DEGENERATIONS OF LOW-DIMENSIONAL COMPLEX LEIBNIZ ALGEBRAS

By

NURUL SHAZWANI BINTI MOHAMED

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To all of my love;
Ibu & Ayah
Faizal Jasri
Hana Einara, Hadif Ezra, Hud Erhan

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chairman: **Sharifah Kartini Binti Said Husain, PhD**
Institute: **Mathematical Research**

Non-commutative analog of Lie algebras are Leibniz algebras. One of the important course of study is the degenerations of Leibniz algebras. Degenerations (or formerly known as contractions) were effectively applied to a wide range of physical and mathematical points of view. This thesis focuses on the degenerations of low-dimensional Leibniz algebras over the field of complex numbers particularly in the algebraic description of the varieties of three-dimensional complex Leibniz algebras and five-dimensional complex filiform Leibniz algebras arising from naturally gradaed non-Lie Leibniz algebras. The first part of this thesis describe the basic concepts and definitions of structural theory of Leibniz algebras and its degenerations. From the classification list, calculation of invariance arguments are collected. As a result, degenerations of algebras have been constructed by using algebraic invariants. The second part of this thesis concentrates on finding some essential degenerations of an arbitrary pair of the algebras of the same dimensions. Existence of degeneration matrices, g_t is needed in order to prove the degenerations. For non degeneration case, it is enough to provide certain reasons to reject the degenerations. The last part of this thesis gives the orbit closure, rigid algebras and irreducible components of an affine algebraic variety of three-dimensional complex Leibniz algebras.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGECUTAN ALGEBRA LEIBNIZ BERDIMENSI RENDAH TERHADAP NOMBOR KOMPLEKS

Oleh

NURUL SHAZWANI BINTI MOHAMED

Februari 2023

Pengerusi: **Sharifah Kartini Binti Said Husain, PhD**
Institut: **Penyelidikan Matematik**

Analog bukan hukum kalis tukar tertib bagi algebra Lie ialah algebra Leibniz. Salah satu cabang pengajian yang penting dalam kajian ialah pengecutan algebra Leibniz. Konsep pengecutan telah digunakan secara efektif pada pelbagai sudut pandangan fizikal dan juga matematik. Tesis ini memberi tumpuan kepada pengecutan algebra Leibniz berdimensi rendah terhadap nombor kompleks khususnya pada perihal algebra jenis algebra Leibniz terhadap nombor kompleks berdimensi tiga dan algebra Leibniz filiform nombor kompleks berdimensi lima yang muncul daripada algebra bukan Lie yang digredkan secara semula jadi. Bahagian pertama tesis ini menerangkan konsep asas dan definisi teori struktur algebra Leibniz dan pengecutannya. Daripada senarai klasifikasi, pengiraan ketakberubahan dikumpul. Akibatnya, pengecutan algebra telah dibina dengan menggunakan kaedah ketakberubahan algebra. Bahagian kedua tesis ini menumpukan pada mencari beberapa pengecutaan penting bagi pasangan algebra dalam dimensi yang sama. Kewujudan matriks pengecutan, g_t diperlukan untuk membuktikan setiap pengecutan. Untuk kes bukan pengecutan, ianya cukup untuk memberi alasan tertentu untuk menolak konsep pengecutan. Bahagian terakhir tesis ini memberikan tutupan orbit, algebra tegar dan unsur tak terturunkan bagi pelbagai algebra afin khususnya bagi algebra Leibniz terhadap nombor kompleks berdimensi tiga.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

The members of the Supervisory Committee were as follows:

Sharifah Kartini Binti Said Husain, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Faridah Binti Yunos, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

Isamiddin S. Rakhimov, PhD
Professor
Faculty of Computer and Mathematical Sciences
Universiti Teknologi MARA
(Member)

Bekbaev Ural, PhD
Associate Professor
Department of Natural and Mathematical Sciences
Turin Polytechnic Universiti of Tashkent
Uzbekistan
(Member)

ZALILAH MOHD SHARIFF, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

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LIST OF SYMBOLS

$Alg_n(K)$	Affine Algebraic Variety Over Field K
$Z(L)$	Center of L
$Der(L)$	Derivation of L
K	Field
$\mathbb{C}$	Field of Complex Numbers
$\mathbb{C}^*$	Field of Complex Numbers without 0
Lb_n	Filiform Leibniz Algebras
FLb_{n+1}	First Class Filiform Leibniz Algebras in dimension $n+1$
$Aut(L)$	Group of Automorphisms of L
L^k	k-th Degree of $L, k \in \mathbb{N}$
$\Im(L)$	Left Annihilator of L
$M_{m\times n}(K)$	$m \times n$ Matrices Over Field K
$n_A(L)$	Maximal Abelian Subalgebra of L
$Com(L)$	Maximal Commutative Subalgebra of L
$Lie(L)$	Maximal Lie Subalgebra of L
$M_n(K)$	$n \times n$ Matrices Over Field K
g_t	Non-singular Linear Operator
$R_a^m(x)$	Operator of Right Multiplication
O_λ	Orbit of λ
$O(L)$	Orbit of L
$\Re(L)$	Right Annihilator of L
SLb_{n+1}	Second Class Filiform Leibniz Algebras in dimension $n+1$
$GL_n(K)$	Set of $n \times n$ Invertible Matrices Over Field K
$SL_2(\mathbb{C})$	Set of 2×2 Matrices Over Complex Numbers with Determinant 1
γ_{ij}^k	Structural Constants
G_λ	Subgroup of Linear Reductive Group
$Lb_n(K)$	Subvariety of Affine Algebraic Variety Over Field K
TLb_{n+1}	Third Class Filiform Leibniz Algebras in dimension $n+1$
V	Vector Space

CHAPTER 1

INTRODUCTION

1.1 Basic Concepts

In this chapter we introduce some definitions and basic concepts of the theory of Leibniz algebras as well as being at ease with abstract algebra. The foundation of this theory including the definitions in Section 1.1 until Section 1.4 can be found in Leibniz algebras book by Ayupov et al. (2019). We begin with the definitions of vector space, algebra, Lie algebra, Leibniz algebra and group action.

Definition 1.1.1 *Let V be a set on which two operations, vector addition and scalar multiplication are defined. For every $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ in V and every scalar $c, d \in \mathbb{R}$, if the listed axioms below are satisfied then V is called a vector space.*

1. $\boldsymbol{u}+\boldsymbol{v}$ *is in* V.
2. $\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$.
3. $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$.
4. V *has a* ***zero vector*** *such that for every* $\boldsymbol{u}$ *in* V, $\boldsymbol{u}+\boldsymbol{0} = \boldsymbol{u} = \boldsymbol{0}+\boldsymbol{u}$.
5. *For every* $\boldsymbol{u}$ *in* V, *there is a vector in* V *denoted by* $-\boldsymbol{u}$ *such that* $\boldsymbol{u}+(-\boldsymbol{u}) = \boldsymbol{0} = -\boldsymbol{u}+\boldsymbol{u}$.
6. $c\boldsymbol{u}$ *is in* V.
7. $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u}+c\boldsymbol{v}$.
8. $(c+d)\boldsymbol{u} = c\boldsymbol{u}+d\boldsymbol{u}$.
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$.
10. $1(\boldsymbol{u}) = \boldsymbol{u}$.

Example 1.1.1 *Let V be $m \times n$ matrices over a field K, written as $M_{m\times n}(K)$ equipped with two maps*

$$f(A,B) = A+B \text{ and } g(k,A) = kA.$$

such that $A, B, A+B, kA$ is in V. By using all conditions in Definition 1.1.1, it is indeed a vector space.

Definition 1.1.2 *f is called a bilinear function such that $f : V \times V \to V$ where $x_1, x_2, y_1, y_2 \in V$ and $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in K$ satisfies the following:*

1. $f(\alpha_1 x_1 + \alpha_2 x_2, \beta) = \alpha_1 f(x_1, \beta) + \alpha_2 f(x_2, \beta)$,

2. $f(\alpha, \beta_1 y_1 + \beta_2 y_2) = \beta_1 f(\alpha, y_1) + \beta_2(\alpha, y_2)$.

Example 1.1.2 *Let $f : (A+V) \times (A+V) \to (A+V)$. We define $(a+u)\cdot(b+v) = a \cdot b + \theta(a,b)$. We check the first condition in Definition 1.1.2 with respect to the first argument as follows:*

$$\begin{aligned} f(\alpha(a+u)+\beta(b+v), c+w) &= f(\alpha a + \alpha u + \beta b + \beta v, c+w), \\ &= f((\alpha a + \beta b) + (\alpha u + \beta v), c+w). \end{aligned}$$

The same way can be checked for the second condition and applied both conditions with respect to the second arguments. Therefore, f is a bilinear map.

Definition 1.1.3 *A is called an algebra over the field K if A is a vector space over K, such that $f : A \times A \to A$ and*

1. $f(\alpha x + \beta y, z) = \alpha f(x,z) + \beta f(y,z)$,

2. $f(x, \alpha y + \beta z) = \alpha f(x,y) + \beta f(x,z)$,

where $x, y, z \in A$ and $\alpha, \beta \in K$.

The dimension of an algebra A is its dimension as a vector space. An algebra A is finite dimensional if A is a finite dimensional vector space.

Example 1.1.3 *Let $V = M_2(K)$ be the vector space of 2×2 matrices over a field K. Introduce a binary operation on $M_2(K)$ as follows:*

$$\lambda\left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right) = \begin{pmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{pmatrix}.$$

It is easy to see that $M_2(K, \lambda)$ is an algebra where $dim_K(M_2(K)) = 2^2 = 4$.

There is a well known class of algebras which is Lie algebras. The notion of Lie algebra arose in the study of Lie groups and now became an object of self-theory.

Definition 1.1.4 *Lie algebra over a field K is an algebra L over K with a bilinear binary operation that satisfies:*

1. *Antisymmetry:* $[x,y] = -[y,x]$ *for all* $x,y \in L$,
2. *Jacobi identity:* $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ *for all* $x,y,z \in L$.

Example 1.1.4 *Let A be an associative algebra and* $x,y \in A$. *Define bilinear operation* $[x,y] = xy - yx$. *Then,* $L = (A,[\cdot,\cdot])$ *is a Lie algebra. Particularly, if A be* $n \times n$ *matrices with the entries from K, denoted by* $M_n(K)$ *is a Lie algebra over a field K where* $dim_K(M_n(K)) = n^2$.

Definition 1.1.5 *An algebra L over a field F is called a Leibniz algebra, if its bilinear operation* $[\cdot,\cdot]$ *satisfies the following Leibniz identity:*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all $x,y,z \in L$.

Example 1.1.5 *Let V be a vector space. Introduce the bilinear operation* $[\cdot,\cdot]$ *on V for the basis vectors* $e_1, e_2, .., e_n$. *Then* $L_n = (V,[\cdot,\cdot])$ *is a Leibniz algebra. For other vectors* $[\cdot,\cdot]$ *is extended by linearity:*

1. $L_2 : [e_1,e_1] = e_2$.
2. $L_3 : [e_1,e_3] = -2e_1,\ [e_2,e_2] = e_1,\ [e_3,e_2] = e_2,\ [e_2,e_3] = -e_2$.
3. $L_4 : [e_1,e_1] = e_2,\ [e_2,e_1] = e_3,\ [e_3,e_1] = e_4$.
4. $L_5 : [e_1,e_1] = e_3,\ [e_i,e_1] = e_{i+1},\ 2 \le i \le 4,\ [e_1,e_2] = e_4 + \lambda e_5,\ [e_2,e_2] = e_4 - e_5,\ [e_3,e_2] = e_5$.

Now we briefly summarize basic facts about some classes of algebras which are closely related to Lie algebras which is Leibniz algebras. Leibniz algebras are non-commutative variation of Lie algebras. Generalization of Leibniz Algebras can be obtained by applying antisymmetry properties $[x,y] = -[y,x]$ from Definition 1.1.4 into the Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, (refer Definition 1.1.5) that will give us $[[x,y],z] - [x,[y,z]] - [[x,z],y] = 0$. Then the Jacoby identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ will be obtained. Clearly, a Lie algebra is a Leibniz algebra.

The set of all bilinear maps $V \otimes V \to V$ form a vector space $Hom(V \otimes V, V)$ of dimensional n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $Alg_n(K) \cong K^{n^3}$. An n-dimensional algebra L over K can be considered as an element $\lambda(L)$ of $Alg_n(K)$ via the bilinear mapping $\lambda : L \otimes L \to L$ defining a binary algebraic operation on L. A linear bijection $GL(V)$ acts on $Alg_n(K)$ by

$$(g * \lambda)(x,y) = g(\lambda(g^{-1}(x), g^{-1}(y))). \tag{1.1}$$

It is called "transport of structure". Let $\{e_1, e_2, ..., e_n\}$ be a basis of the vector space V. Then $x, y \in V$ can be written as follows:

$$x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$

and

$$y = \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n.$$

Therefore,

$$\lambda(x,y) = \sum_{i,j=1}^{n} \alpha_i \beta_j \lambda(e_i, e_j) = \sum_{i,j,k=1}^{n} \alpha_i \beta_j \gamma_{ij}^k e_k.$$

The coefficients γ_{ij}^k, where $i, j, k = 1, 2, ..., n$ of the linear combinations

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k, \tag{1.2}$$

are said to be the structure constants of the algebra L on the basis $\{e_1, e_2, ..., e_n\}$. Therefore, if a basis of the underlying vector space V is fixed then all possible algebra structures over V can be identified by points $\{\gamma_{ij}^k\}$ of n^3- dimensional affine space K^{n^3}.

Definition 1.1.6 *Let G be a group and X be a nonempty set. An action of a group G on a set X is a function $\sigma : G \times X \to X$ that satisfies the following conditions:*

1. *$\sigma(e,x) = x$, for all $x \in X$, where e is the identity element of G.*

2. *$\sigma(g, \sigma(h,x)) = \sigma(gh,x)$, for all $g,h \in G$ and $x \in X$.*

Let K be a field. $K[X] = \{f : X \to K\}$ be the set of all functions f on X. It is an algebra over K with respect to point-wise addition, multiplication and multiplication by scalars. If G is an algebraic group acting on an algebraic variety X, then there is an additional condition that σ is a morphism. Let G be a group which acts on the set X. The orbit of x under the action of G is given as follows:

$$O_G(x) = \{g \cdot x | g \in G\}.$$

1.2 Topology, Zariski Topology on Affine Space and Irreducible

A topological space is a set endowed with a structure, called a topology which allows defining all kinds of continuity.

Let X be a set and $P(X) = \{A \mid A \subset X\}$. For example, $X = \{a,b\}$. $P(X) = \{\{a\},\{b\},\{a,b\},\emptyset\}$. Let τ be a subset of X. Then, τ is a topology if $\tau \subset P(X)$ and the following properties are satisfied:

1. $\emptyset$ and X are both in τ.
2. If $A,B \in \tau$, then $A \cup B \in \tau$.
3. If $A_i \in \tau$ then $\bigcap_{i\in I} A_i \in \tau$, or τ is closed with finite intersection.

The elements of τ are called open sets. (X,τ) is called a topological space. A subset A of (X,τ) is said to be closed in X if its complement $X \setminus A$ is an open subset of (X,τ). $\emptyset$ and X are always both open and closed.

Example 1.2.1 *Let $X = \{1,2,3,4,5\}$ and $P(X) = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},$ $\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\},\{1,2,3\},\{1,2,4\},$ $\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{3,4,5\},\{1,2,3,4\},$ $\{1,2,3,5\},\{2,3,4,5\},\{1,2,3,4,5\},\{\emptyset\}\}$.*

Then, $\tau = \{\emptyset,\{1,2\},\{2,3\},\{2\},\{1,2,3\},X\}$ is a topology.
Elements of τ is indeed an open set e.g. $\{1,2\}$.
Meanwhile, $\{3,4,5\}$ is a closed set because $\{3,4,5\} = X \setminus \{1,2\}$.

Definition 1.2.1 *An algebraic group is an affine variety G equipped with morphisms of varieties $\mu : G \times G \to G$, $\iota : G \to G$ that give G the structure of a group.*

Let K be a fixed algebraically closed field. We define affine space over K denoted by $\mathbb{A}^n$, to be the set of all n-tuples of elements of K. An element $P \in \mathbb{A}^n$ is called a point, and if $P = (a_1,...,a_n)$ with $a_i \in K$, then the a_i are called the coordinates of P.

Let $A = K[x_1,...,x_n]$ be the polynomial ring in n variables over K. We will interpret the elements of A as functions from the affine n-space to K, by defining $f(P) = f(a_1,...,a_n)$ where $f \in A$ and $P \in \mathbb{A}^n$. Thus, if $f \in A$ is a polynomial, the set of zeros of f can be determined by:

$$Z(f) = \{P \in \mathbb{A}^n \mid f(P) = 0\}. \tag{1.3}$$

More generally, if T is any subset of A, we define the zero set of T to be the common zeros of all the elements of T:

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0, \forall f \in T\}. \tag{1.4}$$

The zeros of a polynomial $K[x]$ are all the x-values that make the polynomial equal

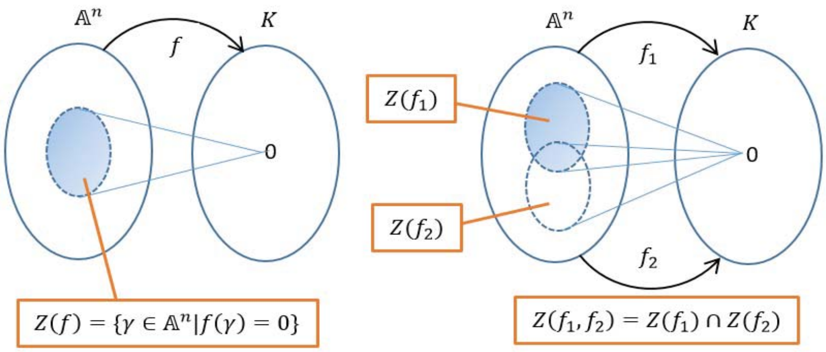

Figure 1.1: Common Zeros of Polynomial

to zero. If we have two or more polynomials, the common zeros of polynomial can be determined by finding the intersection of zeros of polynomials. It can be seen in Figure 1.1 above.

Example 1.2.2 *Consider $n = 3$, $K[x,y,z]$ where $f_1 = x, f_2 = y$.*
Choose a point $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{A}^3$. The zeros of f_1, $Z(f_1) = \{\gamma \in \mathbb{A}^3 \mid f_1(\gamma) = 0\} = \gamma_1$ and the zeros of f_2, $Z(f_2) = \{\gamma \in \mathbb{A}^3 \mid f_2(\gamma) = 0\} = \gamma_2$. It gives that our point $\gamma = (0, 0, \gamma_3)$.

Therefore, the common zeros of polynomials f_1 and f_2, $Z(f_1, f_2) = Z(f_1 \cap f_2) = \{(0, 0, \gamma_3) \mid \gamma_3 \in K\}$. The graph of the polynomial functions can be illustrated in Figure 1.2.

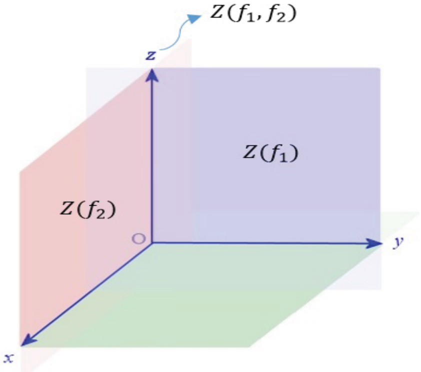

Figure 1.2: Graph of Polynomial Functions $K[x,y,z]$ **where** $f_1 = x, f_2 = y$

Clearly if $\wp$ is the ideal of A generated by T, then $Z(T) = Z(\wp)$. Since A is a Noetherian ring, any ideal $\wp$ has a finite set of generators $f_1, ..., f_r$. Thus, $Z(T)$ can be regarded as the common zeros of the finite set of polynomials $f_1, ..., f_r$.

Definition 1.2.2 *A subset Y of $\mathbb{A}^n$ is an algebraic set if there exists a subset $T \subset A$ such that $Y = Z(T)$.*

The following proposition can be found in Ayupov et al. (2019).

Proposition 1.2.3 *The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty and the whole space are algebraic sets.*

We define the Zariski topology on $\mathbb{A}^n$ by considering the complements of the open subsets. This is indeed a topology if the intersection of two open sets is open sets, and the union of any family of open sets is open sets. Additionally, empty set and whole sets are both open sets.

Definition 1.2.4 *A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one is closed in Y. The empty set is not considered to be irreducible.*

Proposition 1.2.5 *If Y is an irreducible subset of X, then its closure in X is also irreducible.*

Definition 1.2.6 *An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology).*

Remark 1.2.7 *Every set is always contained in its closure i.e. $A \subseteq \bar{A}$.*

Let (X, τ) be a topological space and A be the subset of X. The closure of A is denoted by $\bar{A}$ is the intersection of all closed sets containing A. That is, $\bar{A} = \{x \in X : \forall N(x), N(x) \cap A \neq \emptyset\}$. Or it can be clearly seen in Figure 1.3.

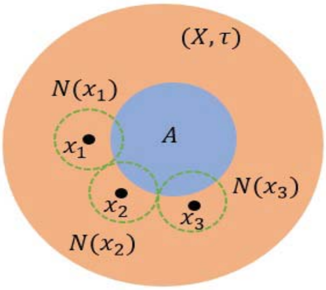

Figure 1.3: Closure of A

Example 1.2.3 *Let $X = \{a,b,c,d\}$ and $P(x) = \{\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\}, \{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\},\emptyset\}$ with topology*

$$\tau = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$$

and $A = \{b,d\}$ be a subset of X.

1. *Open sets are $\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X$.*
2. *Closed sets are $X, \{b,c,d\}, \{a,d\}, \{d\}, \emptyset$.*
3. *Closed set containing A are $X, \{b,c,d\}$.*
4. *$\bar{A} = \{b,c,d\} \cap X = \{b,c,d\}$.*

1.3 Introduction on Leibniz algebra

The theory of Lie algebras is one of the most developed extensions in modern algebra. It has been deeply explored and reviewed by many mathematicians. Due to active investigations on the properties of Lie algebras, a more general object called

Leibniz algebras also has been investigated. In this thesis, we consider Leibniz algebras over the field of complex number. Theorem 1.3.1 states that any n-finite dimensional Leibniz algebra can be written as semidirect sum of solvable Leibniz and semisimple Lie algebra. The proof of the following theorem were given by Barnes (2012).

Theorem 1.3.1 *(Levi's theorem for Leibniz algebras) For a finite-dimensional Leibniz algebra L over a field of characteristic zero there exists a subalgebra S which is a semisimple Lie algebra, such that $L = S \bigoplus R$ where R is the radical of L.*

This thesis focused on the study of Leibniz algebras which was introduced by a French Mathematician, Loday (1993) for "non-antisymmetric" analogue of Lie algebras. If the properties of antisymmetric is applied to the Leibniz identity, then it is equivalent to the Jacobi identity. Hence, any Lie algebra is Leibniz algebra. Leibniz algebras also appear to be related in natural way to several topics such as differential geometry, homological algebra, classical algebraic topology, noncommutative geometry, quantum physics and many more.

A Leibniz algebra L on n-dimensional vector space V over a field K can be written as a pair $L = (V, \lambda)$, where λ is a Leibniz algebra law on V. Let us denote the set of Leibniz algebra laws by LB_n. Let $\{e_1, e_2, e_3, ..., e_n\}$ be a basis of V. The structure constants of $\lambda \in LB_n$, $\gamma_{ij}^k \in K$ can be identified by the law:

$$\sum_{l=1}^{n} \left(\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m \right) = 0, \tag{1.5}$$

where $i, j, k, m = 1, 2, ..., n$. Then LB_n appears as an algebraic variety embedded in the linear space of bilinear mapping on V, isomorphic to K^{n^3}.

Definition 1.3.2 *Two laws λ_1 and λ_2 from LB_n are said to be isomorphic if there is $g \in GL_n(K)$ such that*

$$\lambda_2(x, y) = (g * \lambda_1)(x, y) = g^{-1}(\lambda_1(g(x), g(y)))$$

for all $x, y \in V$.

Within the context of this study, the following is a particular definition of orbit function.

Definition 1.3.3 *A function $f : LB_n \to K$ is said to be invariant (or orbit) function if*

$$f(g * \lambda) = f(\lambda)$$

for all $g \in GL_n(K)$ and $\lambda \in LB_n$.

Let $O(\lambda)$ be the set of laws isomorphic to λ. It is called the orbit of λ with respect to action of $GL_n(K)$. The closure of the orbit with respect to the Zariski topology is denoted by $\overline{O(\lambda)}$.

The lower central series and derived series of a Leibniz algebra L are defined as follows:

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \in \mathbb{N}.$$

$$L^{[1]} = L,\ L^{[s+1]} = [L^{[s]}, L^{[s]}],\ s \in \mathbb{N}.$$

Definition 1.3.4 *A Leibniz algebra L is said to be nilpotent and solvable if there exists an integer $i \in \mathbb{N}$ such that*

$$L^1 \supset L^2 \supset \ldots \supset L^i = 0.$$

$$L^{[1]} \supset L^{[2]} \supset \ldots \supset L^{[i]} = 0.$$

respectively. Any nilpotent algebra is solvable algebra.

If A and B are subspaces of Leibniz algebra L, we define

$$[A,B] = span\{[a,b] | a \in A, b \in B\}$$

and

$$A + B = \{a + b | a \in A, b \in B\}.$$

If $z \in [A,B]$, then there exist $x_1, \ldots, x_r \in A$ and $y_1, \ldots, y_r \in B$ such that $z = \sum_{i=1}^{r} [x_i, y_i]$.

Example 1.3.1 *The table of multiplication of algebra L given as follows:*
$[e_1,e_1] = e_3,\ [e_2,e_1] = e_4,\ [e_3,e_1] = e_5.$

$$\begin{aligned} L^1 &= L = span\{e_1,e_2,e_3,e_4,e_5\}, \\ L^2 &= [L,L] = span\{[a,b] | a,b \in L\} = \{e_3,e_4,e_5\}, \\ L^3 &= [L^2,L] = span\{e_5\}, \\ L^4 &= [L^3,L] = 0. \end{aligned}$$

Therefore, L is nilpotent algebra.

Example 1.3.2 *Consider $g = SL_2(\mathbb{C})$ with trace $= 0$. The basis given*

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

has the relations $[h,e] = 2e, [h,f] = -2f, [e,f] = h$ *and* $L^{[1]} = L^{[2]} = L^{[3]} = \ldots = L^{[k]} = span\{e,f,h\}$. *Therefore g is non solvable Lie algebra.*

Definition 1.3.5 *An n-dimensional Leibniz algebra L is said to be filiform if*

$$dimL^i = n - i,$$

where $2 \leq i \leq n$.

It is clear that filiform Leibniz algebra is always nilpotent. However, nilpotent Leibniz algebras is not necessarily to be filiform. We can illustrate the inclusion of various type of Leibniz algebras in the following Figure 1.4.

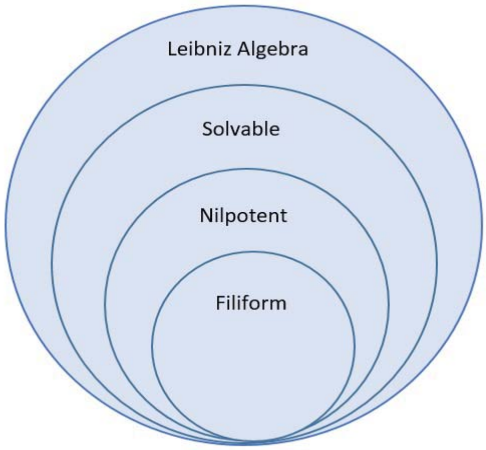

Figure 1.4: The Inclusion of Various Type of Leibniz Algebras

Definition 1.3.6 *A linear transformation* $d : L \to L$ *is said to be a derivation if* $d[x,y] = [d(x),y] + [x,d(y)]$ *for any* $x,y \in L$.

The set of all derivations of an algebra L is denoted by $Der(L)$. A subspace L_0 of a Leibniz algebra L is said to be subalgebra if for $x,y \in L_0$ implies $[x,y] \in L_0$.

Definition 1.3.7 *The subspace* $\Re(L), \Im(L)$ *of an algebra L is defined by*

$$\Re(L) = \{x \in L \mid [L,x] = 0\}$$

and

$$\Im(L) = \{x \in L \mid [x,L] = 0\}$$

is said to be the right annihilator and left annihilator of L, respectively.

In other words, the right and left annihilator of a Leibniz algebra L can be defined as all element in L which product is zero.

Definition 1.3.8 *The center of a Leibniz algebra L defined by*

$$Z(L) = \{x \in L \mid [L,x] = [x,L] = 0\}.$$

Right annihilator, $\Re(L)$ has a left ideal meanwhile, left annihilator $\Im(L)$ has a right ideal and center $Z(L)$ has two sided ideals. All such ideals are ideals of Leibniz algebra L. Any ideal is a subalgebra and the subalgebra is subspace and have its own dimension.

Definition 1.3.9 *The maximal commutative subalgebra of a Leibniz algebra L denoted by $Com(L)$ if $[x,y] = [y,x]$ for all $x,y \in L$.*

Definition 1.3.10 *The maximal abelian subalgebra of L denoted by $n_A(L)$ if $[x,y] = 0$ for all $x,y \in L$ and $n_A(L)$ is the maximal with respect to inclusion.*

Definition 1.3.11 *The maximal Lie subalgebra of L denoted by $Lie(L)$ is defined by $[x,y] = -[y,x]$ for all $x,y \in L$ and $Lie(L)$ is the maximal satisfying the condition.*

Definition 1.3.12 *The k-th degree of L defined by $L^k = [L^{k-1}, L]$ for all $k \in \mathbb{N}$.*

1.4 Degeneration Concepts

Let $L = (V, \lambda)$ be an n-dimensional algebra with an underlying n-dimensional vector space V over a field K and a bilinear map $\lambda : V \times V \to V$. Consider a continuous function $g_t : (0,1] \to GL(V)$. Specifically, g_t is a non-singular linear operator on V for all $t \in (0,1]$. A parameterized family of new isomorphic to L algebra structures on V is determined via the binary operation λ as follows:

$$\lambda_t(x,y) = (g_t * \lambda)(x,y) = g_t^{-1}\lambda(g_t(x), g_t(y)), \qquad (1.6)$$

where $x,y \in V$.

Definition 1.4.1 *If the limit* $\lim_{t\to+0}\lambda_t = \lambda_0$ *exists for any* $x, y \in V$, *then the algebraic structure* λ_0 *is said to be a degeneration of the algebra L.*

We recall the following simple properties of the degeneration. The degenerations can be treated in basis level. Let $\{e_1, e_2, ...e_n\}$ be a basis of an n-dimensional algebra L. If the limit $\lim_{t\to+0}\lambda_t(e_i, e_j) = \lambda_0(e_i, e_j)$ exists, then the algebra (V, λ_0) is a degeneration of L.

Note 1 *If* λ *not isomorphic to* μ, *then the assertion* $\lambda \to \mu$ *is called a proper degeneration.*

Note 2 *A degeneration is called trivial if* $\lambda \cong \mu$, *that is, if* $\mu \in O(\lambda)$.

Note 3 *Degeneration is transitive, that is if* $\lambda \to \mu$ *and* $\mu \to \nu$ *then* $\lambda \to \nu$.

The definition of the degeneration is given as follows.

Definition 1.4.2 *An algebra* λ *is said to degenerate to another algebra* μ, *if* μ *is represented by a structure which lies in the Zariski closure of the* $GL(\mathbb{V})$*-orbit of the structure which represents* λ, *i.e.* $\mu \in \overline{O(\lambda)}$. *We denote this by* $\lambda \to \mu$.

Note that in this case we have $\overline{O(\mu)} \subset \overline{O(\lambda)}$. Hence, definition of degeneration does not depend on the choice of λ and μ. It is easy to see that any algebra degenerates to the algebra with zero multiplication. From now on we use this fact without mentioning it. We write $\lambda \not\to \mu$ if $\mu \notin \overline{O(\lambda)}$. The set of all Leibniz algebra structures on an n-dimensional vector space V over a field K is denoted by $LB_n(K)$ and can be included in the above mentioned n^3-dimensional affine space. The $LB_n(K)$ is closed subset of $Alg_n(K)$. Thus, $LB_n(K)$ is an algebraic set.

A subset of an algebraic set is said to be irreducible if it cannot be written as a union of two non trivial Zariski closed subsets. A maximal irreducible closed subsets of an algebraic set is called irreducible component of the algebraic set. A Leibniz algebra μ is said to be rigid if its orbit $\overline{O(\mu)}$ is an irreducible component of $LB_n(K)$ i.e., $O(\mu)$ is an open subset of the irreducible component. There are only finitely many irreducible components in each dimension. It is interesting but difficult to study the structure of $LB_n(K)$. In particular, one is interested to find the irreducible components of $LB_n(K)$.

1.5 Problem Statement

One of the tools to determine the degenerations is by using some invariance arguments. A few results on degenerations of three-dimensional complex Leibniz algebras has been given by Rakhimov(2012) and Ismailov(2019). However, they used an old list of classification of algebras. To gain a right description of degenerations of three-dimensional complex Leibniz algebras, we need to consider the updated classification list by Rikhsiboev(2012), Casas et.al(2012) and Ayupov(1999).

In this work, we consider degenerations of three-dimensional complex Leibniz algebras and five-dimensional complex filiform Leibniz algebras arising from naturally graded non-Lie Leibniz algebras. Results on degenerations of four-dimensional complex Leibniz algebras was solved by Amir et al. (2022).

It is interesting to find the possible degenerations between the algebras of low-dimensional complex Leibniz algebras. However, constructing the degeneration matrices in order to prove the degenerations are really difficult. In the past work, no clear procedure on finding the degeneration matrices given. To find the dimension of the algebraic variety, it is enough by considering its rigidity as the orbit closure of rigid algebras gives irreducible components of the variety.

1.6 Research Objectives

The objectives of this research are:

1. to compute the invariance arguments for three-dimensional complex Leibniz algebras and five-dimensional complex filiform Leibniz algebras arising from naturally graded non-Lie filiform Leibniz algebras.

2. to construct the degenerations and non degenerations between the algebras for three and five dimensional complex Leibniz algebras.

3. to determine the affine algebraic variety of three-dimensional complex Leibniz algebras by finding the orbit closure, rigidity and its irreducible components.

1.7 Methodology

In this subsection, we give general method structure that have been used in this research in order to verify and corroborate the results.

1. To achieve the first objective, we construct the invariance arguments of Leibniz algebras by using some well-known invariance used in the previous works. One of the most powerful invariance is the derivation of algebras. Given two algebras A and B. We consider the case where, if $A \to B$ and A is not isomorhic to B, then the dimensions of $Der(A) <$ dimensions of $Der(B)$. We collect the relations between those invariance arguments as necessary criteria of degenerations in one theorem.

2. For the second objective, we need to construct families of matrices parametrized by t. Let A and B be two algebras represented by λ and μ from $Alg_n(K)$ respectively. We fix a basis $e_1, e_2, \ldots, e_n$ of V and γ_{ij}^k be the structure constants of μ in this basis. If there exist $c_i^j(t) \in \mathbb{C}$ and $t \in \mathbb{C}^*$ such that $E_i^t = \sum_{j=1}^n c_i^j(t) e_j$ where $i = 1, 2, \ldots, n$ form a basis of V and the structure constants of λ in the basis $E_1^t, \ldots, E_n^t$ are such polynomials $\gamma_{ij}^k(t) \in \mathbb{C}[t]$ that $\gamma_{ij}^k(0) = \gamma_{ij}^k$, then $A \longrightarrow B$. Thus $E_1^t, \ldots, E_n^t$ is called a *parametrized basis* for $A \to B$. Particularly in $LB_3(\mathbb{C})$, we consider $E_i(t)$ where $i = 1, 2, 3$ in this form:

$$
\begin{aligned}
E_1(t) &= \varphi_{11}(t)e_1 + \varphi_{12}(t)e_2 + \varphi_{13}(t)e_3, \\
E_2(t) &= \varphi_{22}(t)e_2 + \varphi_{23}(t)e_3, \\
E_3(t) &= \varphi_{33}(t)e_3.
\end{aligned}
$$

   For another case where the pairs of any Leibniz algebras does not satisfy at least one relation from the list, we prove that degeneration does not exist by providing a reason.

3. For the third objective, in order to find algebraic variety, it is enough to study the properties that deal with rigidity. Algebras whose orbits are open in $LB_n(K)$ are called rigid. The orbits of the rigid algebras give irreducible components of the variety $LB_n(K)$. By the rigidity properties, we obtain the irreducible components in low dimensional complex Leibniz algebras. If L is a rigid algebra in $LB_n(K)$, then there exists an irreducible components, C of $LB_n(K)$ such that $O(L) \cap C$ is non empty open subset of C. The closure of $O(L)$ is contained in C. Then, the dimension of the irreducible component C of $LB_n(K)$ is given by dim $C = n^2-$ dim $Aut(L)$.

1.8 Outline of Thesis

Chapter 1 gives a brief introduction about the motivation of this research. Vector spaces, algebra, Lie algebra, Leibniz algebra, basic definitions and degeneration concepts were introduced in this chapter together with the research problems, research objectives and methodology.

Chapter 2 focuses on the previous works done by many researchers. This chapter reviews the mathematical background that is related to this research.

Chapter 3 consists of computation of invariance arguments. The value of degeneration invariance are collected in table and have been applied to three-dimensional complex Leibniz algebras. This chapter also discusses the possible degenerations of three-dimensional complex Leibniz algebras. We consider some essential degenerations and prove that the possible degenerations are in fact degenerations. Some propositions on degenerations are listed and the parametrized basis for each degeneration also given. Rigid algebras and irreducible component of the variety were concluded in a theorem.

Chapter 4 gives the invariance arguments of five-dimensional complex filiform Leibniz algebras arising from naturally graded non-Lie Leibniz algebras. Some degenerations of five-dimensional complex filiform Leibniz algebras were given and proved by constructing the degeneration matrices. The non-degeneration cases together with some reasons that contradicts the necessary condition of degenerations have been listed into a table.

Finally in Chapter 5 contains the summary of all the contributions that has been made in this thesis together with the future work that can be extended from this research.

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