PURSUIT DIFFERENTIAL GAMES OF MANY PLAYERS ON SURFACE OF CYLINDER AND CONE

By

PIRIATHARISINI A/P KARAPANAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

November 2022

IPM 2022 17

DEDICATIONS

To the love of my life:

Father & Mother

அம்மா அப்பா!

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chairman: Assoc. Prof. Gafurjan Ibragimov, PhD
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In this thesis, pursuit differential game problems taking place on the surface of cylinder and cone are studied. The mobility of the players are expressed by simple differential equations. In the first problem, we consider a pursuit differential game involving two pursuers and one evader on the surface of a cylinder. To solve this problem, we study an equivalent differential game of two groups of countably many pursuers and one group of countably many evaders in $\mathbb{R}^2$ for a fixed duration. We find an estimate for the value of the differential game on the cylinder. We construct strategies for pursuers that are admissible for which they guarantee the estimation for the value of the game. In a different approach, this differential game problem is solved in the half-space. Secondly, we consider a pursuit differential game of many pursuers and evaders on the surface of a cylinder. Similar to the first problem, we consider an equivalent differential game performed by many groups of countably many pursuers and many groups of countably many evaders in $\mathbb{R}^2$. For this problem, we find a sufficient condition that pursuit can be completed if the total resource of the pursuers remains greater than that of the evaders and construct admissible strategies for pursuers which satisfy the completion of pursuit. Furthermore, we investigate a pursuit differential game where many pursuers pursuing one evader on the surface of a cone. We study an equivalent differential game in a sector by unfolding the cone. We prove that pursuit can be completed by the pursuers for a finite time by obtaining a sufficient condition. Admissible pursuit strategies are constructed for the pursuers. One can say that, in brief, the three problems are approached by reducing the differential games in $\mathbb{R}^3$ to equivalent ones in $\mathbb{R}^2$.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PERMAINAN PEMBEZAAN MENGEJAR OLEH RAMAI PEMAIN PADA PERMUKAAN SILINDER DAN KON

Oleh

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Dalam tesis ini, masalah permainan pembezaan mengejar yang berlaku pada permukaan silinder dan kon dikaji. Dalam masalah pertama, kami mempertimbangkan permainan pembezaan mengejar yang melibatkan dua orang pengejar dan seorang pengelak pada permukaan silinder. Untuk menyelesaikan masalah ini, kami mengkaji permainan pembezaan setara yang melibatkan dua kumpulan dengan ramai pengejar dan satu kumpulan dengan ramai pengelak dalam $\mathbb{R}^2$ untuk tempoh masa yang tetap. Kami mendapatkan anggaran untuk nilai permainan pembezaan pada silinder. Kami membina strategi yang boleh diterima untuk pengejar yang menjamin anggaran nilai permainan itu. Dalam pendekatan yang berbeza, masalah permainan pembezaan ini diselesaikan dalam ruang separuh. Kedua, kami menganggap permainan pembezaan mengejar yang melibatkan ramai pengejar dan pengelak pada permukaan silinder. Seperti masalah pertama, kami menganggap permainan pembezaan setara yang dimain oleh banyak kumpulan dengan ramai pengejar dan banyak kumpulan dengan ramai pengelak dalam $\mathbb{R}^2$. Untuk masalah ini, kami mendapati syarat yang mencukupi bahawa pengejaran boleh diselesaikan jika jumlah sumber bagi pengejar adalah lebih besar daripada jumlah sumber bagi pengelak dan membina strategi yang boleh diterima untuk pengejar yang memenuhi penyempurnaan pengejaran. Kami juga menyiasat permainan pembezaan mengejar dengan keadaan ramai pengejar mengejar seorang pengelak pada permukaan kon. Kami mengkaji permainan pembezaan setara dalam sektor dengan membuka lipatan kon itu. Kami membuktikan bahawa pengejaran boleh diselesaikan oleh pengejar untuk masa yang terhad dengan mendapatkan syarat yang mencukupi. Strategi mengejar yang boleh diterima dibina untuk pengejar. Secara ringkas, kita boleh mengatakan bahawa ketiga-tiga masalah ini diselesaikan dengan mengurangkan permainan pembezaan dalam $\mathbb{R}^3$ kepada yang setara dalam $\mathbb{R}^2$.

ACKNOWLEDGEMENTS

All praises to God, the Almighty, for His showers of blessing throughout my research journey.

High gratitude to my supervisor, Dr. Gafurjan, for the continuous guidance and knowledge sharing throughout this journey. Profound appreciation to my co-supervisors, Dr. Idham and Dr. Risman, for giving me endless motivation in completing this thesis.

Thanks a bunch to my parents, Mr. and Mrs. Karapanan Chelvam and my husband, Mr. Nitia Ruban for the constant love and understanding of my life long passion.

Nothing can beat the care and support from my siblings and in laws as they have given me endless support to make this journey better everyday.

Kind regards to my friends who were in the same boat as well as my relatives for their encouragement in this journey.

Deepest appreciation goes to Kementerian Pendidikan Malaysia for offering me MyBrainSc scholarship. Not to forget, INSPEM's staffs who were dedicated on assisting me in administrative processes pertaining to my studies. I really appreciate them.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfillment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

P	Pursuer
E	Evader
$\dot{x}$	First derivative of x
$\dot{y}$	First derivative of y
u_i	Control of the ith pursuer
v_j	Control of the jth evader
z_i	The ith fictitious evader
U_i	Strategy of the ith pursuer
V_j	Strategy of the jth evader
$x(\cdot)$	Function of $x(t),\ 0 \le t \le T$
θ	Duration of the game
γ	Value of the game
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n-dimensional vector space
l_2	Hilbert space
conv	convex hull of a given set
$H(x_0, r)$	Ball of radius r centered at x_0

CHAPTER 1

INTRODUCTION

This chapter introduces the background of differential game problems together with the basic definitions, motivation of the study, objectives of the study, scope and limitation and organisation of the thesis.

1.1 Introduction

What is a game in terms of Mathematics? There are various kind of games from widespread range of explanations. In game theory, dynamic games are comparable with static games. In static games, all players form their strategies simultaneously at a time. In dynamic games, one's strategy is dependent on another's with respect to time. Every game has at least two players, strategies for the players, and a set of outcomes that explain the result of winning or losing for each of the players simultaneously. Basically in game theory, differential game theory is categorised as continuous-time case. It is also extended from the sequential game theory. Differential games are a peculiar type of mathematical problems where they are related to mathematical modeling and analysis of conflicts in the framework of dynamical systems which are driven by differential equations. Optimal control problems are closely relatable with the differential games. In an optimal control problem, a single control function $z(t)$ and a single criterion is to be optimized whereas differential game theory extrapolates this to two control functions, $z_1(t), z_2(t)$ and two criteris, one for each player. A differantial game is commonly played by two players, the pursuer and the evader, with conflicting goals. The pursuer aims to pursue or catch the evader and the evader's goal is to evade or avoid the capture. In other words, the pursuer strives to minimize the distance between himself and the evader while the evader tries to maximize the distance between himself and the pursuer. The players achieve these goals by controlling the state of the system.

Pursuit and evasion differential games are two general types of differential games. In a pursuit differential game problem, conditions for which the pursuer can catch the evader are found. Thus in this game, the strategy is constructed for the pursuer for any behaviour of the evader. Pursuit is considered completed when the positions of the pursuer and the evader are at the same point at some time, $T \geq 0$. In other words, if the energy or total resource of the pursuer remains greater than that of the evader, then pursuit can be completed. On the contrary, evasion game problem is to find the conditions for which the evader can avoid the capture of the pursuer. In this game, the strategy is constructed for the evader with no stricture for the movement of the pursuer. Here it is said that evasion is possible when the positions of the players do not coincide for all time. In other words, if the energy or total resource of the pursuer is less than that of the evader, evasion is considered possible.

In differential game problems, the motions of the players are described by differential equations where the state of the players, $x(t)$ and $y(t)$, thrives continuously with time. The control parameters of the players are usually subjected to constraints. State or phase constraint is a constraint referring to the position of a player at all time. In other words, the player can only move within the specified area, like circle, square, rectangle and so on in a space such as $\mathbb{R}^2, \mathbb{R}^3$, etc. When the control of the player belongs to a subset of $\mathbb{R}^n$, then we say that geometric constraint is imposed on the control. For example, the geometric constraint for the control parameter of the player, $u(t)$ in a subset X of $\mathbb{R}^n$ is written as $u(t) \in X \subset \mathbb{R}^n$. It is also known as constraint of speed. For instance, if $u(t) \leq 1$, then the speed of the player is bounded by 1. Integral constraints are referred to the resources, specifically energy, finance, food and fuel, which are bounded by some positive numbers, and written as $\int_0^\infty |u(t)|^2 dt \leq \rho^2$, where $u(t)$ is the speed of the player and ρ is a positive number. The following are elementary examples from the list of examples of differential games. The conflict between Achilles and a tortoise proves that in general, a runner can catch another (Black (1951)). The lion and man game is known as the common example of differential games. This game was initially presented by Rado in 1925. The first outcome to Besicovitch in 1952 proved that evasion is possible in this game (Littlewood (1953)). To solve the pursuit problem, Rado has introduced radial strategy of pursuit. By assuming that the man moves on the boundary of the circle, it is evinced that the positions of the man and the lion coincide after some time. This problem has been explored deeply by Croft (1964), Flynn (1973), Lewin (1986) and Bollobás et al. (2012).

The contributions of differential games to our real life applications are countless. The conventional applications were in militiary conflicts, economic branches, robotics, engineering, computer science and substantially more. There were numerous early applications of zero-sum games. One of them is to the political problem of parliamentary representation. Recently, in economics, differential game theory is applied largely in fields such as quantity and price competitions, oligopolies, environmental economics and so on. The trajectory planning in robotics involves the theory of differential game theory as well. Besides that, computer science is one of the study which applies differential game theory for game programming, network security and many more.

1.2 Preliminaries

We base the formal definitions, examples and some standard results which will be used in the following chapters.

1.2.1 Differential Game Theory

The following are some important definitions of differential game theory (Theodore and Bernhard (2001)).

Definition 1.1 *(Player) A player is an actor who takes decisions in a game.*

Definition 1.2 *(Rationality) A player is said to be rational if he seeks to play in a way which maximizes his payoff. It is often assumed that the rationality of all players is general knowledge.*

Definition 1.3 *(Perfect information) A game has perfect information when only one player makes a move at any point in time and knows all the actions that have been made until then.*

Definition 1.4 *(Strategy) A strategy is one of the allowed possible actions of a player. In a considerable game, a strategy is a complete plan of options, one for each decision point of the player.*

Definition 1.5 *(Zero-sum game) A game is defined to be zero-sum when the sum of the payoffs to all players is zero for any outcome. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed.*

Definition 1.6 *(Payoff) A payoff is a number, also called utility, that shows the desire of a player for an any outcome. The payoffs are usually weighted with their probabilities when the outcome is random. The expected payoff incorporates the player's attitude towards risk.*

1.2.2 Measurable Functions

Definition 1.7 *The set*

$$\{x = (x_1, x_2, \dots, x_n) |\ a < x_i < b,\ i = 1, \dots, n\} \tag{1.1}$$

is called n-cube.

Definition 1.8 *A subset N of $\mathbb{R}^n$ is called a null set in case N is contained in a countable union of n-cubes whose total n-volume is less than any prescribed number $\varepsilon > 0$.*

Example 1.1 *Any finite or countable infinite set of points in $\mathbb{R}^n$ has measure zero.*

Definition 1.9 *Two functions $g(x)$ and $h(x)$ defined on $[a,b]$ that differ in value only on a null set are said to be equal almost everywhere on $[a,b]$.*

Definition 1.10 *The measurable sets of $\mathbb{R}^n$ are defined as the members of the smallest class of sets of $\mathbb{R}^n$ that contains of all open sets, all null sets of $\mathbb{R}^n$, and every difference, and countable union of its members.*

Definition 1.11 *A real-valued function $h(t)$ on a real interval I is called measurable in case for all real α and β, the set $\{t \mid t \in I,\ \alpha < h(t) < \beta\}$ is measurable on I.*

Definition 1.12 *A real-valued function $f(x)$ is called piecewise continuous, if its domain can be represented as union of finite number of intervals on each the function $f(x)$ is continuous.*

Definition 1.13 *The set ξ of subsets A of $[a,b]$, $a < b$, is called σ-algebra if*

(i) $[a,b] \in \xi$,

(ii) $A \in \xi$, *then* $[a,b] \backslash A \in \xi$,

(iii) for a sequence of subsets $A_i,\ i = 1, 2, \dots,$ *each* $A_i \in \xi$, *then* $\bigcup_{i=1}^{\infty} A_i \in \xi$.

Example 1.2 $\xi_0 = \{[a,b], \emptyset\}$. *This σ-algebra is a subset of any σ-algebra ξ.*

Example 1.3 $\xi^0 = \{A \mid A \subset [a,b]\}$. *Any σ-algebra ξ is a subset of the σ-algebra ξ^0.*

Example 1.4 *The set* $\xi = \{[\alpha,\beta] \mid [\alpha,\beta] \subset [a,b]\}$ *that consisted of only closed intervals* $[\alpha,\beta]$ *of* $[a,b]$ *is not a* σ*-algebra.*

Definition 1.14 *If each element of* σ*-algebra* ξ_1 *is an element of* σ*-algebra* ξ_2*, then we say that* ξ_2 *contains* ξ_1*.*

Definition 1.15 *Let* B *be the minimal* σ*-algebra that contains all the intervals* $[\alpha,\beta] \subset [a,b]$*. Elements of* B *are called Borel measurable sets.*

Definition 1.16 *Let* $f(x)$ *be a function defined on* $[a,b]$*. The function* $f(x)$ *is called measurable if, for any number* c*, the set* $A_c = \{x \mid f(x) < c\}$ *is a Borel measurable set.*

It can be shown that, for a measurable function $f(x)$, the sets $A_{(c,d)} = \{x \mid c < f(x) < d\}$, $A_{(c\,,d]} = \{x \mid c < f(x) \leq d\}$, $A_{[c\,,d)} = \{x \mid c \leq f(x) < d\}$, $A_{[c,d]} = \{x \mid c \leq f(x) \leq d\}$ are also measurable for any real numbers c and d. Further, it can be shown that piecewise continuous function defined on $[a,b]$ is a measurable function.

1.3 Motivation

The continuous development of optimal control theory led recent researchers to focus on the study of differential games involving many players in $\mathbb{R}^n$, Hilbert space, on manifolds and so on. Since there is a dearth of investigation on the differential games involving many players on the two-dimensional surfaces, this thesis contributes outcomes for the simple motion differential games on the surface of a cylinder and the surface of a cone. The idea of unfolding a cylinder in $\mathbb{R}^n$ proposed by Nikulin and Shafarevich (1983) motivated this study in achieving the objectives set for the first two problems. The basic research on finding necessary and sufficient conditions for value function and total resource for the players influenced this study to focus on the construction of the strategies which satisfy those conditions found.

1.4 Scope and Limitation

This study will focus on the construction of strategies for the players involved in pursuit differential games involving many players and limited to games that occur on the surface of cylinder and on the surface of a cone.

1.5 Objectives of the Thesis

The following are objectives of this thesis:

1. *To construct strategies for pursuers in differential game with geometric constraints on the surface of a cylinder.*
2. *To find an estimate for the value of the pursuit differential game for which it is bounded above by a number.*
3. *To construct strategies for pursuers in differential game with integral constraints on the surface of a cylinder and find a suffcient condition for the completion of the differential game.*
4. *To construct strategies for pursuers in differential game with integral constraints on the surface of a cone and find a suffcient condition for the completion of the differential game.*

To achieve the first objective, we construct admissible pursuit strategies for the pursuers involved in the differential game on the surface of cylinder where the control functions of the players are subject to geometric constraints. For the following objective, we find the estimate such that the value of the game is bounded above by a number. For Objective 3, we impose integral constraints on the control functions of the players involved in the differential game on the surface of cylinder and construct pursuit strategies for which they guarantee the completion of pursuit satisfying a sufficient condition. In order to attain Objective 4, we construct pursuit strategies for the pursuers involved in the differential game on the surface of cone for which they guarantee the completion of pursuit satisfying a sufficient condition.

1.6 Methodology

In this study, the solutions to the simple motion pursuit differential games on cylinder and cone begin by reducing the diferential games in $\mathbb{R}^3$ to specific ones in $\mathbb{R}^2$. For each problem, we investigate the game involving one pursuer and one evader and then extend to the case of more than one player. The analysis of the methods applied to each problem is as follows:

Step 1: Define the differential game on the cylinder or cone in $\mathbb{R}^3$. In each problem, the controls of the players are subjected to different constraints.
In problem 1, the controls of the players are subjected to geometric constraints. The duration of the game is fixed.
In problems 2 and 3, the controls of the players are subjected integral constraints. The game occurs for a finite time.

Step 2: Reduce the differential game on the surface of cylinder or cone to a specific game in $\mathbb{R}^2$ by unfolding it. Define the reduced game in $\mathbb{R}^2$.
In problem 1, the differential game of two pursuers and one evader on cylinder is equivalent to differential game of two groups of pursuers and a group of evaders in $\mathbb{R}^2$.
In problem 2, the differential game of m pursuers and k evaders on cylinder is equivalent to differential game of m groups of pursuers and k groups of evaders in $\mathbb{R}^2$.
In problem 3, the differential game of m pursuers and one evader on cone is equivalent to differential game of m pursuers and one evader on a sector in $\mathbb{R}^2$.

Step 3: Study an auxiliary differential game for each problem. The game involves one pursuer and one evader.
In problem 1, this game is first studied in the half-plane and followed by in $\mathbb{R}^2$.
In problems 2 and 3, this game is studied in $\mathbb{R}^2$.

Step 4: Study the differential game involving many players.
In problem 1, the game involves two pursuers and one evader.
In problem 2, the game involves m pursuers and k evaders.
In problem 3, the game involves m pursuers and one evader.
Solve the game by using the results obtained in Step 3.

Step 5: In problem 1, find the estimate for the value of the pursuit differential game and construct strategies for the pursuers.
In problems 2 and 3, find the sufficeint condition for completing the pursuit differential game and construct strategies for the pursuers.

1.7 Thesis Outline

The chapters in this thesis are organised as follows.

Chapter 1 narrates the context of differential games in detail and the basic definitions. Then, the motivation, objectives, scope and limitation, methods used to solve the problems and outline of the thesis are presented.

Chapter 2 briefs the history of the development of differential games and some mathematical background that will be used throughout this thesis. Subsequently, we review the related literature on some significant works.

Chapter 3 introduces differential game of simple motion involving single player. Here, the concept of attainability sets of the players is discussed. In the following section of this chapter, the necessary and sufficient condition for the value function is explicated. Then, we investigate the reduction of differential game on manifolds in $\mathbb{R}^{n+1}$ to a game in $\mathbb{R}^n$. Here, the properties of the multivalued mapping is studied. In the last section, we present the methods used to carry out our research.

Chapter 4 studies pursuit differential game on the surface of a cylinder involving two pursuers and one evader. In the first section, we state the problem to be studied and study an identical differential game in $\mathbb{R}^2$. In the next sections, differential games involving one pursuer versus one evader and two pursuers versus one evader in $\mathbb{R}^2$ are studied respectively. Subsequently, the solution for the game on the cylinder is found and the chapter is concluded in the last section.

Chapter 5 investigates pursuit differential game on the surface of cylinder involving many pursuers and evaders. We give the statement of the problem and study an identical differential game in $\mathbb{R}^2$. In the following section, an auxiliary differential game performed by single player is studied and followed by the game on the cylinder. The chapter is then concluded in the final section.

Chapter 6 analyses pursuit differential game on the surface of cone involving many pursuers and one evader. We present the statement of the problem and study an auxiliary differential game of one pursuer versus one evader in $\mathbb{R}^2$. In the next section, the game involving many pursuers and one evader is examined and followed by the conclusion of this chapter.

Chapter 7 summarizes all our results together with suggestions of potential future works that can be extended from this research.

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