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ON A CONTROL PROBLEM AND A PURSUIT GAME OF TRANSFERRING STATES DESCRIBED BY AN INFINITE THREE-SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we devote to study a pursuit game described by an infinite three-systems of differential equations in Hilbert space. The game involves transferring of the states as the pursuit is said to be completed if the state $\zeta(\cdot)$ of the system is shifted to another non zero state ζ^1 of the system at some finite time. The control functions of the players are constrained by geometric constraints. We first find the control function that transfers the control system's state to the state ζ^1 at some time. We then extend to solve the pursuit problem where an admissible pursuer's strategy is constructed and a guaranteed pursuit time is determined.

Key words: Three-systems of differential equations, control problem, pursuit problem, geometric constraints, Hilbert space.

Mathematics Subject Classification MSC2020: 91A23, 49N70.

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1. Introduction

The continuous profound works in the subject of differential games aim to search solutions for various problems that include two parties with conflicting motives. The solutions are then applied to resolve real-world issues. Initially, the concept of differential games was introduced by Isaacs [1] to propose a framework for modelling and examining dynamic interactions in military conflicts. This, is then, further extended by (see, for example, [2–5]) and found applications in various fields like economics, engineering, political sciences and many more.

The method of decomposition is a well-known and frequently applied method in the field of differential games. This is because it is common for a real-life problem to be modelled as a system of partial differential equations (for instance, [6–8]), which could include hyperbolic and parabolic partial differential equations. Hence, decomposition method is employed to reduce such system to the one defined by an infinite system of ordinary differential equations, as in the works of [9–11]. This provides a way for researchers to study the differential game

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problem of an infinite system of ordinary differential equations in an independent framework.

In the context of pursuit differential game, the completion of pursuit is solely based on the motive of the first player, that is the pursuer. For the case of a pursuit game in an infinite system, the pursuer's conventional objective is to transfer the state of the system into another state of the system, which could be the origin (see, for example, [12–14]) or a non-zero state (see, for instance, [15–19]), to complete the pursuit.

The finding of solution for pursuit game of shifting the state of the system into another non zero state requires both initial state and another non zero state to be present in the construction of sufficient conditions. For example, the work of [15] examined a pursuit game with dynamic of motions of the players as follows:

$$\dot{z_k}(t) + \mu_k z_k(t) = w_k(t), \quad z_k(0) = z_k^0, \quad k = 1, 2, ..., \tag{1.1}$$

where z_k, w_k are real numbers. The game is considered to be restricted by both integral and geometric constraints separately. As the pursuer's aim is to transfer the initial state z^0 of the system into another non zero state z^1, the authors established sufficient conditions of the game. They looked into the control problem of the system and determined an admissible control function. The pursuit was then shown to be completed by developing a strategy for the pursuer.

Later, in the study of [16], every coordinate of control functions of players in a game described by (1.1) is subjected to integral constraints. With that, an admissible strategy was constructed for the pursuer to complete the pursuit, and a formula for guaranteed pursuit time was obtained.

By bearing pursuer of similar objective, differential game theorists continued to investigate pursuit game in a higher level of infinite system. The papers of [17] and [18] aimed to investigate the pursuit game of transferring states for an infinite 2-system of differential equations described by

$$\begin{aligned} \dot{x_k} &= -\alpha_k x_k - \beta_k y_k + u_{k1} - v_{k1}, \quad x_k(0) = x_{k0}, \\ \dot{y_k} &= \beta_k x_k - \alpha_k y_k + u_{k2} - v_{k2}, \quad y_k(0) = y_{k0}, \quad k = 1, 2, ..., \end{aligned} \tag{1.2}$$

where $\alpha_k \geq 0$ and β_k are real numbers. The researchers considered the pursuit under different types of constraints imposed on the players' control functions, with the preceding one focusing on integral constraints and the subsequent one on geometric constraints. In both works, they built the pursuer's strategy by taking states z^0 and z^1 into account, and proved that the pursuit can be completed.

Furthermore, the article [19] delved such pursuit game by using a game model of three-systems of differential equations in l_2 space. The authors designed a control function satisfying integral constraints that steers the state of control system into another non zero state at some finite time. Subsequently, they established a strategy to be used by the pursuer in completing the pursuit.

This paper deals with a pursuer of similar goal and system as in [19], but this time we implement geometric constraints on the control functions of the players.

We determine the sufficient conditions needed to complete the pursuit and develop an equation for guaranteed pursuit time.

2. Preliminaries

The game is described by the following three-systems of differential equations:

$$\begin{aligned} \dot{x_k} &= -\delta_k x_k - u_{k1} + v_{k1}, \quad x_k(0) = x_k^0, \\ \dot{y_k} &= -\mu_k y_k - \lambda_k z_k - u_{k2} + v_{k2}, \quad y_k(0) = y_k^0, \\ \dot{z_k} &= \lambda_k y_k - \mu_k z_k - u_{k3} + v_{k3}, \quad z_k(0) = z_k^0, \end{aligned} \tag{2.1}$$

where $\delta_k, \mu_k \geq 0, \lambda_k \in \mathbb{R}$ and $u_{kj}, v_{kj} \in \mathbb{R}$ for $k = 1, 2, ...,$ and $j = 1, 2, 3$ with $x^0 = (x_1^0, x_2^0, x_3^0, ...) \in l_2$, $y^0 = (y_1^0, y_2^0, y_3^0, ...) \in l_2$, $z^0 = (z_1^0, z_2^0, z_3^0, ...) \in l_2$.

We define the state of the system (2.1) as $\zeta(t)$ that is

$$\begin{aligned} \zeta(t) &= (\zeta_1(t), \zeta_2(t), ...) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t), ...), \\ \zeta_k(t) &= (x_k(t), y_k(t), z_k(t)), \quad \|\zeta_k(t)\| = \left(x_k^2(t) + y_k^2(t) + z_k^2(t)\right)^{\frac{1}{2}}, \\ \|\zeta(t)\| &= \left(\sum_{k=1}^{\infty} x_k^2(t) + y_k^2(t) + z_k^2(t)\right)^{\frac{1}{2}}, \end{aligned} \tag{2.2}$$

where $t \in [0, \vartheta]$.

The initial non zero state of the system is given by

$$\begin{aligned} \zeta^0 &= (\zeta_1^0, \zeta_2^0, ...) = (x_1^0, y_1^0, z_1^0, x_2^0, y_2^0, z_2^0, ...), \\ \zeta_k^0 &= (x_k^0, y_k^0, z_k^0), \quad ||\zeta_k^0|| = \left((x_k^0)^2 + (y_k^0)^2 + (z_k^0)^2\right)^{\frac{1}{2}}, \\ \|\zeta^0\| &= \left(\sum_{k=1}^{\infty} (x_k^0)^2 + (y_k^0)^2 + (z_k^0)^2\right)^{\frac{1}{2}}, \end{aligned} \tag{2.3}$$

and the another non zero state of the system is given by

$$\begin{aligned} \zeta^1 &= (\zeta_1^1, \zeta_2^1, ...) = (x_1^1, y_1^1, z_1^1, x_2^1, y_2^1, z_2^1, ...), \\ \zeta_k^1 &= (x_k^1, y_k^1, z_k^1), \quad \|\zeta_k^1\| = \left((x_k^1)^2 + (y_k^1)^2 + (z_k^1)^2\right)^{\frac{1}{2}}, \\ \|\zeta^1\| &= \left(\sum_{k=1}^{\infty} (x_k^1)^2 + (y_k^1)^2 + (z_k^1)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

The pursuer's aim is to bring the state of the system to another state $\zeta^1 \neq 0$ at some time, and this would guarantee the completion of pursuit in the game. The resources of both pursuer and evader are limited by geometric constraints.

Definition 2.1. A control $w(\cdot) = (w_1(\cdot), w_2(\cdot), ...) \in l_2$, $w : [0, \vartheta] \to l_2$ that complies to condition

$$\sqrt{\sum_{k=1}^{\infty} \sum_{j=1}^{3} |w_{kj}(t)|^2} \leq \bar{\rho}, \quad t \in [0, \vartheta], \tag{2.5}$$

where $\bar{\rho} > 0$ is called an admissible control function, with measurable coordinates $w_k = (w_{k1}, w_{k2}, w_{k3})$ for $k = 1, 2, ...$. Set of all admissible controls is denoted by $S(\bar{\rho})$.

The unique solution $\zeta(\cdot) = (\zeta_1(\cdot), \zeta_2(\cdot), ...)$ that belongs to space of continuous functions C in l_2 space on time interval $[0, \vartheta]$ for three-systems of differential equations,

$$\begin{aligned} \dot{x_k} &= -\delta_k x_k + w_{k1}, \quad x_k(0) = x_k^0, \\ \dot{y_k} &= -\mu_k y_k - \lambda_k z_k + w_{k2}, \quad y_k(0) = y_k^0, \\ \dot{z_k} &= \lambda_k y_k - \mu_k z_k + w_{k3}, \quad z_k(0) = z_k^0, \end{aligned} \tag{2.6}$$

where $\delta_k, \mu_k \geq 0, \lambda_k \in \mathbb{R}$ and $w_{kj} \in \mathbb{R}$ for $k = 1, 2, ...,$ and $j = 1, 2, 3$ is given by

$$\zeta_k(t) = \alpha_k(t)\zeta_k^0 + \int_0^t \alpha_k(t - s) w_k(s)\, ds, \tag{2.7}$$

where

$$\alpha_k(t) = \begin{pmatrix} e^{-\delta_k t} & 0 & 0 \\ 0 & e^{-\mu_k t} \cos \lambda_k t & -e^{-\mu_k t} \sin \lambda_k t \\ 0 & e^{-\mu_k t} \sin \lambda_k t & e^{-\mu_k t} \cos \lambda_k t \end{pmatrix}, \tag{2.8}$$

if $w(\cdot) \in S(\bar{\rho})$ and $\psi_k = min\{\delta_k, \mu_k\}$ (see [20]).

Property 2.1. The matrix $\alpha_k(t), k = 1, 2, ...,$ has the following properties:

1. $\alpha_k(t + s) = \alpha_k(t)\alpha_k(s) = \alpha_k(s)\alpha_k(t)$,
2. $\alpha_k^{-1}(t) = \alpha_k(-t)$,
3. $|\alpha_k(t)\zeta_k| \leq e^{-\psi_k t}|\zeta_k|$.

Definition 2.2. The controls $u(\cdot) = (u_1(\cdot), u_2(\cdot), ...) \in l_2$, $u : [0, \vartheta] \to l_2$ and $v(\cdot) = (v_1(\cdot), v_2(\cdot), ...) \in l_2$, $v : [0, \vartheta] \to l_2$ that comply to conditions

$$\sum_{k=1}^{\infty} \sum_{j=1}^{3} |u_{kj}(t)|^2 \leq \rho^2, t \in [0, \vartheta], \qquad \sum_{k=1}^{\infty} \sum_{j=1}^{3} |v_{kj}(t)|^2 \leq \sigma^2, t \in [0, \vartheta], \tag{2.9}$$

where $\rho, \sigma > 0$ is called admissible control functions of pursuer and that of evader respectively, with measurable coordinates $u_k = (u_{k1}, u_{k2}, u_{k3})$, $v_k = (v_{k1}, v_{k2}, v_{k3})$ for $k = 1, 2, ...$. Set of admissible control of pursuer (of evader) is denoted by $S(\rho)(S(\sigma))$.

Definition 2.3. A function $U(t, v(t)) = (U_1(t, v(t)), U_2(t, v(t)), ...), U : [0, \vartheta] \times l_2 \to l_2$ that complies to condition

$$\sqrt{\sum_{k=1}^{\infty} |U_k(t, v_k(t))|^2} \leq \rho, \quad t \in [0, \vartheta], \quad w(\cdot) \in S(\rho - \sigma), v(\cdot) \in S(\sigma), \tag{2.10}$$

where $\rho > 0$ is called an admissible strategy of the pursuer, with measurable coordinates U_k, v_k, w_k such that $U_k(t, v_k(t)) = v_k(t) - w_k(t)$ for $k = 1, 2, ...$.

Definition 2.4. If, for any admissible evader's control $v(\cdot) \in S(\sigma)$, there exists an admissible pursuer's strategy U such that $\zeta(\tau) = \zeta^1$ for some time $\tau \in [0, \vartheta]$, then the pursuit is said to be completed in game (2.1). The time $\vartheta > 0$ is referred as guaranteed pursuit time.

3. Main Results

3.1. Construction of Control for the System

It is common practice in differential games for the corresponding control problem to be considered before the pursuit differential game is solved. For this, we develop a control $w(\cdot)$ that satisfies geometric constraints. The control is then utilised to bring the state of control system (2.6) into another non zero state ζ^1 at some time.

For a given $\vartheta > 0$, we let

$$\sum_{k=1}^{\infty} 2(\zeta_k^0)^T \gamma_k(\vartheta) \beta_k^{-2}(0, \vartheta) \zeta_k^0 + 2(\zeta_k^1)^T \gamma_k^2(\vartheta) \beta_k^{-2}(0, \vartheta) \zeta_k^1 = \bar{\rho}^2, \tag{3.1}$$

where

$$\beta_k(0, \vartheta) = \begin{pmatrix} \int_0^{\vartheta} e^{2\delta_k s} \, ds & 0 & 0 \\ 0 & \int_0^{\vartheta} e^{2\mu_k s} \, ds & 0 \\ 0 & 0 & \int_0^{\vartheta} e^{2\mu_k s} \, ds \end{pmatrix}, \tag{3.2}$$

and

$$\gamma_k(\vartheta) = \begin{pmatrix} e^{2\delta_k \vartheta} & 0 & 0 \\ 0 & e^{2\mu_k \vartheta} & 0 \\ 0 & 0 & e^{2\mu_k \vartheta} \end{pmatrix}. \tag{3.3}$$

Note that $\beta_k(0, \vartheta) = \int_0^{\vartheta} \alpha_k(-s) \alpha_k^T(-s) ds$ for each k where α^T is transpose of matrix α.

Theorem 3.1. *If equation* (3.1) *is satisfied, then there exists a control given by*

$$w_k(t) = -\alpha_k^T(-t)\big(\beta_k^{-1}(0,\vartheta)\zeta_k^0 - \alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\big), \quad t \in [0,\vartheta], \tag{3.4}$$

that guarantees the equality $\zeta(\vartheta) = \zeta^1$ *in control system* (2.6).

Proof. Prove that the constructed control (3.4) is admissible. The usage of inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ and properties of (2.8) allow us to obtain the following:

$$\sum_{k=1}^{\infty}|w_k(t)|^2 = \sum_{k=1}^{\infty}| - \alpha_k^T(-t)\beta_k^{-1}(0,\vartheta)\zeta_k^0 + \alpha_k^T(-t)\alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1|^2$$

$$\le 2\sum_{k=1}^{\infty}|\alpha_k^T(-t)\beta_k^{-1}(0,\vartheta)\zeta_k^0|^2 + 2\sum_{k=1}^{\infty}|\alpha_k^T(-t)\alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1|^2. \tag{3.5}$$

We calculate the terms on the right side of (3.5) separately.
For $\sum_{k=1}^{\infty}|\alpha_k^T(-t)\beta_k^{-1}(0,\vartheta)\zeta_k^0|^2$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}|\alpha_k^T(-t)\beta_k^{-1}(0,\vartheta)\zeta_k^0|^2 \\
&= \sum_{k=1}^{\infty}\Bigg((x_k^0)^2 e^{2\delta_k t}\Big(\int_0^{\vartheta} e^{2\delta_k s}\,ds\Big)^{-2} + (y_k^0)^2 e^{2\mu_k t}\Big(\int_0^{\vartheta} e^{2\mu_k s}\,ds\Big)^{-2} \\
&\qquad + (z_k^0)^2 e^{2\mu_k t}\Big(\int_0^{\vartheta} e^{2\mu_k s} ds\Big)^{-2}\Bigg) \\
&\le \sum_{k=1}^{\infty}(x_k^0)^2 e^{2\delta_k \vartheta}\Big(\int_0^{\vartheta} e^{2\delta_k s}\,ds\Big)^{-2} + (y_k^0)^2 e^{2\mu_k \vartheta}\Big(\int_0^{\vartheta} e^{2\mu_k s}\,ds\Big)^{-2} \\
&\qquad + (z_k^0)^2 e^{2\mu_k \vartheta}\Big(\int_0^{\vartheta} e^{2\mu_k s}\,ds\Big)^{-2} \\
&= \sum_{k=1}^{\infty}(\zeta_k^0)^T\gamma_k(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^0.
\end{aligned} \tag{3.6}$$

Now, for $\sum_{k=1}^{\infty}|\alpha_k^T(-t)\alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1|^2$, we use the fact that $t \in [0,\vartheta]$ to get that

$$\begin{aligned}
&\sum_{k=1}^{\infty} |\alpha_k^T(-t)\alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1|^2\,ds \\
&= \sum_{k=1}^{\infty} \Bigg((x_k^1)^2 e^{2\delta_k(t+\vartheta)} \Big(\int_0^{\vartheta} e^{2\delta_k s}\,ds \Big)^{-2} + (y_k^1)^2 e^{2\mu_k(t+\vartheta)} \Big(\int_0^{\vartheta} e^{2\mu_k s}\,ds \Big)^{-2} \\
&\qquad + (z_k^1)^2 e^{2\mu_k(t+\vartheta)} \Big(\int_0^{\vartheta} e^{2\mu_k s}\,ds \Big)^{-2} \Bigg) \\
&\le \sum_{k=1}^{\infty} \Bigg((x_k^1)^2 (e^{2\delta_k\vartheta})^2 \Big(\int_0^{\vartheta} e^{2\delta_k s}\,ds \Big)^{-2} + (y_k^1)^2 (e^{2\mu_k\vartheta})^2 \Big(\int_0^{\vartheta} e^{2\mu_k s}\,ds \Big)^{-2} \\
&\qquad + (z_k^1)^2 (e^{2\mu_k\vartheta})^2 \Big(\int_0^{\vartheta} e^{2\mu_k s} ds \Big)^{-2} \Bigg) \\
&= \sum_{k=1}^{\infty} (\zeta_k^1)^T \gamma_k^2(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^1. \qquad (3.7)
\end{aligned}$$

By substituting (3.6) and (3.7) into (3.5), we obtain

$$\sum_{k=1}^{\infty} |w_k(t)|^2 \le \sum_{k=1}^{\infty} 2(\zeta_k^0)^T \gamma_k(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^0 + 2(\zeta_k^1)^T \gamma_k^2(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^1 = \bar{\rho}^2.$$

Thus, the control function satisfies geometric constraints (2.5).

Show that the constructed control (3.4) transfers the state of the system (2.6) into state ζ^1. Indeed,

$$\begin{aligned}
\zeta_k(\vartheta) &= \alpha_k(\vartheta) \Bigg(\zeta_k^0 + \int_0^{\vartheta} \alpha_k(-s) w_k(s)\,ds \Bigg) \\
&= \alpha_k(\vartheta) \Bigg(\zeta_k^0 + \int_0^{\vartheta} \alpha_k(-s)(-\alpha_k^T(-s)\big(\beta_k^{-1}(0,\vartheta)\zeta_k^0 - \alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\big)\,ds \Bigg) \\
&= \alpha_k(\vartheta) \Bigg(\zeta_k^0 - \beta_k(0,\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^0 + \alpha_k(-\vartheta)\beta_k(0,\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1 \Bigg) \\
&= \alpha_k(\vartheta) \Big(\zeta_k^0 - \zeta_k^0 + \alpha_k(-\vartheta)\zeta_k^1 \Big) = \zeta_k^1.
\end{aligned}$$

This ends the proof. □

3.2. Construction of Pursuer's Strategy for Transferring States

We now extend the problem to study the pursuit differential game where the pursuer plays against the evader with motion defined by (2.1) with (2.9). The resource of pursuer is greater than that of evader, $\rho > \sigma$. The pursuer uses its admissible contructed strategy, and thus the solution of the system is expressed as follows:

$$\zeta_k(t) = \alpha_k(t)\zeta_k^0 + \int_0^t \alpha_k(t-s)\Big(- U_k(s, v_k(s)) + v_k(s)\Big)\, ds.$$

We consider the following equation:

$$\sum_{k=1}^{\infty} 2(\zeta_k^0)^T \gamma_k(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^0 + 2(\zeta_k^1)^T \gamma_k^2(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^1 = (\rho-\sigma)^2. \tag{3.8}$$

Theorem 3.2. *The time ϑ in equation* (3.8) *is guaranteed pursuit time for pursuit game* (2.1).

Proof. We first construct strategy for the pursuer in time interval $[0, \vartheta]$ as follows:

$$U_k(t, v_k(t)) = \alpha_k^T(-t)\big(\beta_k^{-1}(0,\vartheta)\zeta_k^0 - \alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\big) + v_k(t), \quad t \in [0,\vartheta], \tag{3.9}$$

with the fact that $v(\cdot) \in S(\sigma)$.

Show that the constructed strategy (3.9) satisfies geometric constraints. We use Minkowski inequality and get

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} |U_k(t, v_k(t))|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{\infty} \Big|\alpha_k^T(-t)\big(\beta_k^{-1}(0,\vartheta)\zeta_k^0 - \alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\big) + v_k(t)\Big|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{\infty} \Big|\alpha_k^T(-t)\big(\beta_k^{-1}(0,\vartheta)\zeta_k^0 - \alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\big)\Big|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} |v_k(t)|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{\infty} 2|\alpha_k^T(-t)\beta_k^{-1}(0,\vartheta)\zeta_k^0|^2 + 2|\alpha_k^T(-t)\alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1|^2\right)^{\frac{1}{2}} + \sigma \\
&= \left(\sum_{k=1}^{\infty} 2(\zeta_k^0)^T\gamma_k(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^0 + 2(\zeta_k^1)^T\gamma_k^2(\vartheta)\beta_k^{-2}(0,\vartheta)\zeta_k^1\right)^{\frac{1}{2}} + \sigma \\
&= \rho - \sigma + \sigma = \rho.
\end{aligned}$$

Prove that pursuer applies strategy (3.9) in the game that is

$$\begin{aligned}
\zeta_k(\vartheta) &= \alpha_k(\vartheta)\left(\zeta_k^0 + \int_0^\vartheta \alpha_k(-s)(-U_k(s, v_k(s)) + v_k(s))\, ds\right) \\
&= \alpha_k(\vartheta)\Bigg(\zeta_k^0 + \int_0^\vartheta \alpha_k(-s)\Big(- \alpha_k^T(-s)\big(\beta_k^{-1}(0,\vartheta)\zeta_k^0 + \alpha_k(-\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\big)\Big)\, ds \\
&\qquad\qquad - \int_0^\vartheta \alpha_{,k}(-s)v_k(s)\, ds + \int_0^\vartheta v_k(s)\, ds\Bigg) \\
&= \alpha_k(\vartheta)\Big(\zeta_k^0 - \beta_k(0,\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^0 + \alpha_k(-\vartheta)\beta_k(0,\vartheta)\beta_k^{-1}(0,\vartheta)\zeta_k^1\Big) \\
&= \alpha_k(\vartheta)(\alpha_k(-\vartheta)\zeta_k^1) = \zeta_k^1.
\end{aligned}$$

The pursuit is shown to be completed at time ϑ. This finishes the proof of the theorem. □

4. Conclusion

In the present research, we have investigated a pursuit differential game of a three-systems of differential equations in Hilbert space l_2. Geometric constraints are imposed on the players' control functions. We have figured out the control function required to bring the initial state ζ^0 of the system into another non zero state ζ^1 of the system at some finite time interval, hence solved the control problem. Moreover, we have proposed an equation used to obtain guaranteed pursuit time ϑ of the game, and developed a strategy for the pursuer. The strategy confirms that the state of the system coincides with state ζ^1 at time ϑ, thus we have proved the completion of pursuit in the game.

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