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Characteristic Polynomial of Power Graph for Dihedral Groups Using Degree-Based Matrices

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Abstract A fundamental feature of spectral graph theory is the correspondence between matrix and graph. As a result of this relation, the characteristic polynomial of the graph can be formulated. This research focuses on the power graph of dihedral groups using degree-based matrices. Throughout this paper, we formulate the characteristic polynomial of the power graph of dihedral groups based on seven types of graph matrices which include the maximum degree, the minimum degree, the greatest common divisor degree, the first Zagreb, the second Zagreb, the misbalance degree, and the Nirmala matrices.

Keywords: Characteristic polynomial, degree-based matrices, power graph, dihedral group.

Introduction

Spectral graph theory begins with a correspondence between matrices and graphs, most prominently the adjacency and Laplacian matrices. The main goals of spectral graph theory are to calculate or estimate the eigenvalues of these matrices, and to create connections between the eigenvalues and structural characteristics of the graph. It turns out that one useful tool when investigating graph theory includes the spectral perspective.

Finite group theory currently has a high level of activity in investigating graphs defined as the elements of a finite group. There has been extensive research on these graphs in the literature. A variety of graphs are associated with groups, for example, Cayley graph, commuting graph, coprime graph and power graph. A power graph of the group G is denoted by Γ_G and defined as a graph whose vertex set is all the elements of G and two distinct vertices v_p and v_q are adjacent if and only if $v_p^x = v_q$ or $v_q^y = v_p$ for positive integers x and y [5].

Recent research has focused on the power graph for some finite groups, as stated in [20]. It is demonstrated that a degree formula can be derived for a vertex in the finite abelian group. Different group has been discussed in [9], they focused on all nilpotent groups and presented the isomorphism of the power graph. The study of the power graph of dihedral groups was performed by [4]. In addition, Kumar *et al*. [10] have discussed a survey of the power graph for some finite groups. The fact that a power graph is always a divisor graph is presented in [22].

In the present work, we focus on the non-abelian dihedral groups of order $2n$, $n \geq 3$, denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ and its elements can be written as a^i and $a^i b$ [3]. Throughout this note, we are concerned with Γ_G of D_{2n} and denoted by $\Gamma_{D_{2n}}$. The spectral property of $\Gamma_{D_{2n}}$ can be expressed in

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adjacency matrix, $A(\Gamma_{D_{2n}}) = [a_{pq}]$ of size $2n \times 2n$, in which $a_{pq} = 1$, if v_p and v_q are adjacent, and otherwise, it is zero. The characteristic polynomial of $A(\Gamma_{D_{2n}})$, $P_{A(\Gamma_{D_{2n}})}(\lambda) = |\lambda I_{2n} - A(\Gamma_{D_{2n}})|$, where I_{2n} is the identity matrix of size $2n \times 2n$.

Many researchers are currently interested in studying the characteristic polynomial of graphs, for instance, the signless Laplacian polynomial for simple graphs [13] and the characteristic polynomial based on the Sombor matrix [14]. Moreover, the Laplacian spectrum of coprime order graph for finite abelian p-group has been presented in [21]. Meanwhile, a discussion of characteristic polynomials based on degree-based matrices applied to commuting and non-commuting graphs for D_{2n} can be seen in [15,16,17,18,19]. These results motivate us to investigate the characteristic polynomial of the power graph of the dihedral graph associated with degree-based matrices.

Degree-based matrices have been introduced that involve the degree formulas of every vertex in a graph. The definition of the maximum degree matrix of a graph can be found in [1]. A study of the minimum degree matrix was performed by [2]. The greatest common divisor degree matrix has been defined in [11]. Moreover, we can see in [12] that the Zagreb matrices are defined as two types of matrices. Meanwhile, some definitions have also been presented, for instance, the misbalance degree [6] and the Nirmala matrices [8].

The methodology involves the construction of degree-based matrices of $\Gamma_{D_{2n}}$, which include the maximum degree, the minimum degree, the greatest common divisor degree, the first Zagreb, the second Zagreb, the misbalance degree, and the Nirmala matrices. The next step is partitioning those matrices into block matrices and formulating the characteristic polynomial. Therefore, we need to present one theorem to simplify the characteristic polynomial of a particular matrix in the first main result.

Preliminaries

We study the power graph for D_{2n}, $\Gamma_{D_{2n}}$. Suppose that d_{v_i} as the degree of v_i. The following theorem presents d_{v_i}, for every v_i on $\Gamma_{D_{2n}}$.

Theorem 2.1. [4] Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}. Then
1. the degree of e on $\Gamma_{D_{2n}}$ is $d_e = 2n - 1$
2. the degree of a^i on $\Gamma_{D_{2n}}$ is $d_{a^i} = n - 1$, for $1 \leq i \leq n - 1$, and
3. the degree of $a^i b$ on $\Gamma_{D_{2n}}$ is $d_{a^i b} = 1$, $1 \leq i \leq n$.

Suppose that $G_1 = \{e\}$, $G_2 = \{a^i : 1 \leq i \leq n - 1\}$, and $G_3 = \{a^i b : 1 \leq i \leq n\}$ such that $D_{2n} = G_1 \cup G_2 \cup G_3$. It is shown that vertex e is adjacent to all other vertices in $\Gamma_{D_{2n}}$. Every vertex in G_2 is adjacent to e and all other members in G_2. Meanwhile, all vertices in G_3 are only adjacent to e [4].

The degree-based matrices are defined in the following definitions. They are the maximum degree, minimum degree, greatest common divisor degree, first Zagreb, second Zagreb, misbalance degree, and Nirmala matrices.

Definition 2.1. [1] The maximum matrix of $\Gamma_{D_{2n}}$, denoted by $Max(\Gamma_{D_{2n}}) = [max_{pq}]$ whose (p, q) –th entry is

$$max_{pq} = \begin{cases} max\{d_{v_p}, d_{v_q}\}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. [2] The minimum matrix of $\Gamma_{D_{2n}}$, denoted by $Min(\Gamma_{D_{2n}}) = [min_{pq}]$ whose (p, q) –th entry is

$$min_{pq} = \begin{cases} min\{d_{v_p}, d_{v_q}\}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. [11] The greatest common divisor degree matrix of $\Gamma_{D_{2n}}$, denoted by $GCDD(\Gamma_{D_{2n}}) = [g_{pq}]$ whose (p, q) –th entry is

$$g_{pq} = \begin{cases} g.c.d\{d_{v_p}, d_{v_q}\}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4. [12] The first Zagreb matrix of $\Gamma_{D_{2n}}$, denoted by $Z_1(\Gamma_{D_{2n}}) = \left[z_{1_{pq}}\right]$ whose (p,q) −th entry is

$$z_{1_{pq}} = \begin{cases} d_{v_p} + d_{v_q}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.5. [12] The second Zagreb matrix of $\Gamma_{D_{2n}}$, denoted by $Z_2(\Gamma_{D_{2n}}) = \left[z_{2_{pq}}\right]$ whose (p,q) −th entry is

$$z_{2_{pq}} = \begin{cases} d_{v_p} \cdot d_{v_q}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6. [6] The misbalance degree matrix of $\Gamma_{D_{2n}}$, denoted by $MD(\Gamma_{D_{2n}}) = \left[md_{pq}\right]$ whose (p,q) −th entry is

$$md_{pq} = \begin{cases} \left|d_{v_p} - d_{v_q}\right|, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7. [8] The Nirmala matrix of $\Gamma_{D_{2n}}$, denoted by $N(\Gamma_{D_{2n}}) = \left[n_{pq}\right]$ whose (p,q) −th entry is

$$n_{pq} = \begin{cases} \sqrt{d_{v_p} + d_{v_q}}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

To formulate the characteristic polynomial of degree-based matrices of $\Gamma_{D_{2n}}$, we need the following result.

Theorem 2.2. [7] If a square matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be partitioned into four blocks, where $|A| \neq 0$, then

$$|M| = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.$$

For the formulation of the characteristic polynomial, elementary row and column operations must be performed on a square matrix M. Now suppose that R_i be the i-th row and R_i' be the new i-th row resulting from a row operation on M. Also, let the i-th column as C_i and C_i' is the new i-th column obtained from a column operation of M.

Main Results

We begin with the simple form of the characteristic polynomial of a square matrix that very useful for our main results in this section.

Theorem 3.1. For a, b, c are real numbers, the characteristic polynomial of the $2n \times 2n$ matrix of

$$M = \begin{bmatrix} 0 & aJ_{1\times(n-1)} & bJ_{1\times n} \\ aJ_{(n-1)\times 1} & c(J-I)_{n-1} & 0_{(n-1)\times n} \\ bJ_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}$$

can be simplified as

$$P_M(\lambda) = \lambda^{n-1}(\lambda + c)^{n-2}\left(\lambda^3 - c(n-2)\lambda^2 - (b^2n + a^2(n-1))\lambda + b^2cn(n-2)\right).$$

Proof.
Suppose that M is a $2n \times 2n$ matrix as follows:

$$M = \begin{bmatrix} 0 & aJ_{1\times(n-1)} & bJ_{1\times n} \\ aJ_{(n-1)\times 1} & c(J-I)_{n-1} & 0_{(n-1)\times n} \\ bJ_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix} = \begin{bmatrix} 0 & a & a & \cdots & a & b & b & \cdots & b \\ a & 0 & c & \cdots & c & 0 & 0 & \cdots & 0 \\ a & c & 0 & \cdots & c & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a & c & c & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Below is a determinant that represents the characteristic polynomial for M,

$$P_M(\lambda)=\begin{vmatrix}\lambda & -aJ_{1\times(n-1)} & -bJ_{1\times n}\\ -aJ_{(n-1)\times 1} & (\lambda+c)I_{n-1}-cJ_{n-1} & 0_{(n-1)\times n}\\ -aJ_{n\times 1} & 0_{n\times(n-1)} & \lambda I_n\end{vmatrix}.$$

By applying the row and column operations:

1. $R'_{n+1+i}=R_{n+1+i}-R_{n+1}$, for $1\le i\le n-1$;
2. $C'_{n+1}=C_{n+1}+C_{n+1}+\cdots+C_{2n}$;
3. $C'_1=C_1+\frac{b}{\lambda}C_{n+1}$;
4. $R'_{2+i}=R_{2+i}-R_2$, for $1\le i\le n-2$;
5. $C'_2=C_2+C_3+\cdots+C_n$.

Then we obtain

$$P_M(\lambda)=\begin{vmatrix}\frac{\lambda^2-b^2n}{\lambda} & -a(n-1) & -aJ_{1\times(n-2)} & -bn & -bJ_{1\times(n-1)}\\ -a & \lambda-c(n-2) & -cJ_{1\times(n-2)} & 0 & 0_{1\times(n-1)}\\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda+c)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)}\\ 0 & 0 & 0_{1\times(n-2)} & \lambda & 0_{1\times(n-1)}\\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(n-2)} & 0_{(n-1)\times 1} & \lambda I_{n-1}\end{vmatrix}. \tag{7}$$

By Theorem 2.2 with $A=\begin{vmatrix}\frac{\lambda^2-b^2n}{\lambda} & -a(n-1)\\ -a & \lambda-c(n-2)\end{vmatrix}_{2\times 2}$, $B=\begin{vmatrix}-aJ_{1\times(n-2)} & -bn & -bJ_{1\times(n-1)}\\ -cJ_{1\times(n-2)} & 0 & 0_{1\times(n-1)}\end{vmatrix}_{2\times(2n-2)}$,

$C=0_{(2n-2)\times 2}$, $D=\begin{vmatrix}(\lambda+c)I_{n-2} & 0_{(n-2)\times n}\\ 0_{n\times(n-2)} & \lambda I_n\end{vmatrix}_{(2n-2)\times(2n-2)}$, then we have

$$\begin{aligned}P_M(\lambda)=|A||D|&=\begin{vmatrix}\dfrac{\lambda^2-b^2n}{\lambda} & -a(n-1)\\ -a & \lambda-c(n-2)\end{vmatrix}(\lambda+c)^{n-2}\lambda^n\\ &=\lambda^{n-1}(\lambda+c)^{n-2}\big(\lambda^3-c(n-2)\lambda^2-(b^2n+a^2(n-1))\lambda+b^2cn(n-2)\big).\end{aligned}$$

□

The following results present the characteristic polynomial of maximum degree, minimum degree, greatest common divisor degree, first Zagreb, second Zagreb, misbalance degree, and Nirmala matrices of $\Gamma_{D_{2n}}$.

Theorem 3.2. The characteristic polynomial of the maximum matrix of $\Gamma_{D_{2n}}$ is
$P_{Max(\Gamma_{D_{2n}})}(\lambda)=\lambda^{n-1}(\lambda+n-1)^{n-2}\big(\lambda^3-(n-2)(n-1)\lambda^2-(2n-1)^2\lambda+n(2n-1)^2(n-2)(n-1)\big)$.

Proof.
Let $\Gamma_{D_{2n}}$ be a power graph of $D_{2n}=G_1\cup G_2\cup G_3$, where $G_1=\{e\}$, $G_2=\{a,a^2,\cdots,a^{n-1}\}$, and $G_3=\{b,ab,a^2b,\cdots,a^{n-1}b\}$. We know that vertex e is adjacent to all other vertices in $\Gamma_{D_{2n}}$. Every vertex in G_2 is adjacent to e and all other members in G_2. Meanwhile, all vertices in G_3 are only adjacent to e. From Theorem 2.1 we have $d_e=2n-1$, $d_{a^i}=n-1$, $1\le i\le n-1$, and $d_{a^ib}=1$, $1\le i\le n$. Based on this information and Definition 2.1, the maximum matrix for $\Gamma_{D_{2n}}$ is a $2n\times 2n$ matrix as follows:

$$Max(\Gamma_{D_{2n}})=\begin{array}{cc} & \begin{array}{c:cccc:cccc} e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b\end{array}\\ \begin{array}{c} e\\ a\\ a^2\\ \vdots\\ a^{n-1}\\ b\\ ab\\ \vdots\\ a^{n-1}b\end{array} & \left[\begin{array}{c:cccc:cccc} 0 & 2n-1 & 2n-1 & \cdots & 2n-1 & 2n-1 & 2n-1 & \cdots & 2n-1\\ \hdashline 2n-1 & 0 & n-1 & \cdots & n-1 & 0 & 0 & \cdots & 0\\ 2n-1 & n-1 & 0 & \cdots & n-1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2n-1 & n-1 & n-1 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \hdashline 2n-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 2n-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2n-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\end{array}\right]\end{array}.$$

The maximum matrix of $\Gamma_{D_{2n}}$ can be expressed as the block matrix

$$Max(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & (2n-1)J_{1\times(n-1)} & (2n-1)J_{1\times n} \\ (2n-1)J_{(n-1)\times 1} & (n-1)(J-I)_{n-1} & 0_{(n-1)\times n} \\ (2n-1)J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

By Theorem 3.1 with $a = b = 2n-1$ and $c = n-1$, we derive the characteristic polynomial of $Max(\Gamma_{D_{2n}})$ as given below:

$$P_{Max(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+n-1)^{n-2}\left(\lambda^3-(n-2)(n-1)\lambda^2-(2n-1)^2\lambda+n(2n-1)^2(n-2)(n-1)\right).$$

□

Theorem 3.3. The characteristic polynomial of the minimum matrix of $\Gamma_{D_{2n}}$ is

$$P_{Min(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+n-1)^{n-2}\left(\lambda^3-(n-2)(n-1)\lambda^2-(n+(2n-1)^2(n-1))\lambda+n(n-2)(n-1)\right).$$

Proof.

The minimum matrix definition from Definition 2.2 gives the minimum matrix for $\Gamma_{D_{2n}}$ as a $2n\times 2n$ matrix:

$$Min(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{c|cccc|cccc} 0 & n-1 & n-1 & \cdots & n-1 & 1 & 1 & \cdots & 1 \\ \hline n-1 & 0 & n-1 & \cdots & n-1 & 0 & 0 & \cdots & 0 \\ n-1 & n-1 & 0 & \cdots & n-1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & n-1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

The minimum matrix of $\Gamma_{D_{2n}}$ can be stated as the partition matrix

$$Min(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & (n-1)J_{1\times(n-1)} & J_{1\times n} \\ (n-1)J_{(n-1)\times 1} & (n-1)(J-I)_{n-1} & 0_{(n-1)\times n} \\ J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

By using Theorem 3.1 with $a = n-1, b = 1$, and $c = n-1$, therefore, we obtain:

$$P_{Min(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+n-1)^{n-2}\left(\lambda^3-(n-2)(n-1)\lambda^2-(n+(n-1)^3)\lambda+n(n-2)(n-1)\right).$$

□

Theorem 3.4. The characteristic polynomial of the greatest common divisor degree matrix of $\Gamma_{D_{2n}}$ is

$$P_{GCDD(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+n-1)^{n-2}\left(\lambda^3-(n-2)(n-1)\lambda^2-(2n-1)\lambda+n(n-2)(n-1)\right).$$

Proof.

According to Definition 3.3, the greatest common divisor degree matrix for $\Gamma_{D_{2n}}$ is a $2n\times 2n$ matrix as follows:

$$GCDD(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{c|cccc|cccc} 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \hline 1 & 0 & n-1 & \cdots & n-1 & 0 & 0 & \cdots & 0 \\ 1 & n-1 & 0 & \cdots & n-1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n-1 & n-1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

The greatest common divisor degree matrix of $\Gamma_{D_{2n}}$ can be expressed as

$$GCDD(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & (n-1)(J-I)_{n-1} & 0_{(n-1)\times n} \\ J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

Therefore, by Theorem 3.1 with $a = b = 1$, and $c = n-1$, we derive the following formula:

$$P_{GCDD(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+n-1)^{n-2}\left(\lambda^3-(n-2)(n-1)\lambda^2-(2n-1)\lambda+n(n-2)(n-1)\right).$$

□

Theorem 3.5. The characteristic polynomial of the first Zagreb matrix of $\Gamma_{D_{2n}}$ is

$$P_{Z_1(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda + 2n - 2)^{n-2}\left(\lambda^3 - 2(n-2)(n-1)\lambda^2 - (4n^3 + (3n-2)^2(n-1))\lambda + 8n^3(n-2)(n-1)\right).$$

Proof.

By Definition 2.4, we can construct he first Zagreb matrix for $\Gamma_{D_{2n}}$ as a $2n \times 2n$ matrix as follows:

$$Z_1(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{c:cccc:cccc} e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{c:cccc:cccc} 0 & 3n-2 & 3n-2 & \cdots & 3n-2 & 2n & 2n & \cdots & 2n \\ \hdashline 3n-2 & 0 & 2n-2 & \cdots & 2n-2 & 0 & 0 & \cdots & 0 \\ 3n-2 & 2n-2 & 0 & \cdots & 2n-2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n-2 & 2n-2 & 2n-2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hdashline 2n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 2n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

The first Zagreb matrix of $\Gamma_{D_{2n}}$ can be partitioned into block matrices

$$Z_1(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & (3n-2)J_{1\times(n-1)} & 2nJ_{1\times n} \\ (3n-2)J_{(n-1)\times 1} & (2n-2)(J-I)_{n-1} & 0_{(n-1)\times n} \\ 2nJ_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

By using Theorem 3.1 with $a = 3n-2$, $b = 2n$, and $c = 2n-2$, thus we get:

$$P_{Z_1(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda + 2n - 2)^{n-2}\left(\lambda^3 - 2(n-2)(n-1)\lambda^2 - (4n^3 + (3n-2)^2(n-1))\lambda + 8n^3(n-2)(n-1)\right).$$

□

Theorem 3.6. The characteristic polynomial of the second Zagreb matrix of $\Gamma_{D_{2n}}$ is

$$P_{Z_2(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda + (n-1)^2)^{n-2}\left(\lambda^3 - (n-2)(n-1)^2\lambda^2 - ((2n-1)^2(n + (n-1)^3))\lambda + n(2n-1)^2(n-1)^2(n-2)\right).$$

Proof.

By using Definition 2.5, the second Zagreb matrix for $\Gamma_{D_{2n}}$ is a $2n \times 2n$ matrix as follows:

$$Z_2(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{c:cccc:cccc} e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{c:cccc:cccc} 0 & (2n-1)(n-1) & (2n-1)(n-1) & \cdots & (2n-1)(n-1) & 2n-1 & 2n-1 & \cdots & 2n-1 \\ \hdashline (2n-1)(n-1) & 0 & (n-1)^2 & \cdots & (n-1)^2 & 0 & 0 & \cdots & 0 \\ (2n-1)(n-1) & (n-1)^2 & 0 & \cdots & (n-1)^2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (2n-1)(n-1) & (n-1)^2 & (n-1)^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hdashline 2n-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 2n-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

The next step is to express the second Zagreb matrix of $\Gamma_{D_{2n}}$as the block matrix as given below:

$$Z_2(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & (2n-1)(n-1)J_{1\times(n-1)} & (2n-1)J_{1\times n} \\ (2n-1)(n-1)J_{(n-1)\times 1} & (n-1)^2(J-I)_{n-1} & 0_{(n-1)\times n} \\ (2n-1)J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

Consequently, by applying Theorem 2.1 with $a = (2n-1)(n-1)$, $b = 2n-1$, and $c = (n-1)^2$, we can obtain:

$$P_{Z_2(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda + (n-1)^2)^{n-2}\left(\lambda^3 - (n-2)(n-1)^2\lambda^2 - ((2n-1)^2(n + (n-1)^3))\lambda + n(2n-1)^2(n-1)^2(n-2)\right).$$

□

Theorem 3.7. The characteristic polynomial of the misbalance degree matrix of $\Gamma_{D_{2n}}$ is

$$P_{MD(\Gamma_{D_{2n}})}(\lambda) = \lambda^{2n-2}(\lambda^2 - n(n-1)(5n-4)).$$

Proof.

The misbalance degree matrix for $\Gamma_{D_{2n}}$ is a $2n \times 2n$ matrix can be provided following Definition 3.6. It is

$$MD(\Gamma_{D_{2n}}) = \begin{array}{c|c|cccc|cccc} & e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \\ \hline e & 0 & n & n & \cdots & n & 2n-2 & 2n-2 & \cdots & 2n-2 \\ \hline a & n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a^2 & n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline b & 2n-2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ ab & 2n-2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & 2n-2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}.$$

The misbalance degree matrix of $\Gamma_{D_{2n}}$ can be expressed as the block matrix

$$MD(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & nJ_{1\times(n-1)} & (2n-2)J_{1\times n} \\ nJ_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ (2n-2)J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

By Theorem 3.1 with $a = n$, $b = 2n-2$ and $c = 0$, we get the following formua:

$$P_{MD(\Gamma_{D_{2n}})}(\lambda) = \lambda^{2n-2}(\lambda^2 - n(n-1)(5n-4)).$$

□

Theorem 3.8. The characteristic polynomial of the Nirmala matrix of $\Gamma_{D_{2n}}$ is

$$P_{N(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda + \sqrt{2n-2})^{n-2}\left(\lambda^3 - \sqrt{2n-2}(n-2)\lambda^2 - (2n^2 + (3n-2)(n-1))\lambda + 2n^2\sqrt{2n-2}(n-2)\right).$$

Proof.
According to Definition 2.7, we obtain the Nirmala matrix for $\Gamma_{D_{2n}}$ as a $2n \times 2n$ matrix:

$$N(\Gamma_{D_{2n}}) = \begin{array}{c|c|cccc|cccc} & e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \\ \hline e & 0 & \sqrt{3n-2} & \sqrt{3n-2} & \cdots & \sqrt{3n-2} & \sqrt{2n} & \sqrt{2n} & \cdots & \sqrt{2n} \\ \hline a & \sqrt{3n-2} & 0 & \sqrt{2n-2} & \cdots & \sqrt{2n-2} & 0 & 0 & \cdots & 0 \\ a^2 & \sqrt{3n-2} & \sqrt{2n-2} & 0 & \cdots & \sqrt{2n-2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & \sqrt{3n-2} & \sqrt{2n-2} & \sqrt{2n-2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline b & \sqrt{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ ab & \sqrt{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & \sqrt{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}.$$

The Nirmala matrix of $\Gamma_{D_{2n}}$ can be expressed as the following block matrices

$$N(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & \sqrt{3n-2}J_{1\times(n-1)} & \sqrt{2n}J_{1\times n} \\ \sqrt{3n-2}J_{(n-1)\times 1} & \sqrt{2n-2}(J-I)_{n-1} & 0_{(n-1)\times n} \\ \sqrt{2n}J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

Hence, by Theorem 3.1 with $a = \sqrt{3n-2}$, $b = \sqrt{2n}$ and $c = \sqrt{2n-2}$, we get

$$P_{N(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda + \sqrt{2n-2})^{n-2}\left(\lambda^3 - \sqrt{2n-2}(n-2)\lambda^2 - (2n^2 + (3n-2)(n-1))\lambda + 2n^2\sqrt{2n-2}(n-2)\right).$$

□

Conclusion

In this note, we present the characteristic polynomial of a square matrix to simplify the process for formulating the determinant. We have shown the characteristic polynomial of the power graph of D_{2n}, $n \geq 3$, based on the maximum degree, the minimum degree, the greatest common divisor degree, the first Zagreb, the second Zagreb, the misbalance degree, and the Nirmala matrices.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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