.7877
(0.6, 0.6)	0.9600	1.0013	1.0138	1.0066
(0.7, 0.7)	1.1900	1.2131	1.2333	1.2336
(0.8, 0.8)	1.4400	1.4463	1.4561	1.4624
(0.9, 0.9)	1.7100	1.7066	1.7082	1.7130
(1, 1)	2	2	2	2

Example 4.2. Consider the following TFTEs [55]:

$$\begin{cases} \dfrac{\partial^{\alpha}\varphi(x,\, t)}{\partial t^{\alpha}} + \dfrac{\partial^{\alpha-1}\varphi(x,\, t)}{\partial t^{\alpha-1}} - \dfrac{\partial^{2}\varphi(x,\, t)}{\partial x^{2}} + \varphi(x,\, t) = x^2 + t - 1, \\ \varphi(x,\, 0) = x^2,\, \varphi(x,\, 1) = x^2 + 1,\, 0 \leqslant x \leqslant 1, \\ \varphi(0,\, t) = t,\, \varphi(1,\, t) = t + 1,\, 0 < t \leqslant 1. \end{cases}$$

For $\alpha = 2$, $\varphi(x, t) = x^2 + t$.A comparative investigation was carried out to determine the efficacy of the CHNN's solutions against the analytic solutions. Table 2 shows the findings for α values of 1.1, 1.7, and 1.9 , respectively. Figure 4 provides a visual comparison of the analytic and CHNN solutions for different vale of α, while figure 5 illustrates the comparison between the CHNN solution and the solution obtained through the method in [55] .

Example 4.3. Let consider the following TFTEs [56]:

$$\begin{cases} \dfrac{\partial^{1.5}\varphi(x,\, t)}{\partial t^{1.5}} + \dfrac{\partial^{0.5}\varphi(x,\, t)}{\partial t^{0.5}} - \dfrac{\partial^{2}\varphi(x,\, t)}{\partial x^{2}} + 2\varphi(x,\, t) = f(x,\, t), \\ \varphi(x,\, 0) = e^{x^2},\, \varphi(x,\, 1) = e^{x^2+1},\, 0 \leqslant x \leqslant 1, \\ \varphi(0,\, t) = e^{t},\, \varphi(1,\, t) = e^{t+1},\, 0 < t \leqslant 1. \end{cases}$$

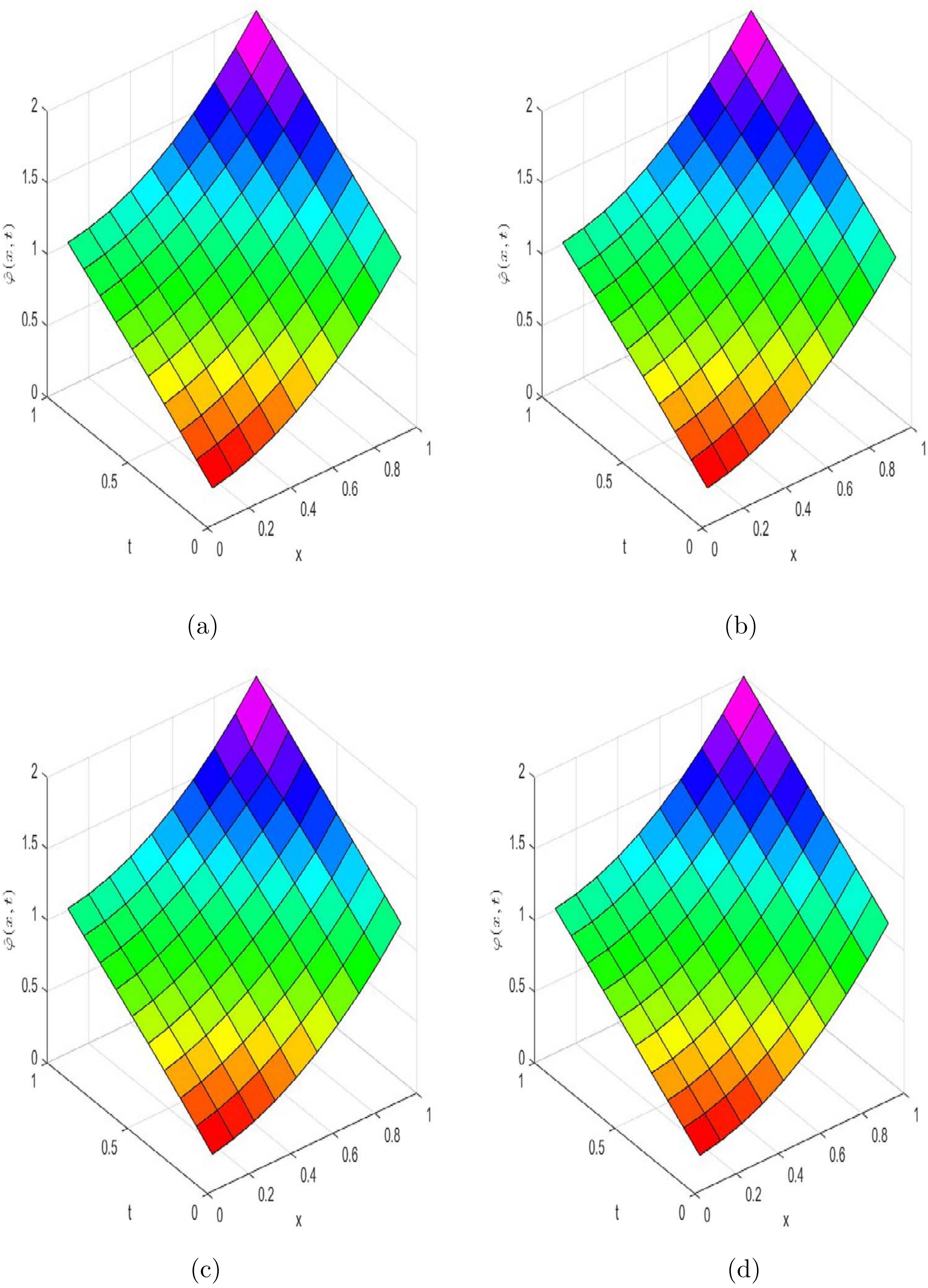
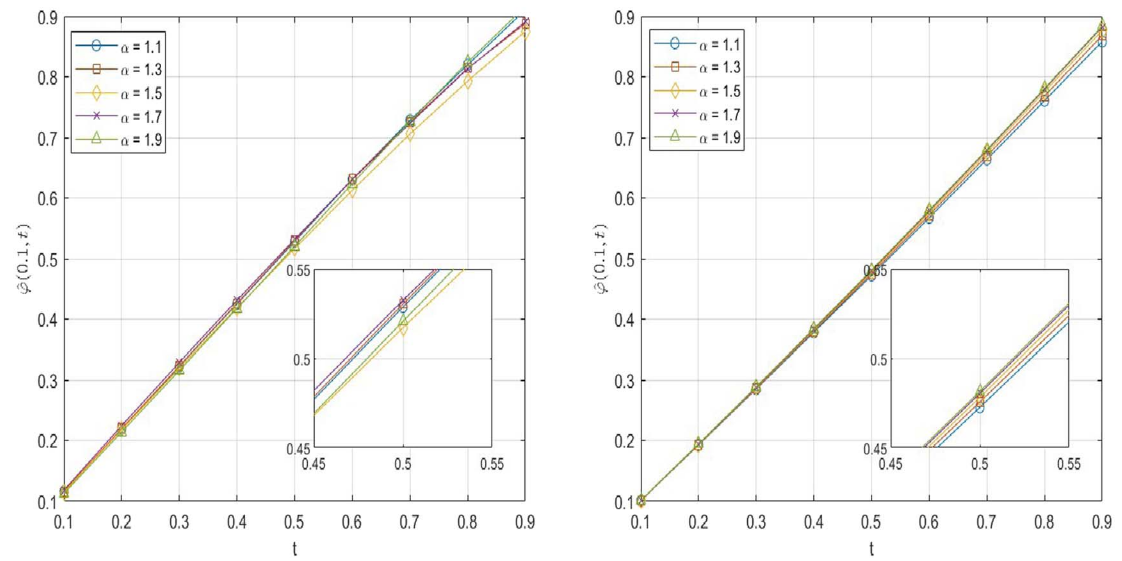


Figure 4. The CHNN solution is illustrated in (a), (b) and (c) with $\alpha = 1.3, 1.5, 1.99$ respectively and analytic solution is illustrated in (d) for Example 4.2.

Where $f(x,\, t) = (\dfrac{\sqrt{\pi}}{\Gamma(1/2)}\, erf\,(\sqrt{t}) - 4x^2)e^{x^2+t}$, and the error function, erf, is described as $erf\,(x) = \dfrac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, and $\varphi(x,\, t) = e^{x^2+t}$.To evaluate the performance of the analytic solutions and the CHNN, a comparative analysis was conducted. The results are presented in table 3. The analytic and CHNN solutions are visually compared in figure 6. These comparisons validate the CHNN model's ability to capture the complexities of the given scenario.

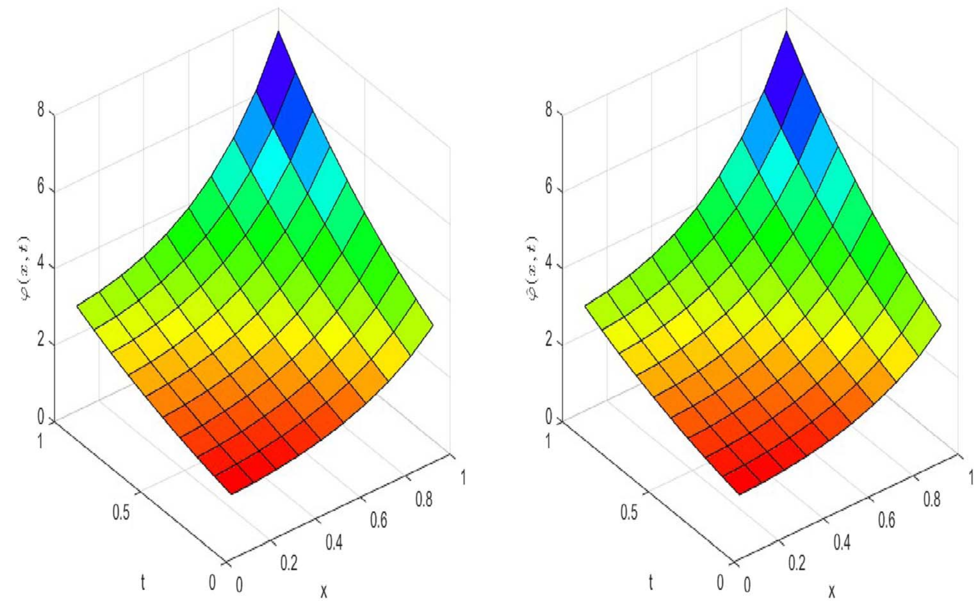
5. Conclusion

This paper presents a novel technique for solving TFTEs by utilizing the CHNN, where the traditional hidden layer is substituted with Chebyshev polynomials. The CHNN is trained using the Adam optimizer to determine the optimal parameters. This technique incorporates the Taylor series in activation, granting the CHNN full



(a) (b)

Figure 5. Comparison of CHNN solution in (a) and the method in [55] in (b) when $x = 0.1, \alpha = 1.1, 1.3, 1.5, 1.7, 1.9$ in Example 4.2.



(a) (b)

Figure 6. The analytic solution is illustrated in (a), and the CHNN solution is illustrated in (b) for Example 4.3.

Table 3. Comparison of analytic and CHNN solutions for Example 4.3.

(x, t)	Analytic solution	CHNN
(0.1, 0.1)	1.1163	1.1135
(0.2, 0.2)	1.2712	1.2591
(0.3, 0.3)	1.4769	1.4494
(0.4, 0.4)	1.7506	1.7060
(0.5, 0.5)	2.1170	2.0582
(0.6, 0.6)	2.6116	2.5432
(0.7, 0.7)	3.2870	3.2143
(0.8, 0.8)	4.2206	4.1596
(0.9, 0.9)	5.5289	5.4968
(1, 1)	7.3891	7.3891

authority to select the activation function, thus facilitating effective management of TFTEs. This enhancement improves the versatility and efficacy of the procedure. The results demonstrate the efficacy and dependability of the technique in resolving TFTEs. The proposed approach exhibits outstanding performance in solving intricate problems, as indicated by the precision of the outcomes and the effectiveness of the approach. Future research could focus on improving the proposed method by extending it to a multi-layer architecture and applying this method to solve FPDEs with different boundary conditions or systems of FPDEs.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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