CLASSIFICATION OF MULTI-QUBIT STATES WITH HIGHER ORDER SINGULAR VALUE DECOMPOSITION AND CONCURRENCY OF THREE LINES

By

CHOONG PAK SHEN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To my beloved ones.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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December 2022

Chairman : Assoc. Prof. Hishamuddin Zainuddin, PhD
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This research studied the classification problem of multipartite states. Since the Hilbert space of a multipartite system has a tensor product structure, we made use of a tensor decomposition, called higher order singular value decomposition (HOSVD) to solve the problem. We focused on finding the solutions to the set of all-orthogonality conditions from HOSVD to obtain a complete classification of three qubits. Based on the relationship between the set of first n-mode singular values, $\sigma_1^{(n)2}$, we identified three possible cases that contain all the entanglement classes of three qubits. An entanglement polytope was illustrated to demonstrate how the entanglement classes of three qubits change with respect to $\sigma_1^{(n)2}$, which is in accordance with the existing literature. The geometrical significance of our classification method was studied by finding the stabilizer dimensions of all the entanglement classes of three qubits. We found that different entanglement classes of three qubits have different stabilizer dimensions. Furthermore, by making use of the concurrency of three lines, we generalized our classification approach for multi-qubit states, which is computationally simple and yet capable of producing a finite number of family of states. As a demonstration, we classified four-qubit states with our proposal and found four possible cases that contain the genuinely entangled four-qubit states.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENCIRIAN KELAS KETERBELITAN BERBILANG QUBIT DENGAN HURAIAN NILAI SINGULAR TERTIB TINGGI DAN KESERENTAKAN TIGA GARISAN

Oleh

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Penyelidikan ini mengkaji pengkelasan keterbelitan untuk keadaan kuantum berbilang parti. Oleh sebab ruang Hilbert untuk sistem berbilang parti mempunyai struktur hasildarab tensor, kami menggunakan huraian tensor yang dipanggil huraian nilai singular tertib tinggi (HOSVD) untuk menyelesaikan masalah ini. Kami fokus kepada penentuan penyelesaian untuk syarat ortogon keseluruhan daripada HOSVD supaya kami memperolehi satu pengkelasan keterbelitan yang lengkap untuk keadaan tiga qubit. Berdasarkan hubungan antara satu set yang mengandungi nilai singular n-mod pertama, $\sigma_1^{(n)2}$, kami mengenalpasti tiga kes yang mengandungi semua kelas keterbelitan tiga qubit. Satu politop keterbelitan telah dilakarkan untuk menunjukkan bagaimana kelas keterbelitan keadaan tiga qubit berubah dengan $\sigma_1^{(n)2}$. Hasil kajian ini menyetujui dengan rujukan lain yang ada. Kepentingan cara pengkelasan kami dari segi geometri telah dikaji dengan dimensi penstabil untuk semua kelas keterbelitan tiga qubit. Kami mendapati bahawa kelas keterbelitan yang berbeza mempunyai dimensi penstabil yang berbeza. Tambahan pula, dengan menggunakan keserentakan tiga garisan, kami menbuat pengitlakan cara pengkelasan kami untuk keadaan kuantum berbilang qubit. Pengitlakan ini adalah mudah secara komputasi, namun masih mampu menghasilkan bilangan keluarga keadaan yang terhingga. Sebagai demonstrasi, kami mengkelaskan keadaan empat qubit dengan cadangan kaedah di atas dan berjaya mengenalpasti empat kes yang mengandungi keadaan empat qubit yang terbelit secara tulen.

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LIST OF SYMBOLS

$\mathscr{H}$	Hilbert space
$\mathbb{C}$	Complex vector space
$\lvert\psi\rangle$	Ket vector
$\langle\psi\rvert$	Bra vector
Ψ	Tensor of the multiparite state
ψ_{ijk}	Probability amplitudes/tensor elements (of three qubits)
$\bar{t}$	Complex conjugate
M^T	Matrix transpose
$M^\dagger$	Conjugate transpose
ρ	Density matrix
ρ_n	n-th one-body reduced density matrix
ρ_n^d	n-th diagonalized one-body reduced density matrix
$\otimes$	Tensor product
SU(n)	Special unitary group of degree n
V^{I_n}	n-th vector space of dimension I_n
$\mathscr{X}$	Arbitrary tensor
$X_{(n)}$	n-th matrix unfolding of an arbitrary tensor
$\chi_{i_1 i_2 \ldots i_N}$	N-th order tensor elements
$\mathscr{T}$	Core tensor
$t_{i_1 i_2 \ldots i_N}$	Elements of the core tensor $\mathscr{T}$
$\mathscr{T}_{i_n=\alpha}$	Subtensor of the core tensor $\mathscr{T}$ when the n-th index is fixed to α
$\sigma_i^{(n)}$	i-th n-mode singular value

LIST OF ABBREVIATIONS

HOSVD	Higher Order Singular Value Decomposition
LOCC	Local Operation and Classical Communication
LU	Local Unitary
SLOCC	Stochastic Local Operation and Classical Communication
SVD	Singular Value Decomposition

CHAPTER 1

INTRODUCTION

1.1 Motivation and basic concepts

Tying to the fundamental understanding of our physical world, quantum entanglement has been in the limelight among the quantum research community. Historically, a thought experiment proposed by Einstein et. al. (1935) questioned the completenesss of quantum theory as a physical theory. With a pair of entangled particles, they showed that the measurement outcome of a particle can be predicted if the other particle is measured, thus concluding that information can be transmitted faster than the speed of light. In order to resolve this contradiction with relativity theory which stated that nothing can travel faster than the speed of light, Bell (1964) devised a statistical constraint that a local hidden variable theory must obey, and showed that quantum theory does not fit into such a local hidden variable theory, hence entanglement is seen as a type of non-local correlation. The statistical constraint was known as Bell inequality later on. Multiple more convenient forms of Bell-type inequalities and experimental setups had been derived later on, such as the CHSH inequality (Clauser et al., 1969) and GHZ-Mermin experiments (Greenberger et al., 1990; Mermin, 1990).

Meanwhile, a series of breakthroughs in cryptography and computer science sparked new interests in quantum theory. The successes in making use of quantum properties to detect eavesdroppers, for instance the BB84 (Bennett and Brassard, 1984) and E91 (Ekert, 1991) quantum key distribution protocols, plus the quantum algorithms such as Deutsch-Jozsa algorithm (Deutsch and Jozsa, 1992) and Shor's algorithm (Shor, 1994), had shown that quantum computers are capable to perform certain computational tasks more efficiently than classical computers. These achievements encouraged physicists to look at quantum theory from a more operational perspective. In this sense, entanglement is a resource in quantum information processing tasks under the local operation and classical communication (LOCC) protocols (Chitambar and Gour, 2019). One prominent example of this perspective is the expense of entanglement between a maximally entangled qubit pair after a round of LOCC procedure in quantum teleportation (Bennett et al., 1993).

Motivated by the two reasons above, entanglement is mainly studied from two different approaches. The density matrix formalism introduced by Von Neumann (2018) allowed us to extend terminologies from classical statistical mechanics, particularly ideas from the pioneering work by Shannon and Weaver (1964) on information theory. From here, we can quantify entanglement by using entanglement measures, such as the von Neumann entropy (Nielsen and Chuang, 2010) and negativity

(Życzkowski et al., 1998). Qualitatively, one would like to understand the available entanglement classes under operations that preserve entanglement. This is important because ideally, multipartite states from the same entanglement class can be converted to each other without any loss of entanglement between the subsystems, hence having the same resources to perform the same quantum informational tasks. Our work concerns with the latter approach.

In quantum theory, quantum states are represented by normalized state vectors belonging to the Hilbert space $\mathscr{H} = \mathbb{C}^d$ modulo with a multiplicative constant, known as the global phase factor. Since only the direction of state vectors are important in distinguishing between them, we can projectivize the global phase from the state vectors by defining an equivalence relation,

$$|\psi\rangle \sim |\phi\rangle \text{ if and only if } |\psi\rangle = c\,|\phi\rangle\,, \tag{1.1}$$

where $c = e^{i\theta}$, $\theta \in [0, 2\pi)$. Quantum states are now more precisely defined as rays in the complex projective Hilbert space. Geometrically, complex projective space is a kähler manifold with three mutually compatible geometrical structures, i.e. complex, symplectic and Riemannian structures (Hou and Hou, 1997; Bengtsson et al., 2002).

In addition, the Hilbert space of a multipartite quantum system is a tensor product of its constituents' Hilbert spaces (Nielsen and Chuang, 2010). As a counterpart to classical information theory, only two-level (generally d-level) quantum systems are important from the quantum information perspective. Therefore, the Hilbert space of such n-partite two-level quantum systems can be explicitly written as the tensor product of $\mathbb{C}^2$ (or $\mathbb{C}^d$),

$$\mathscr{H} = \bigotimes_{i=1}^{n} \mathbb{C}_i^2. \tag{1.2}$$

This tensor product structure of the Hilbert space for multipartite systems bestows tensorial properties to the multipartite state vectors (Lathauwer et al., 2000). As a side note, we call two-level quantum systems as qubits and d-level quantum systems as qudits.

Owing to the rich mathematical structures of the Hilbert space and complex projective space for multipartite quantum states, many mathematical tools were used to study the classification problem of quantum states, such as local symmetries and invariants (Carteret and Sudbery, 2000; Sudbery, 2001), tensor decomposition (Lathauwer et al., 2000; Liu et al., 2012; Li and Qiao, 2013; Li et al., 2014) and geometry of quantum states (Kuś and Życzkowski, 2001; Sinolecka et al., 2002). It is worth noting that extending the classification problem to multipartite states is not straightforward at all. For bipartite states, Schmidt decomposition can be used to simplify the classification problem since Schmidt rank is an entanglement monotone (Terhal and Horodecki, 2000). However, its counterpart in multipartite states do not generally exist (Peres, 1995). Multiple generalizations to the classification of multipartite states had been proposed, such as writing down the Schmidt canonical form (Acín et al., 2001) or characterizing multipartite entanglement by a vector of Schmidt numbers (Huber and de Vicente, 2013), where each solution emphasized different facets to the same problem.

In this thesis, we attempted to offer another solution to the classification of multipartite states by using a generalized version of singular value decomposition (SVD), called higher order singular value decomposition (HOSVD) (Lathauwer et al., 2000). Previously, HOSVD has seen its applications in numerical analysis (Xu, 2017; Kempf et al., 2022), image processing (Rajwade et al., 2012; Miao et al., 2023), signal processing (Kreimer and Sacchi, 2012; Li et al., 2020) and the construction of the projected entangled simplex states (PESS) (Xie et al., 2014). In our work, we identified the special states of three qubits by HOSVD and showed that our results correspond to the local unitary (LU) classification of three qubits (Carteret and Sudbery, 2000). Additionally, we proved that the special states can be identified by the eigenvalues of one-body reduced density matrices and constructed an entanglement polytope comparable to the work by Walter et al. (2013), where they used symplectic geometry and geometric invariant theory for their construction. Lastly, we proposed a method to generalize the identification of special states for multi-qubit systems by using a simple geometrical result called the concurrency of three lines. We demonstrated our proposal with four-qubit states.

1.2 Problem statements

Based on singular value decomposition (SVD), Schmidt decomposition offers a way to rewrite bipartite states into the Schmidt canonical form, whereby the Schmidt ranks are used as an entanglement measure. However, Schmidt decomposition for multipartite states does not generally exist. While it is still possible to write three-qubit states into the Schmidt canonical form (Acín et al., 2001), its generalization to multipartite states is unclear. The closest alternative is a generalized version of SVD, called higher order singular value decomposition (HOSVD). HOSVD is a tensor decomposition that relaxes the matrix diagonalization of SVD into orthogonality on the subtensors and has been well-studied for its possibility to classify multipartite states by local symmetries (Liu et al., 2012; Li and Qiao, 2013; Li et al., 2014).

Here, we would like to consider the classification of multipartite states by HOSVD from a different perspective. Since the decomposition provides a set of conditions that needs to be satisfied, we are interested to see if it is possible to classify multipartite states by finding all the states that automatically satisfy the all-orthogonality conditions. Specifically, we aim to answer the following two major questions:-

2. How to classify multipartite states with HOSVD?

1.3 Research objectives

The objectives of this research are

1. to improve the definition of higher order singular value decomposition (HOSVD) from the quantum information perspective
2. to classify three-qubit states with HOSVD by satisfying the all-orthogonality conditions
3. to construct the generalization of the classification procedure to multi-qubit states
4. to give a geometric interpretation of the classification of multi-qubit states with HOSVD and the concurrency of three lines

1.4 Scope of current work

In this work, we did not consider an entanglement measure to complement our classification results. Since the core tensors of higher order singular value decomposition (HOSVD) are invariant from local unitary (LU) actions, our classification is based on LU properties. Our proposed classification procedure to multipartite states is currently limited only to two-level subsystems.

1.5 Organization of thesis

This thesis is structured as follows. We will first review relevant works on the classification of quantum states in Chapter 2. There are two important types of operations, i.e. local unitary (LU) and stochastic local operation and classical communication (SLOCC). While LU classification is finer than the SLOCC counterpart, SLOCC group is the complexification of LU group from group-theoretic viewpoint. Therefore, we find that it is important to understand some geometrical results regarding to the classification of multipartite states. In Chapter 3, we will introduce some basic concepts of quantum theory and group theory. The ideas of tensors and multilinear algebra will be explained with quantum information terminologies. At the end of the chapter, we will prove that higher order singular value decomposition (HOSVD) simultaneously diagonalizes the one-body reduced density matrices for multipartite states. The results of our work will be separated into three chapters. Chapter 4 will discuss our LU classification of three qubits by HOSVD. In Chapter 5, we propose to use the concurrency of three lines to simplify some calculations demonstrated in Chapter 4. Ultimately, we are able to identify all the special states of three qubits. Such simplification is needed so that we can generalize the classification of quantum states by HOSVD to multi-qubit states. Using our proposal, we will identify special states of four qubits in Chapter 6. Finally, we discuss some of the limitations of our proposal and possible future improvements in Chapter 7.

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