IJMMS 2003:41, 2635–2643
PII. S0161171203211522
http://ijmms.hindawi.com

ON THE FRESNEL INTEGRALS AND THE CONVOLUTION

ADEM KILIÇMAN and BRIAN FISHER

Received 24 November 2002

The Fresnel cosine integral $C(x)$, the Fresnel sine integral $S(x)$, and the associated functions $C_+(x)$, $C_-(x)$, $S_+(x)$, and $S_-(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of the Fresnel cosine integral and its associated functions with x_+^r and x^r are evaluated.

2000 Mathematics Subject Classification: 33B10, 46F10.

The *Fresnel cosine integral* $C(x)$ is defined by

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 \, du, \tag{1}$$

(see [3]) and the associated functions $C_+(x)$ and $C_-(x)$ are defined by

$$C_+(x) = H(x)C(x), \qquad C_-(x) = H(-x)C(x). \tag{2}$$

The *Fresnel sine integral* $S(x)$ is defined by

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 \, du, \tag{3}$$

(see [3]) and the associated functions $S_+(x)$ and $S_-(x)$ are defined by

$$S_+(x) = H(x)S(x), \qquad S_-(x) = H(-x)S(x), \tag{4}$$

where H denotes Heaviside's function.

We define the function $I_r(x)$ by

$$I_r(x) = \int_0^x u^r \cos u^2 \, du \tag{5}$$

for $r = 0, 1, 2, \ldots$. In particular,

$$I_0(x) = \sqrt{\frac{\pi}{2}} C(x), \qquad I_1(x) = \frac{1}{2} \sin x^2, \qquad I_2(x) = \frac{1}{2} x \sin x^2 - \frac{\sqrt{\pi}}{2\sqrt{2}} S(x). \tag{6}$$

We define the functions $\cos_+ x$, $\cos_- x$, $\sin_+ x$, and $\sin_- x$ by

$$\begin{aligned} \cos_+ x &= H(x) \cos x, & \cos_- x &= H(-x) \cos x, \\ \sin_+ x &= H(x) \sin x, & \sin_- x &= H(-x) \sin x. \end{aligned} \tag{7}$$

If the classical convolution $f*g$ of two functions f and g exists, then $g*f$ exists and

$$f*g=g*f. \tag{8}$$

Further, if $(f*g)'$ and $f*g'$ (or $f'*g$) exist, then

$$(f*g)'=f*g' \quad (\text{or } f'*g). \tag{9}$$

The classical definition of the convolution can be extended to define the convolution $f*g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see [2].

Definition 1. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f*g$ is defined by the equation

$$\langle (f*g)(x),\varphi(x)\rangle = \langle f(y),\langle g(x),\varphi(x+y)\rangle\rangle \tag{10}$$

for arbitrary φ in $\mathcal{D}'$, provided that f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f*g$ exists by this definition, then (6) and (8) are satisfied.

Theorem 2. *The convolution* $(\cos_+ x^2)*x_+^r$ *exists and*

$$(\cos_+ x^2)*x_+^r=\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}I_{r-i}(x)x_+^i \tag{11}$$

for $r=0,1,2,\ldots$. *In particular,*

$$\begin{aligned}(\cos_+ x^2)*H(x)&=\sqrt{\frac{\pi}{2}}C_+(x),\\ (\cos_+ x^2)*x_+&=-\frac{1}{2}\sin_+ x^2+\sqrt{\frac{\pi}{2}}C(x)x_+.\end{aligned} \tag{12}$$

Proof. It is obvious that $(\cos_+ x^2)*x_+^r=0$ if $x<0$. When $x>0$, we have

$$\begin{aligned}(\cos_+ x^2)*x_+^r&=\int_0^x \cos t^2(x-t)^r\,dt\\ &=\sum_{i=0}^{r}\binom{r}{i}\int_0^x x^i(-t)^{r-i}\cos t^2\,dt\\ &=\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}I_{r-i}(x)x^i,\end{aligned} \tag{13}$$

proving (11). Equations (12) follow on using (6). □

Corollary 3. *The convolution* $(\cos_- x^2) * x_-^r$ *exists and*

$$(\cos_- x^2) * x_-^r = -\sum_{i=0}^{r} \binom{r}{i} I_{r-i}(x) x_-^i \tag{14}$$

for $r = 0, 1, 2, \ldots$. *In particular,*

$$\begin{aligned} (\cos_- x^2) * H(-x) &= -\sqrt{\frac{\pi}{2}} C_-(x), \\ (\cos_- x^2) * x_- &= -\frac{1}{2}\sin_- x^2 - \sqrt{\frac{\pi}{2}} S(x) x_-. \end{aligned} \tag{15}$$

Proof. Equations (14) and (15) follow on replacing x by $-x$ in (11) and (12), respectively, and noting that

$$I_r(-x) = (-1)^{r+1} I_r(x). \tag{16}$$

□

Theorem 4. *The convolution* $C_+(x) * x_+^r$ *exists and*

$$C_+(x) * x_+^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1}(x) x_+^i \tag{17}$$

for $r = 0, 1, 2, \ldots$. *In particular,*

$$\begin{aligned} C_+(x) * H(x) &= -\frac{1}{\sqrt{2\pi}} \sin_+ x^2 + C(x) x_+, \\ C_+(x) * x_+ &= \frac{1}{2\sqrt{2\pi}} \sin x^2 x_+ - \frac{1}{\sqrt{2\pi}} \sin_+ x^2 - \frac{1}{4} S_+(x) + \frac{1}{2} C(x) x_+^2. \end{aligned} \tag{18}$$

Proof. It is obvious that $C_+(x) * x_+^r = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned} \sqrt{\frac{\pi}{2}} C_+(x) * x_+^r &= \int_0^x (x-t)^r \int_0^t \cos u^2 \, du \, dt \\ &= \int_0^x \cos u^2 \int_u^x (x-t)^r \, dt \, du \\ &= \frac{1}{r+1} \int_0^x \cos u^2 (x-u)^{r+1} \, du \\ &= \frac{1}{r+1} \int_0^x \cos u^2 \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-u)^{r-i+1} \, du \\ &= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1}(x) x_+^i. \end{aligned} \tag{19}$$

Equation (17) follows. Equations (18) follow on using (6). □

Corollary 5. *The convolution $C_-(x) * x_-^r$ exists and*

$$C_-(x) * x_-^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} I_{r-i+1}(x) x_-^i \tag{20}$$

for $r = 0,1,2,\ldots$. In particular,

$$\begin{aligned} C_-(x) * H(-x) &= \frac{1}{\sqrt{2\pi}} \sin_- x^2 + C(x)x_-, \\ C_-(x) * x_- &= -\frac{1}{2\sqrt{2\pi}} \sin x^2 x_- + \frac{1}{\sqrt{2\pi}} \sin_- x^2 - \frac{1}{4} S_-(x) + \frac{1}{2} C(x) x_-^2. \end{aligned} \tag{21}$$

Proof. Equations (20) and (21) follow on replacing x by $-x$ in (17) and (18), respectively, and using (16). □

Definition 1 was extended in [1] with the next definition but first of all we let τ be a function in $\mathcal{D}$ having the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$, for $|x| \le 1/2$,
(iv) $\tau(x) = 0$, for $|x| \ge 1$.

The function τ_ν is now defined for $\nu > 0$ by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \le \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu. \end{cases} \tag{22}$$

Definition 6. Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. The *neutrix convolution product* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided that the limit h exists in the sense that

$$\underset{\nu \to \infty}{N\text{-}\lim} \langle f_\nu * g, \varphi \rangle = \langle h, \varphi \rangle, \tag{23}$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [5], with its domain N' the positive real numbers, with negligible functions finite linear sums of the functions

$$\nu^\lambda \ln^{r-1} \nu, \quad \ln^r \nu, \quad \nu^r \sin \nu^2, \quad \nu^r \cos \nu^2 \quad (\lambda \ne 0,\ r = 1,2,\ldots) \tag{24}$$

and all functions which converge to zero in the normal sense as ν tends to infinity.

Note that in this definition the convolution product $f_\nu * g$ is defined in Gel'fand and Shilov's sense, since the distribution f_ν has bounded support.

It was proved in [1] that if $f * g$ exists in the classical sense or by Definition 1, then $f \circledast g$ exists and

$$f \circledast g = f * g. \tag{25}$$

The following theorem was also proved in [1].

Theorem 7. *Let f and g be distributions in $\mathscr{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'. \tag{26}$$

We need the following lemma.

Lemma 8. *If $I_r = N\text{-}\lim_{\nu\to\infty} I_r(\nu)$, then*

$$\begin{aligned} I_{4r} &= \frac{(-1)^r(4r)!\sqrt{\pi}}{2^{4r+1}(2r)!\sqrt{2}}, \\ I_{4r+1} &= 0, \\ I_{4r+2} &= \frac{(-1)^r(4r+1)!\sqrt{\pi}}{2^{4r+2}(2r)!\sqrt{2}}, \\ I_{4r+3} &= \frac{(-1)^{r+1}(2r)!}{2} \end{aligned} \tag{27}$$

for $r = 0, 1, 2, \ldots$.

Proof. It is easily proved that

$$I_3(x) = \frac{1}{2}x^2 \sin x^2 - \frac{1}{2} + \frac{1}{2}\cos x^2 \tag{28}$$

and it follows from (6) and (28) that (27) hold when $r = 0$, since

$$S(\infty) = C(\infty) = \frac{1}{2}, \tag{29}$$

see Olver [4].

We also have

$$\begin{aligned} I_{2r}(x) &= \frac{1}{2}x^{2r-1}\sin x^2 + \frac{2r-1}{4}x^{2r-3}\cos x^2 - \frac{(2r-1)(2r-3)}{4}I_{2r-4}(x), \\ I_{2r+1}(x) &= \frac{1}{2}x^{2r}\sin x^2 + \frac{r}{2}x^{2r-2}\cos x^2 - r(r-1)I_{2r-3}(x) \end{aligned} \tag{30}$$

and it follows that

$$\begin{aligned} N\text{-}\lim_{\nu\to\infty} I_{2r}(\nu) &= -\frac{(2r)!(r-2)!}{2^4(2r-4)!r!}\, N\text{-}\lim_{\nu\to\infty} I_{2r-4}(\nu), \\ N\text{-}\lim_{\nu\to\infty} I_{2r+1}(\nu) &= -\frac{r!}{(r-2)!}\, N\text{-}\lim_{\nu\to\infty} I_{2r-3}(\nu). \end{aligned} \tag{31}$$

Equations (27) now follow by induction. □

Theorem 9. *The neutrix convolution* $(\cos_+ x^2) * x^r$ *exists and*

$$(\cos_+ x^2) \circledast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} I_{r-i} x^i \tag{32}$$

for $r = 0, 1, 2, \ldots$. *In particular,*

$$\begin{aligned}(\cos_+ x^2) \circledast 1 &= \frac{\sqrt{\pi}}{2\sqrt{2}},\\ (\cos_+ x^2) \circledast x &= \frac{\sqrt{\pi}}{2\sqrt{2}} x.\end{aligned} \tag{33}$$

Proof. We put $(\cos_+ x^2)_\nu = (\cos_+ x^2)\tau_\nu(x)$. Then the convolution $(\cos_+ x^2)_\nu * x^r$ exists and

$$(\cos_+ x^2)_\nu * x^r = \int_0^\nu \cos t^2 (x-t)^r \, dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) \cos t^2 (x-t)^r \, dt. \tag{34}$$

□

Now,

$$\begin{aligned}\int_0^\nu \cos t^2 (x-t)^r \, dt &= \sum_{i=0}^{r} \binom{r}{i} \int_0^\nu x^i (-t)^{r-i} \cos t^2 \, dt\\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} I_{r-i}(\nu) x^i\end{aligned} \tag{35}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_0^\nu \cos t^2 (x-t)^r \, dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} I_{r-i} x^i. \tag{36}$$

Further, it is easily seen that, for each fixed x,

$$\lim_{\nu\to\infty} \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) \cos t^2 (x-t)^r \, dt = 0 \tag{37}$$

and (32) follows from (34), (36), and (37). Equations (33) follow immediately.

Corollary 10. *The neutrix convolution* $\cos_- x^2 \circledast x^r$ *exists and*

$$(\cos_- x^2) \circledast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i+1} I_{r-i} x^i \tag{38}$$

for $r = 0, 1, 2, \ldots$. *In particular,*

$$\begin{aligned}(\cos_- x^2) \circledast 1 &= -\frac{\sqrt{\pi}}{2\sqrt{2}},\\ (\cos_- x^2) \circledast x &= -\frac{\sqrt{\pi}}{2\sqrt{2}} x.\end{aligned} \tag{39}$$

Proof. Equation (38) follows on replacing x by $-x$ in (32) and noting that I_r must be replaced by

$$N\text{-}\lim_{\nu\to\infty} I_r(-\nu) = (-1)^{r-1} N\text{-}\lim_{\nu\to\infty} I_r(\nu) = (-1)^{r-1} I_r. \tag{40}$$

Equations (33) follow. □

Corollary 11. *The convolution* $(\cos x^2) \circledast x^r$ *exists and*

$$(\cos x^2) \circledast x^r = 0 \tag{41}$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (41) follows from (32) and (38) on noting that $\cos x^2 = \cos_+ x^2 + \cos_- x^2$. □

Theorem 12. *The neutrix convolution* $C_+(x) \circledast x^r$ *exists and*

$$C_+(x) \circledast x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1} x^i \tag{42}$$

for $r = 0, 1, 2, \ldots$. *In particular*

$$C_+(x) \circledast 1 = 0, \tag{43}$$

$$C_+(x) \circledast x = \frac{1}{8}. \tag{44}$$

Proof. We put $[C_+(x)]_\nu = C_+(x)\tau_\nu(x)$. Then the convolution product $[C_+(x)]_\nu * x^r$ exists and

$$[C_+(x)]_\nu * x^r = \int_0^\nu C(t)(x-t)^r \, dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) C(t)(x-t)^r \, dt. \tag{45}$$

We have

$$\begin{aligned}
\sqrt{\frac{\pi}{2}} & \int_0^\nu C(t)(x-t)^r \, dt \\
&= \int_0^\nu (x-t)^r \int_0^t \cos u^2 \, du \, dt \\
&= \int_0^\nu \cos u^2 \int_u^\nu (x-t)^r \, dt \, du \\
&= -\frac{1}{r+1} \int_0^\nu \cos u^2 \left[(x-\nu)^{r+1} - (x-u)^{r+1}\right] du \\
&= -\frac{1}{r+1} \int_0^\nu \sum_{i=0}^{r} \binom{r+1}{i} x^i \left[(-\nu)^{r-i+1} - (-u)^{r-i+1}\right] \cos u^2 \, du
\end{aligned} \tag{46}$$

and it follows that

$$N\text{-}\lim_{\nu\to\infty}\int_0^{\nu} C(t)(x-t)^r\,dt = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i+1}I_{r-i+1}x^i. \tag{47}$$

Further, it is easily seen that, for each fixed x,

$$\lim_{\nu\to\infty}\int_{\nu}^{\nu+\nu^{-\nu}} \tau_\nu(t)C(t)(x-t)^r\,dt = 0 \tag{48}$$

and (42) now follows immediately from (45), (47), and (48). □

Corollary 13. *The neutrix convolution $C_-(x) \circledast x^r$ exists and*

$$C_-(x)\circledast x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}I_{r-i+1}x^i \tag{49}$$

for $r = 0,1,2,\ldots$. In particular,

$$C_-(x)\circledast 1 = 0, \tag{50}$$

$$C_-(x)\circledast x = -\frac{1}{8}. \tag{51}$$

Proof. Equation (49) follows on replacing x by $-x$ and I_r by $(-1)^{r-1}I_r$ in (42). Equations (50) and (51) follow. □

Corollary 14. *The neutrix convolution $C(x)\circledast x^r$ exists and*

$$C(x)\circledast x^r = 0 \tag{52}$$

for $r = 0,1,2,\ldots$.

Proof. Equation (52) follows from (43) and (50) on noting that $C(x) = C_+(x)+C_-(x)$. □

Acknowledgment. This research was carried out during the second author's visit to the University of Putra Malaysia (UPM) and partially supported under the Grant IRPA 09-02-04-0259-EA001. The authors greatly acknowledge the partial support.

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Adem Kılıçman: Department of Mathematics, University of Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

Current address: Institute of Advanced Technology, University of Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

E-mail address: akilic@fsas.upm.edu.my

Brian Fisher: Department of Mathematics, University of Leicester, Leicester, LE1 7RH, England

E-mail address: fbr@le.ac.uk