QUANTUM CIRCUIT COMPLEXITY IN RIEMANNIAN GEOMETRY OF 3-QUBIT QUANTUM FOURIER TRANSFORM

By

CHEW KANG YING

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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DEDICATION

To my beloved

Dad, Mom, Sister, and new-born Niece

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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Chair : Nurisya Mohd Shah, PhD
Institute : Mathematical Research

Quantum computation is a physical concept of using the quantum algorithm in quantum computer to solve a problem. With quantum computation, it showed great promises in solving problems faster due to the utilization of qubits instead of conventional bits. In addition, the complexity of a quantum algorithm will determine the difficulty in performing computation for that algorithm to solve complex problem. This thesis will describe quantum gates as element of $\mathbf{SU}(2^{\mathrm{n}})$ Lie group in operator representation. Its respective $\mathfrak{su}(2^n)$ Lie algebra will be constructed, and is next represented in Pauli basis such that the properties of the quantum gates can be studied. The generation of $\mathbf{SU}(\mathbf{8})$ Lie group from its respective generator in $\mathfrak{su}(8)$ will also be demonstrated. The representation of $\mathfrak{su}(8)$ in Pauli basis also will be shown to be similar to change of basis. Frequency of structure constant for $\mathfrak{su}(8)$ will also be computed, analyzed and compared with $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$. Since $\mathbf{SU}(2^{\mathrm{n}})$ Lie group spanned a differential manifold, quantum circuits complexity problem can be cast into geometry problem and can be studied using the Riemannian geometry. Alongside with the concept of superoperator, this thesis will be able to choose a Riemannian metric for $\mathbf{SU}(2^{\mathrm{n}})$ manifold with penalty parameter s and q. Following up, general Levi-Civita connection and the associated geodesic equation for quantum algorithm will be constructed. Finally, the Riemannian metric and differential form of geodesic equation for 3-qubit Quantum Fourier Transform circuit will be generated. This work aspired to provide an organize structures to compute and solve geodesic equation for quantum algorithm of the quantum circuit complexity optimization.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KERUMITAN LITAR QUANTUM DALAM GEOMETRI RIEMANNIAN 3-QUBIT JELMAAN FOURIER KUANTUM

Oleh

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Pengiraan kuantum adalah konsep fizikal yang menggunakan algoritma kuantum dalam kuantum komputer untuk menyelesaikan masalah. Bersama pengiraan kuantum, ia memberi harapan dalam penyelesaian masalah yang lebih cepat disebabkan oleh penggunaan qubit dan bukannya bit yang lazim. Tambahan pula, kerumitan sesebuah algoritma kuantum akan menentukan kesulitan dalam melaksanakan pengiraan bagi algoritma tersebut untuk menyelesaikan sesuatu masalah yang rumit. Tesis ini akan menerangkan get quantum sebagai elemen kumpulan Lie $\mathbf{SU}(2^n)$ dalam perwakilan operator. Aljabar Lie $\mathfrak{su}(2^n)$ masing-masing juga akan dibina dan seterusnya diwakili dalam asas Pauli supaya sifat get kuantum dapat dikaji. Penjanaan $\mathbf{SU}(8)$ oleh penjana masing-masing dari $\mathfrak{su}(8)$ juga akan didemonstrasikan. Perwakilan $\mathfrak{su}(8)$ dalam asas Pauli juga akan dipaparkan seiras dengan asas pertukaran. Kekerapan pemalar struktur untuk $\mathfrak{su}(8)$ juga akan dikira, dianalisis dan dibandingkan dengan $\mathfrak{su}(2)$ dan $\mathfrak{su}(4)$. Memandangkan kumpulan Lie $\mathbf{SU}(2^n)$ membentangkan manifold bezaan, masalah kerumitan litar kuantum boleh dikaji dalam geometri Riemmanan sebagai masalah geometri. Bersama dengan konsep superoperator, tesis ini akan memilih metrik Riemannan untuk manifold $\mathbf{SU}(2^n)$ dengan parameter dendaan s dan q. Seterusnya, kaitan Levi-Civita am dan persamaan geodesi bagi algorithma kuantum yang akan dibina. Akhirnya, metrik Riemannan dan persamaan geodesi bentuk bezaan untuk litar jelmaan Fourier kuantum tiga-qubit akan dijana. Kerja ini berhasrat untuk menyediakan struktur yang teratur untuk mengira dan menyelesaikan persamaan geodesi bagi algoritma kuantum untuk pengoptimuman kerumitan litar kuantum.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Declaration by Members of the Supervisory Committee

This is to confirm that:

- the research and the writing of this thesis were done under our supervision;
- supervisory responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2015-2016) are adhered to.

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CHAPTER 1

INTRODUCTION

1.1 Research background

Classical circuit computation and quantum circuit computation

Quantum circuit computation is the process of performing calculation utilizing the collectives properties of quantum states involving superposition and entanglement. Similarly to classical computer which used to solve tedious problems, quantum computer potentially could provide problem-solving in exponentially lesser time due to quantum properties. Quantum computation utilizes qubits instead of classical bits. Bits is the portmanteau for binary digits, which each digit only consists of two possible number which are 1 and 0. There are differences in qubits and bits. Classical bits in classical computer is determined to be either 0 (off) or 1 (on) of the states of a transistor, while quantum bits (qubits) can be a superposition states from 0 states to 1 states which can be represented in one axis of a Bloch sphere. Subsequently, increase of n number of classical bits, the string of bits could stores up to 2^n data but at any instance the string of bits can only represent one of the 2^n possible state of data. On the other hand, a single qubit can be represented by linear combination of two orthonormal basis state vector which denoted by $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1. \tag{1.1}$$

Thus n number of qubits can be in superposition in 2^n different states simultaneously, interfere with each other in forms of wave functions. This implies the qubits string could contain exponentially more data than the classical bits string. In classical computer, classical bits is represented by individual transistors which act as switches controlling the voltage in the circuits, while in quantum computer, quantum bits is represented by any two-level quantum system. Some of the examples are individual electron states, spin up or spin down, and photon polarization states, vertical or horizontal polarization. In classical computer, computational functions are implemented using logic gates, mainly for adjoint function (AND gate), disjoint function (OR gate) and negation (NOT gate). The evolution of the systems must preserves the properties of stochastic matrices, which the probability distribution of each state add up to one. In quantum computer the computational functions are implemented with quantum gates, The evolution of quantum systems must preserves the properties of unitary matrices, which the sum of squares of probability distribution of each state add up to one. With that, it implies that the coefficient of the states (probability distribution) for classical bits must be positive-definite real number, while for qubits it can contain complex numbers. This additional parameter from the emergent of phase of states due to i in complex numbers brings advantages to the quantum computer. Currently, there are three classes of quantum algorithm that could provide major speed up for calculation over classical algorithm. These are quantum Fourier transform based algorithms, quantum search algorithms, and quantum simulation. (Nielsen and Chuang, 2002).

Classical algorithm and quantum algorithm

With the basic units of computation ready, the computation will require algorithm to bring the input problems and provide output solutions. Classical algorithm is a set of instructions performed step-by-step in order to solve a problem. The instructions are performed by combination of logic gates both in series and parallel. Similarly, quantum algorithm composed of set of instructions performed by combinations of quantum gates both in series and parallel. Any classical algorithm can be run in quantum computer as representation of quantum algorithm due to unitary matrices in quantum gates as being the generalized form for the stochastic matrices in classical logic gates. The length of a classical algorithm is determined by the number of logic gates applied in series in solving a problem. Similarly, the length of a quantum algorithm is determined by the number of quantum gates applied in series in solving a problem. Every step performed will incur implementation time thus, the time required to complete solving the problem is proportional to the length of the algorithm itself. This arise to be a field of study itself, called time complexity which studies the efficiency of an algorithm and estimation of time taken for an algorithm to complete its run in forms of scaling in n number of inputs. In classical computation, there exists deterministic algorithm as well as probabilistic algorithm. In quantum computation, it does exist deterministic algorithm which it performs much similar with the classical counter parts in terms of time taken. However, most of the quantum algorithm that provide major speed up are probabilistic algorithm. It utilizes the superposition between the states and alter the probability distribution between states by quantum gates. To obtain the result, the answer will be measured with certain probability determined by the probability amplitude of the state.

Computational complexity

Computational complexity represent the classification of computational problem solving by the resource usage. Quantum algorithm are difficult to have deterministic outputs thus the class of decision problems solvable in polynomial time by quantum algorithm with high probability of correctness is the bounded-error quantum polynomial time (BQP). In particular for our studies we are looking into circuit complexity which taking account of the number of quantum gates applied for an algorithm in a circuit. This is analogous to the class of bounded-error probabilistic polynomial time in classical algorithm run in classical computer. For a given algorithm, optimization of the method could reduce the time complexity thus making a more efficient algorithm.

Physical Implementation of Quantum Computation

In the year of 1980, Paul Benioff extended the concept of Turing machines into a theoretical concept of Quantum computation (Benioff, 1980). Then the Yoshihisa Yamamoto and his team in year 1988 came out with the proposal on first realizable quantum computation model using single atoms with photon fields (Igeta and Yamamoto, 1988). Physical realizable model of quantum computation came into spotlights when Shor's Algorithm is introduced which posed a challenge to our current data encryption method (Shor, 1994). At the same time quantum computation also bring along promises to match the rising demands of computation power in our development of technologies. There are five requirements to realize a physical quantum computation as stated in (DiVincenzo, 2000):

1. Well characterized qubits in a scalable physical system
2. To be able to initialize the qubits state to a reference state.
3. Having decoherence times relatively much longer than gate operation time.
4. A "universal" set of quantum gates
5. Capability to perform qubit-specific measurement.

Based on the requirements there are many groups of research strives to make attempts in realizing a physical, scalable and consistent quantum computer. Each approaches utilize different concepts and properties of natures to represent two-level system (qubit). To highlight a few approaches, we have cold trapped ions (Cirac and Zoller, 1995), nuclear magnetic resonance (Chuang et al., 1998), coupled quantum dots (Loss and DiVincenzo, 1998), optical lattices with neutral atoms (Brennen et al., 1999), quantum cluster state (Raussendorf and Briegel, 2001), photons (O'Brien et al., 2003), solid lattice structure defect, electrons spin in semiconductors (Kane, 2005), (Jelezko et al., 2004), anyons in topological quantum computatation utilized in Microsoft quantum computer (Gibney, 2016), Josephson junction in superconducting qubits utilized by IBM (Steffen et al., 2011), Google and Amazon in their quantum computer with comprehensive gate-based quantum computation technique (Kwon et al., 2021). A quick overall review of the development and concepts of physical systems for quantum computation can be referred to this work (Ladd et al., 2010). Thus it become increasingly more important to optimize quantum algorithm to propose a theoretical limit of speeds up for physical realizable model to gauge on their system.

1.2 Quantum Fourier Transform

Quantum Fourier Transform (QFT) is a popular quantum circuit used to solve mathematically hard problem. One of the popular example would be the Shor Algorithm that utilizes Quantum Fourier Transform as part of the algorithm for polynomial time solving prime factorization problem. In this work we take this circuit as the choice to construct geodesic equation because quantum Fourier transform is well known to be efficient quantum circuit in the context of complexity.

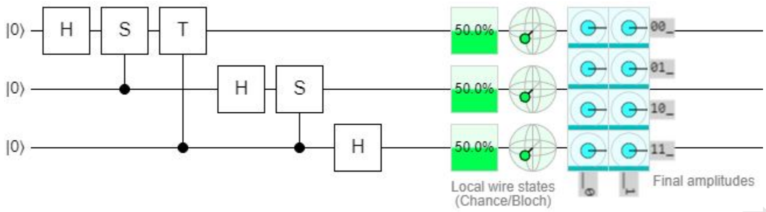

Figure 1.1: 3-qubit Quantum Fourier Transform

for which the Hadamard (H), S-gate and T-gate take the following matrix forms

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \quad \mathrm{S} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \quad \mathrm{T} = \begin{pmatrix}1 & 0\\ 0 & e^{\frac{\pi i}{4}}\end{pmatrix},$$

Both S and T gates are elements from phase shift gates, namely

$$P(\varphi) = \begin{pmatrix}1 & 0\\ 0 & e^{i\varphi}\end{pmatrix},$$

such that $S = P(\frac{\pi}{2})$ and $T = P(\frac{\pi}{4})$. In quantum Fourier transform, together S and T gates are applied as controlled gates which involved two respective qubits. For 3-qubit quantum Fourier transform, the evolution from the initial state can be expressed as follows,

$$(I\otimes I\otimes H)(I\otimes CS)(CT)(I\otimes H\otimes I)(CS\otimes I)(H\otimes I\otimes I)\,|000\rangle \tag{1.2}$$

$$= \frac{1}{\sqrt{8}}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & e^{\frac{i\pi}{4}} & i & ie^{\frac{i\pi}{4}} & -1 & -e^{\frac{i\pi}{4}} & -i & -ie^{\frac{i\pi}{4}}\\
1 & i & -1 & -i & 1 & i & -1 & -i\\
1 & ie^{\frac{i\pi}{4}} & -i & e^{\frac{i\pi}{4}} & -1 & -ie^{\frac{i\pi}{4}} & i & -e^{\frac{i\pi}{4}}\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
1 & -e^{\frac{i\pi}{4}} & i & -ie^{\frac{i\pi}{4}} & -1 & e^{\frac{i\pi}{4}} & -i & ie^{\frac{i\pi}{4}}\\
1 & -i & -1 & i & 1 & -i & -1 & i\\
1 & -ie^{\frac{i\pi}{4}} & -i & -e^{\frac{i\pi}{4}} & -1 & ie^{\frac{i\pi}{4}} & i & e^{\frac{i\pi}{4}}
\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{8}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}, \tag{1.3}$$

$$= \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle). \tag{1.4}$$

CS is denoted as the control-S gate and CT is denoted as the control-T gate which are

$$CS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\frac{i\pi}{2}} \end{pmatrix}; \qquad CT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{i\pi}{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{i\pi}{4}} \end{pmatrix}.$$

The control-T gate only acts on first and third qubit. The representation of the gate is in $2^3 \times 2^3$ due to unable to decompose the identity gate applied for second qubit using tensor product. This is one among the permutation of qubit representation, we can always perform a swap operation between qubits and be able to decompose it using tensor product. For this work we chose this representation.

1.3 Problem statement

Quantum computation utilizes successive application of quantum gates to initial quantum state of qubit to reach the desired computation results. Quantum gates for n-qubits system are elements of simple unitary group $\mathbf{SU}(2^n)$, which also belongs to Lie group. Thus its Lie group and Lie algebra properties can be studied. Furthermore Lie group is able to span its manifold which also enable us to study the properties of the quantum gate evolution under differential geometry perspective. For 3-qubit quantum circuit problem, $\mathbf{SU}(8)$ Lie group is to be used. It's Lie algebra, $\mathfrak{su}(8)$, can be represented in generalized Gell-Mann matrices. For quantum circuit problem, it would be more convenient to study $\mathfrak{su}(8)$ in tensor product of Pauli matrices forming the linearly independent generator (Pauli basis). This project worked on to check on the ability of $\mathfrak{su}(8)$ to be expressed in Pauli basis and the respective sets of $\mathbf{SU}(8)$ Lie Group each basis generated.

Studies of quantum complexity can help to create more efficient quantum algorithm. This project proposed a means of optimization of the complexity for quantum algorithm by restructuring the problem into differential geometry problem. Along with the properties of $\mathbf{SU}(8)$ Lie group and $\mathfrak{su}(8)$ Lie algebra, we are able to span the manifold. Complexity for implementing single qubit gates, two qubit gates and three or more-qubit gates are vastly different, thus by introducing penalty parameter s and q, we can deform the manifold and construct the penalized Riemmanian metric for three qubit Quantum Fourier Transform.

With the construction of Riemannian metric and Levi-Civita connection, we will be working on computing the geodesic equation for three qubit Quantum Fourier Transform without the exact solution. Our choice of for Quantum Fourier Transform is because major portion of quantum algorithm that exists utilizes Quantum Fourier Transform as base or part of its algorithm, this piqued our interest to study its geodesic via our method. This project proposed an organized structure for future work to optimization for quantum algorithm.

1.4 Objective of research

The objectives of this work are

1. to verify the generator of Lie group or Lie algebra of $\mathbf{SU}(8)$ that can be expressed in tensor product of Pauli matrices.
2. to construct the penalized Riemannian metric for three qubits Quantum Fourier Transform circuit.
3. to generate a geodesic equation for three qubits Quantum Fourier Transform circuit.

1.5 Research Scope

There are three quantum computation model, quantum circuit models, adiabatic quantum computation, and topological quantum computation. The scope of this research focused only on quantum circuit model. In quantum circuit model, each qubit started with an initial state and evolves with quantum gates. In our research, we focus on studying the properties of quantum gates utilizing groups, algebra and Riemannian geometry tools. In Riemmanian geometry we emphasize on its metric, Levi-Civita connection and geodesics.

1.6 Conventions

Notations

This subsection is intended for quick reference only. The individual notations will be properly introduced in chapter 3 accompanied along with its theory.

1. The spaces of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Generally we shall denote them by $\mathbb{F}$.
2. Imaginary unit $i = \sqrt{-1}$.
3. Vector spaces, groups and Lie groups are denoted by capital letter $G, H, U, V, W, \ldots$
   Elements in a space or group such as U are denoted by u.
4. Respective Lie algebra of a Lie group are denoted by Fraktur font, such as for Lie group G, the respective Lie algebra will be denoted as $\mathfrak{g}$.
5. The Hermitian adjoint of an operator A will be denoted by $A^{\dagger}$; if $a_{jk} \in \mathbb{C}$ are the matrix elements of A then $\bar{a}_{kj}$, are the elements of $A^{\dagger}$.
6. The "bra-ket" notation, we follow the usual physics convention when writing the inner product in the bra and ket notation. The quantity $\langle f|g\rangle$ is conjugate linear in the first entry and linear in the second.
7. The Kronecker delta function δ_{jk} is defined as

$$\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}$$

8. Following the usual convention, we denote a Lie group in general by $\mathbf{G}$ and its Lie algebra by $\mathfrak{g}$, i.e., using the corresponding small Gothic letter. For example, $\mathbf{SU}(n)$ is denoted as simple unitary group, while $\mathfrak{su}(n)$ denoted as its Lie algebra.
9. In most cases, we denote operators, polynomials, and physical observables which relate to standard quantum mechanics using capital letters, e.g. U representing the Unitary operator.

Layout

The numbering of definitions, theorems, equations and remarks etc., follows sequentially within each Chapter. For example, Chapter 3 starts with **Definition 3.1** followed by **Definition 3.2**, etc.

1.7 Organization of thesis

The thesis is subdivided into six chapters, Bibliography, and Appendix. Chapter 2 will begin with some review of literature on quantum computational complexity, quantum Fourier transform and past history of work towards optimizing quantum algorithm using different mathematical tools such as Riemannian geometry, complexity theory and optimal control theory. These include application of solutions and physical realizable models of scalable quantum computer. Chapter 3 provides relevant theories and definitions. There are quick references for the main parts of the thesis to clear out doubts when readers go through the proof. Together including in this chapter will be the quick overview of methodology used for this work. In Chapter 4, it consist of detail construction of Lie algebra of 3-qubit quantum circuit from its Lie group $SU(2^3)$, representation of the Lie algebra in Pauli basis as well as its properties. In Chapter 5, readers will continue with the geometry side of the work, construction of Riemannian metric, Levi-Civita connection and ultimately geodesic equation for 3-qubit quantum circuit. An example will also be constructed using 3-qubit Quantum Fourier Transform. Furthermore, the link between algebra and geometry forms will be enlightened to enable the continuation of idea for readers to go through the original work. Chapter 6 will be conclusion on the works that have been done and suggestion for future work. These come with problem encountered to give insight for reader to further extend the work. Appendix includes some additional derivations and proofs for interested readers to look out.

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