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Fractional Block Method for the Solution of Fractional Order Differential Equations

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Abstract

The construction of the fourth-order 2-point Fractional Block Backward Differentiation Formula (2FBBDF(4)) to solve the fractional order differential equations (FDEs) is presented in this paper. The method is developed using the fractional linear multistep method (FLMM) linked with the linear difference operator. This paper aims to approximate the fractional order problems via 2FBBDF(4), which is normally used to solve ordinary differential equations. The criteria for the stability of the method are analyzed in order to solve FDE problems. Consequently, the method is determined to be A-stable for different values of α within the interval $(0, 1)$. Then, Newton's iteration is implemented in this method to solve the problems. Multiple numerical examples of linear, nonlinear, and system FDEs are provided for the scenario where the order α lies within the range of 0 and 1. Ultimately, the numerical results confirm that the proposed method performs at a similar level to the existing methods.

Keywords: linear multistep method; fractional block method; fractional order; single order FDEs; stability.

1 Introduction

Fractional calculus is a branch of mathematics that deals with the study and applications of arbitrary order integrals and derivatives. Fractional calculus and fractional differential equations (FDEs) have recently been used in a variety of real-world applications, such as financial markets, insurance, epidemiology, biological reactive system, drugs concentration, and other fields of science and technology [3]. In this article, we examine the initial condition of FDEs expressed as:

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha} y(t) &= f(t, y(t)), \qquad t \geq t_0, \\ y(t_0) &= y_0, \end{aligned} \tag{1}$$

where $0 < \alpha < 1$ is the fractional order. The symbol ${}^{C}D_{t_0}^{\alpha}$ is used to denote as the Caputo α derivative operator which is defined as,

$${}^{C}D_{t_0}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y'(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function [6].

According to the research done by Garrappa [13], ${}^{C}D_{t_0}^{\alpha} y(t) = {}^{RL}D_{t_0}^{\alpha}(y(t) - y(t_0))$ where ${}^{RL}D_{t_0}^{\alpha}(y(t))$ where the expression is the Riemann-Liouville derivative operator which can be defined as,

$${}^{RL}D_{t_0}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_{t_0}^{t} \frac{y(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{3}$$

Let us assume that the function $f(t, y(t))$ fulfills the Lipschitz conditions required for the existence and uniqueness of the solution of Equation (1), as explained in [10].

In literature, Diethelm and Ford [10] discussed the existence, uniqueness, and structural stability of solutions to nonlinear FDEs. Researchers have studied various numerical techniques for solving FDEs. Galeone and Garrappa [12] investigated the explicit and implicit approaches of the fractional Backward Differentiation Formula (BDF) for solving FDEs. Rehman and Khan [29] introduced the Legendre wavelet method as an approach to estimate the solution of FDEs. Tong et al. [31] have demonstrated the efficacy of the Euler numerical method in approximating solutions for ordinary differential equations (ODEs) and other types of equations. Therefore, they enhanced the classical Euler's method by developing the improved Euler's method, which is used to solve functional differential equations [1, 33]. In 2015, Biala and Jator [4, 5] introduced the k-step Implicit Adams Methods (IAMs) and the k-step Continuous Fractional BDF method. These methods utilise derivatives of the Caputo type to approximate the FIVP (1).

Zabidi et al. [34] developed the fractional linear multistep method (FLMM) by applying the principles of Adams Methods to solve fractional differential equations (FDEs). Hattaf [16, 17] introduced a novel mixed fractional derivative that incorporates both singular and non-singular kernels. This derivative is utilised to solve FDEs. In recent times, there has been an increased focus on solving FDEs in the field of research [11, 32] because of the presence of stiff problems. Thus, it was imperative to develop efficient techniques for solving stiff systems of FDEs [2]. Comparatively, FDEs are more practical for modelling application problems than integer order equations. This is because FDEs allow for the description of memory effects, as stated in [15]. Mathematical models that employ FDEs include the Prey-Predator Model [14], the SIR Model [15], the Corona Virus Disease Model [28], the HIV Model [26], and the Pharmacokinetics Model[27].

According to Lambert's research study [22], the implicit method is more precise than the explicit method when it comes to solving stiff problems. Consequently, numerous research studies on the numerical solution of stiff systems have been published, with the implicit linear multistep method being commonly used. The block method of the BDF method is the widely recognised as the most popular numerical method used to solve stiff Ordinary Differential Equations (ODEs), as opposed to the Euler and Adam-Moulton methods. Therefore, the objective of this paper is to address the FDEs by adapting the BBDF method, which has demonstrated efficacy in solving stiff ODE problems [19]. The 2-point BBDF was first introduced by Ibrahim et al. [19], where the r-point BBDF is developed for solving first-order ODEs. In 2013, Suleiman et al. [30] studied the 2-point BBDF by increasing the order of the method from third order to fourth order for solving stiff ODEs. Musa et al. [25] introduced a novel block method that incorporates a free parameter to regulate the stability of the method. The 2-point BBDF method was adapted to a 3-point BBDF method for the purpose of solving stiff ODEs. Presently, multiple investigations are being conducted on the BBDF technique for resolving stiff ODEs [24, 8]. Due to the widespread popularity and demonstrated effectiveness of the BBDF method in solving ODEs, we are motivated to explore its potential for solving FDEs by modifying the BBDF method. Moreover, up until now, there have been no studies that have employed the BBDF method for solving the FDEs.

There has been a lot of focus on developing FBDF for the numerical approximations of FDEs over the past 10 years. This is because FDEs occur in several models. The FBDF methods were proposed and investigated in [7, 18, 35]. Most of the methods have been developed using convolution quadratures independently, as proposed by Lubich [23] in the form:

$$y_n = f(t_n) + h^\alpha \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} g\left(t_j, y_j\right) + h^\alpha \sum_{j=0}^{n} \omega_{nj}^{(\alpha)} g\left(t_j, y_j\right), \tag{4}$$

where h is the step size, $\omega_{n-j}^{(\alpha)}$ is the convolution weight, and $\omega_{nj}^{(\alpha)}$ is the starting quadrature weights.

One of the most challenging aspects of formulating the proposed method is determining the convolution weights for the initial point, as it requires consideration of a future point. Hence, inspired by the prior studies, our objective is to develop a novel numerical technique for estimating the solution of FDEs using an alternative approach other than the Lubich's method (4). The numerical method is derived from the FLMM by Galeone and Garrappa [12], and it relies on the utilization of Taylor's series expansion. This numerical method is limited in its ability to approximate the solution for fractional orders, α in between 0 and 1 only.

The paper is structured in the following manner: In Section 2, we will outline the derivation of the proposed method by building upon the concept of the BBDF method. In Section 3, we will examine the stability properties of the derived method to confirm its convergence and stability. The utilization of the Newton's iteration technique is succinctly explained in Section 4. Subsequently, we provide numerical illustrations to clarify our theoretical findings, and the examination of these results will be presented in Section 5. Finally, Section 6 presents the conclusion of this paper.

2 Derivation of the Method

The construction of the fourth-order 2-point Fractional Block Backward Differentiation Formula (2FBBDF(4)) is demonstrated in this section. The method is devised with a fixed step size, h, and involves using three back points, x_{n-2}, x_{n-1} and x_n to estimate the values of y_{n+1} and y_{n+2} at point x_{n+1} and x_{n+2} simultaneously. Then, the method is constructed using the general formula of FLMM by Galeone and Garrappa [12],

$$\sum_{j=0}^{n} \gamma_j y_{n-j} = h^{\alpha} \sum_{i=0}^{n} \beta_i f\left(t_{n-j}, y_{n-j}\right), \tag{5}$$

and the classical BBDF by Ijam et al. [20] in the form of,

$$\sum_{j=0}^{4} \gamma_{j,i} y_{n+j-2} = h^{\alpha} \beta_i f_{n+i}, \tag{6}$$

where $\gamma_{j,i}$ and β_i are the real parameters, h^{α} is the step length, and $i = 1, 2$ for y_{n+1} and y_{n+2}, respectively. The procedure of the derivation will involve the FLMM (6) associated with the linear difference operator L_h [12], which is defined by,

$$\begin{aligned} L_h\left[y(t), t, \alpha\right] &= \sum_{j=0}^{4} \gamma_{j,i} y_{n+j-2} - h^{\alpha} \beta_i f_{n+i} \\ &= \sum_{j=0}^{4} \gamma_{j,i} y_{n+j-2} - h^{\alpha} \beta_i^C D_{t_0}^{\alpha} y_{n+i} \\ &= \gamma_{0,i} y_{n-2} + \gamma_{1,i} y_{n-1} + \gamma_{2,i} y_n + \gamma_{3,i} y_{n+1} + \gamma_{4,i} y_{n+2} - h^{\alpha} \beta_i^C D_{t_0}^{\alpha} y_{n+i} \\ &= 0, \end{aligned} \tag{7}$$

where $i = 1, 2$. Expanding y_{n-2}, y_{n-1}, y_n, y_{n+1}, y_{n+2}, ${}^C D_{t_0}^{\alpha} y_{n+1}$, and ${}^C D_{t_0}^{\alpha} y_{n+2}$ using Taylor's series expansion [12] about t_n in Equation (7) are given below

$$\begin{aligned} y_{n-2} &= y_n + (-2h) y_n' + \frac{(-2h)^2}{2!} y_n'' + \frac{(-2h)^3}{3!} y_n''' + \frac{(-2h)^4}{4!} y_n^{(4)} + \cdots, \\ y_{n-1} &= y_n + (-h) y_n' + \frac{(-h)^2}{2!} y_n'' + \frac{(-h)^3}{3!} y_n''' + \frac{(-h)^4}{4!} y_n^{(4)} + \cdots, \\ y_n &= y_n, \\ y_{n+1} &= y_n + h y_n' + \frac{(h)^2}{2!} y_n'' + \frac{(h)^3}{3!} y_n''' + \frac{(h)^4}{4!} y_n^{(4)} + \cdots, \\ y_{n+2} &= y_n + (2h) y' t_n + \frac{(2h)^2}{2!} y_n'' + \frac{(2h)^3}{3!} y_n''' + \frac{(2h)^4}{4!} y_n^{(4)} + \cdots, \\ {}^C D_{t_0}^{\alpha} y_{n+1} &= \frac{(h)^{1-\alpha}}{\Gamma(2-\alpha)} y_n' + \frac{(h)^{2-\alpha}}{\Gamma(3-\alpha)} y_n'' + \frac{(h)^{3-\alpha}}{\Gamma(4-\alpha)} y_n''' + \frac{(h)^{4-\alpha}}{\Gamma(5-\alpha)} y_n^{(4)} + \cdots, \\ {}^C D_{t_0}^{\alpha} y_{n+2} &= \frac{(2h)^{1-\alpha}}{\Gamma(2-\alpha)} y_n' + \frac{(2h)^{2-\alpha}}{\Gamma(3-\alpha)} y_n'' + \frac{(2h)^{3-\alpha}}{\Gamma(4-\alpha)} y_n''' + \frac{(2h)^{4-\alpha}}{\Gamma(5-\alpha)} y_n^{(4)} + \cdots. \end{aligned} \tag{8}$$

To obtain the formula for the first point, y_{n+1}, we substitute Equation (8) into Equation (7) when $i = 1$;

$$\begin{aligned}
&\gamma_{0,1}\left[y_n + (-2h)y'_n + \frac{(-2h)^2}{2!}y''_n + \frac{(-2h)^3}{3!}y'''_n + \frac{(-2h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&+ \gamma_{1,1}\left[y_n + (-h)y'_n + \frac{(-h)^2}{2!}y''_n + \frac{(-h)^3}{3!}y'''_n + \frac{(-h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&+ \gamma_{2,1}\,(y_n) + \gamma_{3,1}\left[y_n + hy'_n + \frac{(h)^2}{2!}y''_n + \frac{(h)^3}{3!}y'''_n + \frac{(h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&+ \gamma_{4,1}\left[y_n + (2h)y't_n + \frac{(2h)^2}{2!}y''_n + \frac{(2h)^3}{3!}y'''_n + \frac{(2h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&- h^\alpha\beta_1\left[\frac{(h)^{1-\alpha}}{\Gamma(2-\alpha)}y'_n + \frac{(h)^{2-\alpha}}{\Gamma(3-\alpha)}y''_n + \frac{(h)^{3-\alpha}}{\Gamma(4-\alpha)}y'''_n + \frac{(h)^{4-\alpha}}{\Gamma(5-\alpha)}y_n^{(4)} + \cdots\right] = 0.
\end{aligned} \tag{9}$$

To eliminate the arbitrary nature of the coefficients, we then normalize the value of $\gamma_{3,1} = 1$, which result in,

$$\begin{aligned}
&\gamma_{0,1}\left[y_n + (-2h)y'_n + \frac{(-2h)^2}{2!}y''_n + \frac{(-2h)^3}{3!}y'''_n + \frac{(-2h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&+ \gamma_{1,1}\left[y_n + (-h)y'_n + \frac{(-h)^2}{2!}y''_n + \frac{(-h)^3}{3!}y'''_n + \frac{(-h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&+ \gamma_{2,1}\,(y_n) + (1)\left[y_n + hy'_n + \frac{(h)^2}{2!}y''_n + \frac{(h)^3}{3!}y'''_n + \frac{(h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&+ \gamma_{4,1}\left[y_n + (2h)y't_n + \frac{(2h)^2}{2!}y''_n + \frac{(2h)^3}{3!}y'''_n + \frac{(2h)^4}{4!}y_n^{(4)} + \cdots\right] \\
&- h^\alpha\beta_1\left[\frac{(h)^{1-\alpha}}{\Gamma(2-\alpha)}y'_n + \frac{(h)^{2-\alpha}}{\Gamma(3-\alpha)}y''_n + \frac{(h)^{3-\alpha}}{\Gamma(4-\alpha)}y'''_n + \frac{(h)^{4-\alpha}}{\Gamma(5-\alpha)}y_n^{(4)} + \cdots\right] = 0.
\end{aligned} \tag{10}$$

Then, by factorizing, we gather all of the coefficients of y_n, y'_n, y''_n, y'''_n, $\ldots$ in Equation (10) and collecting terms gives,

$$L_h\left[y(t), t, \alpha\right] = C_{0,1}y_n + \sum_{k=1}^{m} h^k C_{k,1} y_n^{(k)} + h^{m+1}R_{m+1}, \qquad k = 1, 2, 3, \ldots, \tag{11}$$

where the remainder R_{m+1} is derived from Taylor's expansions and the constant;

$$\begin{aligned}
C_{0,1} &:= \gamma_{0,1} + \gamma_{1,1} + \gamma_{2,1} + 1 + \gamma_{4,1} = 0, \\
C_{1,1} &:= -2\gamma_{0,1} - \gamma_{1,1} + 1 + 2\gamma_{4,1} - \left(\frac{(1)^{1-\alpha}}{\Gamma(2-\alpha)}\right)\beta_1 = 0, \\
C_{2,1} &:= \frac{(-2)^2}{2!}\gamma_{0,1} + \frac{(-1)^2}{2!}\gamma_{1,1} + \frac{(1)^2}{2!} + \frac{(2)^2}{2!}\gamma_{4,1} - \left(\frac{(1)^{2-\alpha}}{\Gamma(3-\alpha)}\right)\beta_1 = 0, \\
C_{3,1} &:= \frac{(-2)^3}{3!}\gamma_{0,1} + \frac{(-1)^3}{3!}\gamma_{1,1} + \frac{(1)^3}{3!} + \frac{(2)^3}{3!}\gamma_{4,1} - \left(\frac{(1)^{3-\alpha}}{\Gamma(4-\alpha)}\right)\beta_1 = 0, \\
C_{4,1} &:= \frac{(-2)^4}{4!}\gamma_{0,1} + \frac{(-1)^4}{4!}\gamma_{1,1} + \frac{(1)^4}{4!} + \frac{(2)^4}{4!}\gamma_{4,1} - \left(\frac{(1)^{4-\alpha}}{\Gamma(5-\alpha)}\right)\beta_1 = 0.
\end{aligned} \tag{12}$$

The systems of (12) are solved simultaneously to obtain the coefficient values of $\gamma_{0,1}$, $\gamma_{1,1}$, $\gamma_{2,1}$, $\gamma_{4,1}$ and β_1.

$$\begin{aligned}
\gamma_{0,1} &= \frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}, \qquad \gamma_{1,1} = -\frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72}, \\
\gamma_{2,1} &= \frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}, \qquad \gamma_{4,1} = -\frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}, \\
\beta_1 &= \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}.
\end{aligned} \tag{13}$$

Next, we substitute all the values in Equation (13) into Equation (7), we get

$$\begin{aligned}
&\frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n-2} - \frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72}y_{n-1} \\
&\quad + \frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_n + y_{n+1} - \frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n+2} \\
&\quad + \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}h^{\alpha C}D^{\alpha}_{t_0}y_{n+1} = 0.
\end{aligned} \tag{14}$$

The approximation solution for the first point of the method, y_{n+1}, is then obtained by rearranging Equation (14). The same procedure is applied to obtain the coefficient values for the second point, y_{n+2}, when $i=2$ and $\gamma_{4,2}=1$. Hence, the general corrector formula of the 2FBBDF(4) method is obtained as follows:

$$\begin{aligned}
y_{n+1} &= -\frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n-2} + \frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72}y_{n-1} \\
&\quad - \frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_n + \frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n+2} \\
&\quad - \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}h^{\alpha C}D^{\alpha}_{t_0}y_{n+1}, \\
y_{n+2} &= \frac{\alpha\left(\alpha^2-7\alpha-12\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-2} - \frac{8\alpha\left(\alpha^2-5\alpha-8\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1} \\
&\quad + \frac{12\left(5\alpha^2-35\alpha+12\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_n + \frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n+1} \\
&\quad + \frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3-11\alpha^2+16\alpha+144}h^{\alpha C}D^{\alpha}_{t_0}y_{n+2}.
\end{aligned} \tag{15}$$

3 Stability and Its Properties

This section examines the convergence and stability of method (15) as the fractional order, α, varies between 0 and 1. This paper primarily examines the values of α, specifically 0.7, 0.8, 0.9, and 1.0. Hence, the following definitions will be considered in the analysis of the method.

Theorem 3.1. [22] *The method (5) is convergent if and only if it satisfies both consistency and zero stability.*

Definition 3.1. [22] *The FLMM* (5) *is considered consistent if its order is greater than or equal to* p*, where* $p \geq 1$.

Definition 3.2. [12, 22] *The difference operator* (8) *and the corresponding FLMM* (5) *are considered to have an order of* p *if, in* (12)*,* $C_0 = C_1 = \cdots = C_p = 0$, $C_{p+1} \neq 0$ *where,*

$$\begin{aligned} C_0(n,\alpha) &= \sum_{j=0}^{n} \gamma_j, \\ C_p(n,\alpha) &= \frac{1}{p!}\sum_{j=0}^{n}(j-2)^p\gamma_j - \frac{1}{\Gamma(p+1-\alpha)}\sum_{j=0}^{n}(j-2)^{p-\alpha}\beta_j, \qquad p = 2,3,\ldots. \end{aligned} \tag{16}$$

Next, we will demonstrate the procedure for determining the order of the method in equation (6) when $n = 4$;

$$\begin{aligned} C_0(4,\alpha) &= \sum_{j=0}^{4}\gamma_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_1(4,\alpha) &= \sum_{j=0}^{4}(j-2)\gamma_j - \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n}(j-2)^{1-\alpha}\beta_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_2(4,\alpha) &= \frac{1}{2!}\sum_{j=0}^{4}(j-2)^2\gamma_j - \frac{1}{\Gamma(3-\alpha)}\sum_{j=0}^{n}(j-2)^{2-\alpha}\beta_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_3(4,\alpha) &= \frac{1}{3!}\sum_{j=0}^{4}(j-2)^3\gamma_j - \frac{1}{\Gamma(4-\alpha)}\sum_{j=0}^{n}(j-2)^{3-\alpha}\beta_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_4(4,\alpha) &= \frac{1}{4!}\sum_{j=0}^{4}(j-2)^4\gamma_j - \frac{1}{\Gamma(5-\alpha)}\sum_{j=0}^{n}(j-2)^{4-\alpha}\beta_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_5(4,\alpha) &= \frac{1}{5!}\sum_{j=0}^{4}(j-2)^5\gamma_j - \frac{1}{\Gamma(6-\alpha)}\sum_{j=0}^{n}(j-2)^{5-\alpha}\beta_j = \begin{bmatrix}e_1\\e_2\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}. \end{aligned} \tag{17}$$

According to the computation in Equation (17), it has been proven that the method is of fourth order due to the presence of the error constant at C_5. Therefore, the method (15) is consistent (Refer Definition 3.1), and the table below displays the error constants for each value of α.

Table 1: Error constant for the method (15) for $\alpha = 0.7, 0.8, 0.9$, and 1.0.

α	e_1	e_2
0.7	$\frac{113323}{4041140}$	$-\frac{644854}{10760965}$
0.8	$\frac{3679}{102620}$	$-\frac{1462}{20545}$
0.9	$\frac{39961}{868380}$	$-\frac{284578}{3421655}$
1.0	$\frac{3}{50}$	$-\frac{12}{125}$

Definition 3.3. [22] *The FLMM (5) is considered to be zero stable when all roots of the initial characteristics of the polynomial have moduli equal to or greater than one, and each root with a modulus of one is a simple root.*

The achievement of perfect stability is the fundamental challenge in the numerical solution of stiff systems. In this section, the stability properties of the proposed method will be investigated by considering the following linear test problem [34];

$$\begin{aligned} {}^{C}D^{\alpha}y(t) &= \lambda y(t), \qquad \lambda \in \mathbb{C}, \\ y(t_0) &= y_0, \end{aligned} \tag{18}$$

where λ is the eigenvalue, the true solution is denoted by $y(t) = E_\alpha\left(\lambda\left(t-t_0\right)^\alpha\right)y_0$, and $E_\alpha(\cdot)$ represents the Mittag-Leffler function.

Definition 3.4. [21] *The Mittag-Leffler function is formally defined by the following expression:*

$$E_\alpha(t) = \sum_{k=0}^{\infty}\left(\frac{t^k}{\Gamma(\alpha k+1)}\right), \qquad \alpha \in \mathbb{C}. \tag{19}$$

$E_\alpha(t)$ will be transformed into e^t when $\alpha = 1$.

Equation (18) is then substituted into the numerical method in Equation (15), yielding the following form;

$$\begin{aligned} y_{n+1} &= -\frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n-2} + \frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72}y_{n-1} \\ &\quad - \frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_n + \frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n+2} \\ &\quad - \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}h^\alpha\lambda y_{n+1}, \\ y_{n+2} &= \frac{\alpha\left(\alpha^2-7\alpha-12\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-2} - \frac{8\alpha\left(\alpha^2-5\alpha-8\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1} \\ &\quad + \frac{12\left(5\alpha^2-35\alpha+12\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_n + \frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n+1} \\ &\quad + \frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3-11\alpha^2+16\alpha+144}h^\alpha\lambda y_{n+2}. \end{aligned} \tag{20}$$

By letting the real values $h^{\alpha}\lambda = \bar{h}$ equation (20) will be,

$$
\begin{aligned}
y_{n+1} &= -\frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n-2} + \frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72}y_{n-1} \\
&\quad - \frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_n + \frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)}y_{n+2} \\
&\quad - \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}\bar{h}y_{n+1}, \\
y_{n+2} &= \frac{\alpha\left(\alpha^2-7\alpha-12\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-2} - \frac{8\alpha\left(\alpha^2-5\alpha-8\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1} \\
&\quad + \frac{12\left(5\alpha^2-35\alpha+12\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_n + \frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n+1} \\
&\quad + \frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3-11\alpha^2+16\alpha+144}\bar{h}y_{n+2}.
\end{aligned}
\tag{21}
$$

Rearranging equation (21) into a matrix form yields,

$$
\begin{aligned}
&\begin{bmatrix} 1+\frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}\bar{h} & -\frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ -\frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144} & 1-\frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3-11\alpha^2+16\alpha+144}\bar{h} \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72} & -\frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ -\frac{8\alpha\left(\alpha^2-5\alpha-8\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1} & \frac{12\left(5\alpha^2-35\alpha+12\right)}{\alpha^3-11\alpha^2+16\alpha+144} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & -\frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ 0 & \frac{\alpha\left(\alpha^2-7\alpha-12\right)}{\alpha^3-11\alpha^2+16\alpha+144} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix},
\end{aligned}
\tag{22}
$$

which is equivalent to $AY_m = BY_{m-1} + CY_{m-2}$ where,

$$
\begin{aligned}
A &= \begin{bmatrix} 1+\frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3-22\alpha^2+77\alpha-72}\bar{h} & -\frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ -\frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144} & 1-\frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3-11\alpha^2+16\alpha+144}\bar{h} \end{bmatrix}, \qquad Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, \\
B &= \begin{bmatrix} \frac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72} & -\frac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ -\frac{8\alpha\left(\alpha^2-5\alpha-8\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1} & \frac{12\left(5\alpha^2-35\alpha+12\right)}{\alpha^3-11\alpha^2+16\alpha+144} \end{bmatrix}, \qquad Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, \\
C &= \begin{bmatrix} 0 & -\frac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ 0 & \frac{\alpha\left(\alpha^2-7\alpha-12\right)}{\alpha^3-11\alpha^2+16\alpha+144} \end{bmatrix}, \qquad Y_{m-2} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix}.
\end{aligned}
\tag{23}
$$

From Equation (23), we calculate for the stability polynomial of the method by using the formula,

$$\pi\left(r;\bar{h}\right)=\det\left(Ar^2-Br-C\right), \tag{24}$$

where r denotes the stability polynomial's root, yielding,

$$\begin{aligned}
\pi(r;\bar{h}) &= \det(Ar^2-Br-C)\\
&= \det\left(\begin{bmatrix} 1+\dfrac{3\Gamma(5-\alpha)}{2\alpha^3-22\alpha^2+77\alpha-72}\bar{h} & -\dfrac{\alpha\left(\alpha^2-10\alpha+27\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ -\dfrac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144} & 1-\dfrac{12(2^{\alpha-1})\Gamma(5-\alpha)}{\alpha^3-11\alpha^2+16\alpha+144}\bar{h}\end{bmatrix} r^2\right.\\
&\quad -\begin{bmatrix} \dfrac{\alpha\left(2\alpha^2-14\alpha+21\right)}{2\alpha^3-22\alpha^2+77\alpha-72} & -\dfrac{3\left(5\alpha^2-35\alpha+48\right)}{2\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ -\dfrac{8\alpha\left(\alpha^2-5\alpha-8\right)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1} & \dfrac{12\left(5\alpha^2-35\alpha+12\right)}{\alpha^3-11\alpha^2+16\alpha+144}\end{bmatrix} r \\
&\quad \left. -\begin{bmatrix} 0 & -\dfrac{\alpha\left(\alpha^2-8\alpha+13\right)}{4\left(2\alpha^3-22\alpha^2+77\alpha-72\right)} \\ 0 & \dfrac{\alpha\left(\alpha^2-7\alpha-12\right)}{\alpha^3-11\alpha^2+16\alpha+144}\end{bmatrix}\right).
\end{aligned} \tag{25}$$

From Equation (25), we determined the zero stability by substituting $\bar{h}=0$ which yield:

1. when $\alpha=0.7$,
$$\pi\left(r;0\right)=\frac{660}{235189649}r\left(422927r^3-310164r^2-113841r+1078\right), \tag{26}$$
and the roots, r_s, are $0, 1, -0.2758$ and 0.0092.
2. when $\alpha=0.8$,
$$\pi\left(r;0\right)=\frac{30}{430271}r\left(18254r^3-13413r^2-4902r+61\right), \tag{27}$$
and the roots, r_s, are $0, 1, -0.2772$ and 0.0120.
3. when $\alpha=0.9$,
$$\pi\left(r;0\right)=\frac{620}{17675769}r\left(39768r^3-29781r^2-10154r+167\right), \tag{28}$$
and the roots, r_s, are $0, 1, -0.2669$ and 0.0157.
4. when $\alpha=1.0$,
$$\pi\left(r;0\right)=\frac{197}{125}r^4-\frac{153}{125}r^3-\frac{9}{25}r^2+\frac{1}{125}r, \tag{29}$$
and the roots, r_s, are $0, 1, -0.2442$ and 0.0208.

Based on the calculation above, it is proved that the method is zero stable at various values of α since all the roots lie within $|r_s|\leq 1$. Hence, the method (15) is convergent (Refer Theorem 3.1). Next, we plotted the stability regions of the method for different values of α by using Maple software and presented them as in Figure 1. Referring to the following definitions, we investigated the stability region of the method.

Definition 3.5. [22] *The method (15) is deemed to be absolutely stable in a region, R for a given $\bar{h}$ if each of the roots, r_s of the stability polynomial, $\pi\left(r;\bar{h}\right)=0$, satisfies the condition $|r_s|\leq 1$, where $s=1,2,\ldots,k$.*

Definition 3.6. [22] *The method (15) is said to be A-stable if the region of absolute stability encompasses the complete left-hand half-plane $R\left(\bar{h}\right)<0$.*

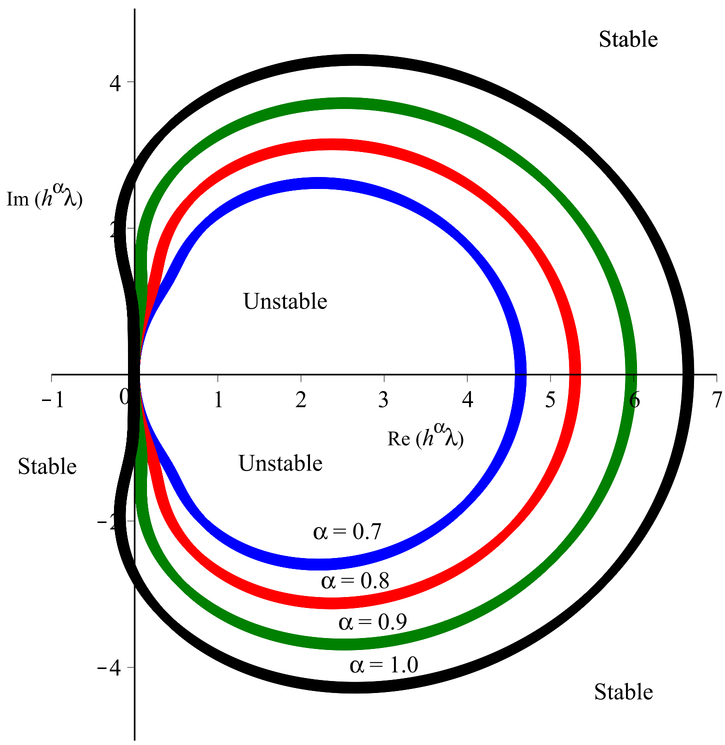


Figure 1: Stability regions for 2FBBDF(4) with different fractional order, α.

Next, we conducted numerical tests to confirm the stability region of the graphs in Figure 1. Referring to the graphs, the stable regions lie beyond the boundaries defined by the blue, red, green, and black lines for $\alpha=0.7$, $\alpha=0.8$, $\alpha=0.9$, and $\alpha=1.0$, respectively. Based on the definition provided in Definition 3.6, it can be concluded that the method (15) with $\alpha=0.7, 0.8$, and 0.9 are A-stable and $\alpha=1.0$ is almost A-stable. This condition is valid for $\alpha\in(0,1)$. In addition, from the graphs, we also found that as the value α decreased, the regions of absolute stability became larger. Hence, the restriction on the chosen starting points is wider.

4 Execution of the Method

In this section, the execution of the 2FBBDF(4) method (15) using Newton's iteration is presented. It also introduces the following notation,

$$e_{n+j}^{(i+1)}=y_{n+j}^{(i+1)}-y_{n+j}^{(i)},\qquad j=1,2, \tag{30}$$

where i is used to define the iteration, $y_{n+j}^{(i+1)}$ is denoted as the $(i+1)^{th}$ iteration values of y_{n+j} and $e_{n+j}^{(i+1)}$ is the differences between iteration values of y_{n+j}. Consequently, the Newton iteration has the following form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F_j\left(y_{n+j}^{(i)}\right)\right)\left(F_j'\left(y_{n+j}^{(i)}\right)\right)^{-1}, \quad j = 1, 2. \tag{31}$$

Substituting Equation (30) into Equation (31) yields the following formula;

$$\begin{aligned} F_1 &= y_{n+1} - \frac{\alpha\left(\alpha^2 - 10\alpha + 27\right)}{4\left(2\alpha^3 - 22\alpha^2 + 77\alpha - 72\right)} y_{n+2} + \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3 - 22\alpha^2 + 77\alpha - 72} h^{\alpha} f_{n+1} - \varsigma_1, \\ F_2 &= y_{n+2} - \frac{8\alpha\left(\alpha^2 - 13\alpha + 48\right)}{\alpha^3 - 11\alpha^2 + 16\alpha + 144} y_{n+1} - \frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3 - 11\alpha^2 + 16\alpha + 144} h^{\alpha} f_{n+2} - \varsigma_2, \end{aligned} \tag{32}$$

where ς_1 and ς_2 are the bakcvalues. Hence,

$$\begin{aligned} e_{n+1}^{(i+1)} &= -\frac{\left(y_{n+1} - \frac{\alpha\left(\alpha^2 - 10\alpha + 27\right)}{4\left(2\alpha^3 - 22\alpha^2 + 77\alpha - 72\right)} y_{n+2} + \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3 - 22\alpha^2 + 77\alpha - 72} h^{\alpha} f_{n+1} - \varsigma_1\right)}{\left(1 + \frac{3\Gamma\left(5-\alpha\right)}{2\alpha^3 - 22\alpha^2 + 77\alpha - 72} h^{\alpha} \frac{\partial f_{n+1}}{\partial y_{n+1}}\right)} \\ e_{n+2}^{(i+1)} &= -\frac{\left(y_{n+2} - \frac{8\alpha\left(\alpha^2 - 13\alpha + 48\right)}{\alpha^3 - 11\alpha^2 + 16\alpha + 144} y_{n+1} - \frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3 - 11\alpha^2 + 16\alpha + 144} h^{\alpha} f_{n+2} - \varsigma_2\right)}{\left(1 - \frac{12(2^{\alpha-1})\Gamma\left(5-\alpha\right)}{\alpha^3 - 11\alpha^2 + 16\alpha + 144} h^{\alpha} \frac{\partial f_{n+2}}{\partial y_{n+2}}\right)}. \end{aligned} \tag{33}$$

For simplicity, Equation (33) is arranged in matrix form,

$$\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{bmatrix} = -\begin{bmatrix} y_{n+1}^{(i)} + \frac{3\Gamma(5-\alpha)}{2\alpha^3-22\alpha^2+77\alpha-72} h^{\alpha} f_{n+1}^{(i)} - \varsigma_1 & -\frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4(2\alpha^3-22\alpha^2+77\alpha-72)} y_{n+2} \\ -\frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144} y_{n+1}^{(i)} & y_{n+2}^{(i)} - \frac{12(2^{\alpha-1})\Gamma(5-\alpha)}{\alpha^3-11\alpha^2+16\alpha+144} h^{\alpha} f_{n+2}^{(i)} - \varsigma_2 \end{bmatrix} \begin{bmatrix} 1 + \frac{3\Gamma(5-\alpha)}{2\alpha^3-22\alpha^2+77\alpha-72} h^{\alpha} \frac{\partial f_{n+1}}{\partial y_{n+1}} & 0 \\ 0 & 1 - \frac{12(2^{\alpha-1})\Gamma(5-\alpha)}{\alpha^3-11\alpha^2+16\alpha+144} h^{\alpha} \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}^{-1}. \tag{34}$$

The derived method is utilised in predictor-corrector computation, specifically in the PECE mode. In this mode, P represents predictor, C represents corrector, and E represents evaluation of the function f, given x and y. Following are the approximate calculations in PECE for y_{n+1} and y_{n+2};

1. P (Predict): Compute the predictor formula, $y_{n+j}^{(p)}$.
2. E (Evaluate): $D^{\alpha} y_{n+j} = f\left(x_{n+j}, y_{n+j}^{(p)}\right)$.
3. C (Correct): Compute the corrector formula, $y_{n+j}^{(c)}$.
4. E (Evaluate): $D^{\alpha} y_{n+j} = f\left(x_{n+j}, y_{n+j}^{(c)}\right)$.

This computational process involved two-stage Newton's iteration to approximate $y_{n+1,n+2}^{(c)}$;

Step 1: Compute $e^{(i+1)}_{n+1,n+2} = A^{-1}B$, where

$$A = \begin{bmatrix} 1 + \frac{3\Gamma(5-\alpha)}{2\alpha^3-22\alpha^2+77\alpha-72} h^\alpha \frac{\partial f_{n+1}}{\partial y_{n+1}} & 0 \\ 0 & 1 - \frac{12(2^{\alpha-1})\Gamma(5-\alpha)}{\alpha^3-11\alpha^2+16\alpha+144} h^\alpha \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix},$$

$$B = \begin{bmatrix} -y^{(i)}_{n+1} - \frac{3\Gamma(5-\alpha)}{2\alpha^3-22\alpha^2+77\alpha-72} h^\alpha f^{(i)}_{n+1} + \varsigma_1 & \frac{\alpha\left(\alpha^2-10\alpha+27\right)}{4(2\alpha^3-22\alpha^2+77\alpha-72)} y_{n+2} \\ \frac{8\alpha\left(\alpha^2-13\alpha+48\right)}{\alpha^3-11\alpha^2+16\alpha+144} y^{(i)}_{n+1} & -y^{(i)}_{n+2} + \frac{12(2^{\alpha-1})\Gamma(5-\alpha)}{\alpha^3-11\alpha^2+16\alpha+144} h^\alpha f^{(i)}_{n+2} + \varsigma_2 \end{bmatrix}.$$

Step 2: Using the value $e^{(i+1)}_{n+1,n+2}$ from Step 1, the corrected value for $y^{(i+1)}_{n+1,n+2}$ are computed.

Step 3: Then, $e^{(i+1)}_{n+1,n+2} = A^{-1}B$ are solved for second stage iteration.

Step 4: The updated values of $y^{(i+1)}_{n+1,n+2}$ are obtained from the last stage of iteration (2nd stage).

5 Numerical Experiments

The numerical examples of fractional order differential equations are solved in this section using the 2FBBDF(4) method (15) which was previously introduced. In order to perform a significant quantitative comparison, we will assess the obtained results in relation to several methods described in the existing literature. C programming was used to compute the numerical results and the absolute error, ABERR is evaluated using the formula:

$$ABERR_i = |y_i(t) - y_i(t_n)|, \tag{35}$$

where $y_i(t)$ is the exact solution and $y_i(t_n)$ is the approximate solution. The notations below are considered in the following tables;

h	:	Step size
ABERR	:	Absolute error
Method	:	Method of comparison
2FBBDF(4)	:	Fourth-order 2-point Fractional Block Backward Differentiation Formula
FDE12	:	Fractional Differential Equation code (FDE12.m) - available in MathWorks
ADM	:	Adomian Decomposition method [2]
FDTM	:	Fractional Differential Transform method [2]
NPSM	:	Numerical Power Series method [9]
FEAM3	:	Fractional Explicit Adams Method order 3 [34]
FRPS	:	Fractional Residual Power Series method [11]
RKHS	:	Reproducing Kernel Hilbert Space method [11]

Therefore, the following examples are considered to validate the proposed numerical method.

Example 5.1. *A basic linear fractional order problem* [34] *is given as follows,*

$$D^\alpha y(t) = -y(t), \qquad y(0) = 1, \qquad t \in [0, 2],$$

with the given true solution: $y(t) = E_\alpha(-t)^\alpha$, *where* $E_\alpha(z)$ *is:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \left(\frac{z^k}{\Gamma(\alpha k + 1)} \right),$$

known as the Mittag-Leffler function.

Example 5.2. *An initial value problem of fractional order equations [34] is considered as follows,*

$$D^{\alpha}y(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^{\alpha} - \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + \left(t^{2\alpha} - t^2\right)^4 - y^4(t), \qquad y(0) = 0, \qquad t \in [0,1],$$

with the given exact solution is

$$y(t) = t^{2\alpha} - t^2.$$

Example 5.3. *Consider the following nonlinear Abel FDEs [2],*

$$D^{\alpha}y(t) = 4y - y^3, \qquad y(0) = \frac{1}{2}, \qquad t \in [0, 0.4],$$

with the given exact solution when $\alpha = 1$ is

$$y(t) = 2\left(\frac{e^{8t}}{e^{8t}+15}\right)^{\frac{1}{2}}.$$

Example 5.4. *An application problem of Riccati FDEs is considered as follows [34],*

$$D^{\alpha}y(t) = -y^2 + 1, \qquad y(0) = 0, \qquad t \in [0,1],$$

with the given exact solution when $\alpha = 1$ is

$$y(t) = \frac{e^{2t}-1}{e^{2t}+1}.$$

Example 5.5. *The problem of non-linear stiff system of fractional order equations by [11] are given as follows,*

$$\begin{aligned} D^{\alpha_1}y_1(t) &= -1002y_1 + 1000y_2^2, \qquad && y_1(0) = 1, \\ D^{\alpha_2}y_2(t) &= y_1 - y_2 - y_2^2, && y_1(0) = 1, \qquad t \in [0,2], \end{aligned}$$

with the given exact solution of the system when $\alpha_1 = \alpha_2 = 1$ as,

$$y_1 = e^{-2t}, \qquad y_2 = e^{-t}.$$

Table 2 displays the absolute error, ABERR, of 2FBBDF(4) (Example 5.1) for various fractional order values, α, and step sizes, h. According to the table, the absolute error reaches its maximum value as h declines, with a tolerance ranging between E-02 and E-03. Nevertheless, the inaccuracy remains acceptable as it diminishes proportionally to the reduction in h. Figure 2 displays the plotted approximation solutions for 2FBBDF(4) and FDE12, with $\alpha = 0.9$ and the time interval, t, ranges from 0 to 2. The curves indicate that 2FBBDF(4) converges to 0, which is the exact solution, in contrast to FDE12. ABERR is subsequently computed based on the solution shown in Figure 2. The performance of the methods is then evaluated by plotting the efficiency curves of log(ABERR) versus t in Figure 3, which is derived from the solution in Figure 2. Both methods yield errors that are less than zero, indicating that the solutions are comparable to the exact solution.

Table 2: Absolute error of 2FBBDF(4) for Example 5.1 as $\alpha = 0.7, 0.8$, and 0.9.

h	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
1.00E-02	3.25090E-02	1.98974E-02	1.02975E-02
1.00E-04	2.73835E-02	1.90526E-02	9.69568E-03
1.00E-06	2.73391E-02	1.90451E-02	9.68725E-03

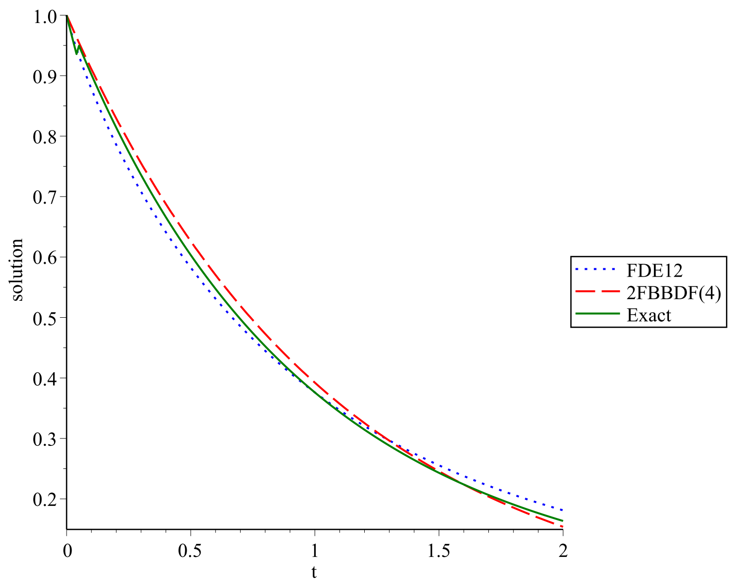


Figure 2: Graphs of the approximate solution for Example 5.1 when $\alpha = 0.9$.

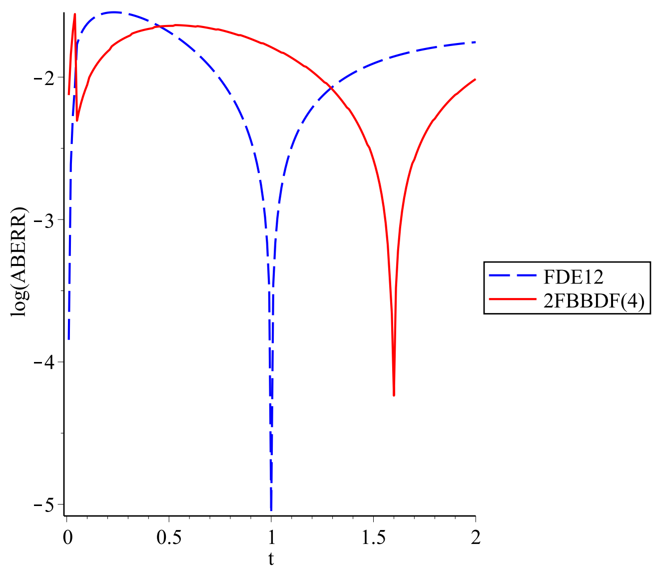


Figure 3: Efficiency curves of Example 5.1 when $\alpha = 0.9$.

The value of $\alpha = 0.7$, 0.8, and 0.9 are used to approximate the solution of Example 5.2, and the absolute error of 2FBBDF(4) is displayed in Table 3. According to the table, the value of ABERR reduces as the value of h lowers. Next, we plot the behavior of the graphs, as depicted in Figure 3, to demonstrate the performance of the techniques with $\alpha = 0.9$. The plots demonstrate that as t grows, the approximation 2FBBDF(4) closely matches the exact answer.

Table 3: Absolute error of 2FBBDF(4) for Example 5.2 as $\alpha = 0.7, 0.8$, and 0.9.

h	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
1.00E-02	1.01710E-02	4.77856E-03	9.62624E-04
1.00E-04	4.01658E-04	7.54718E-05	8.79391E-06
1.00E-06	1.46376E-05	2.36251E-06	1.09551E-07

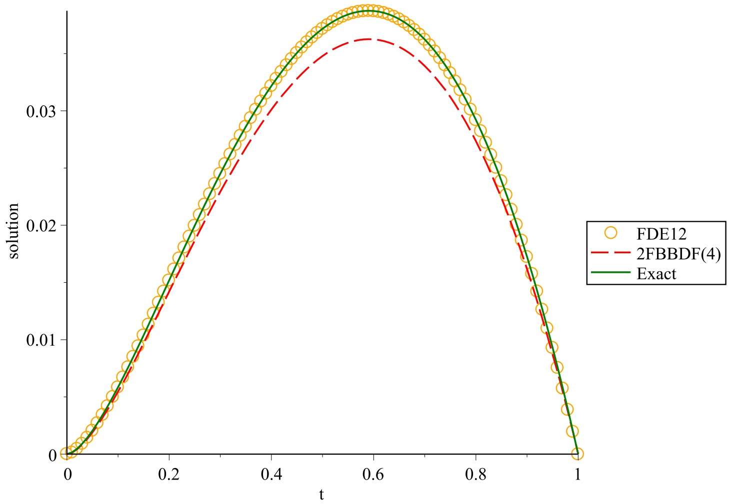


Figure 4: Graphs of the approximate solution for Example 5.2 when $\alpha = 0.9$.

Table 4 displays the approximated solution of Example 5.3 using the ADM, FDTM, and 2FBBDF(4) techniques with $\alpha = 0.9$, as well as the ABERR of the solution with $\alpha = 1.0$. From the table, we illustrate the performance of the methods when $\alpha = 0.9$ and $\alpha = 1.0$ in Figures 5 and 6, respectively. According to Figure 5, we observed that the 2FBBDF(4) method closely approximates the actual solution, in comparison to the ADM and FDTM methods. Meanwhile, in Figure 6, the 2FBBDF(4) and NPSM exhibit a consistent ABERR value in comparison to the FDTM method. The ABERR for the FDTM method increases as t increases.

Table 4: Numerical results for Example 5.3.

t	Method	Solution (when $\alpha = 0.9$)	ABERR (when $\alpha = 1.0$)
0.00	ADM	5.0000E-01	-
	FDTM	5.0000E-01	0.0000E+00
	2FBBDF(4)	5.0000E-01	0.0000E+00
	NPSM	-	0.0000E+00
0.10	ADM	8.05543E-01	-
	FDTM	8.05406E-01	1.14000E-12
	2FBBDF(4)	7.03189E-01	1.66416E-07
	NPSM	-	1.57000E-04
0.20	ADM	1.15315E+00	-
	FDTM	1.14926E+00	7.77000E-08
	2FBBDF(4)	9.58533E-01	1.86205E-07
	NPSM	-	1.85000E-04
0.30	ADM	1.47702E+00	-
	FDTM	1.47387E+00	4.76000E-05
	2FBBDF(4)	1.24360E+00	1.88182E-07
	NPSM	-	1.69000E-04
0.40	ADM	1.62231E+00	-
	FDTM	1.83959E+00	-
	2FBBDF(4)	1.51164E+00	2.05255E-07
	NPSM	-	1.20000E-04

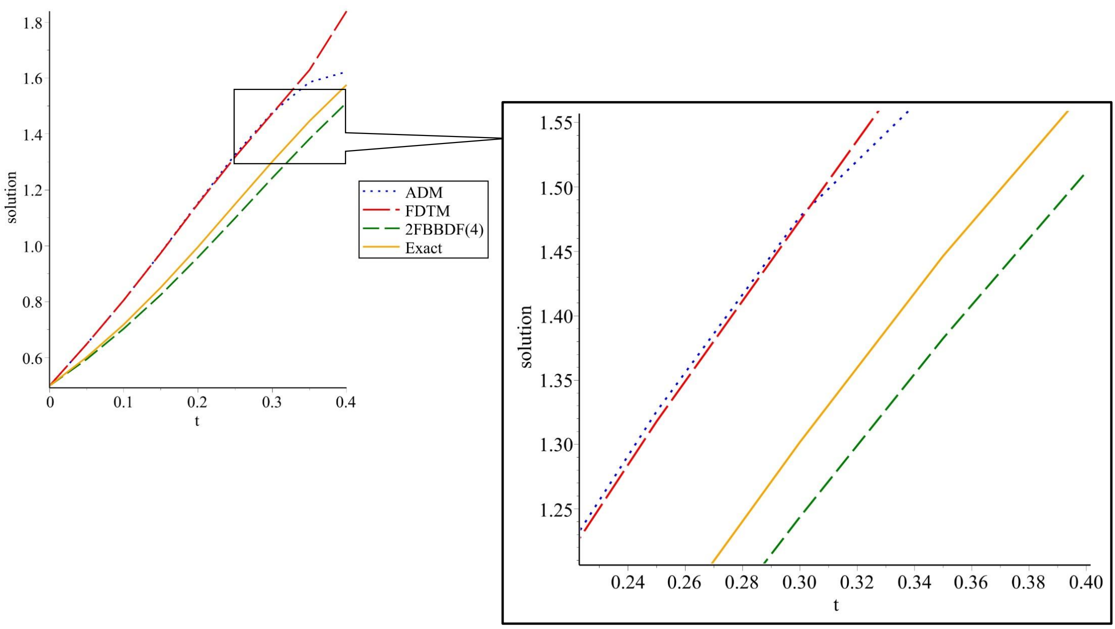


Figure 5: Graph of the approximate solution of Example 5.3 when $\alpha = 0.9$.

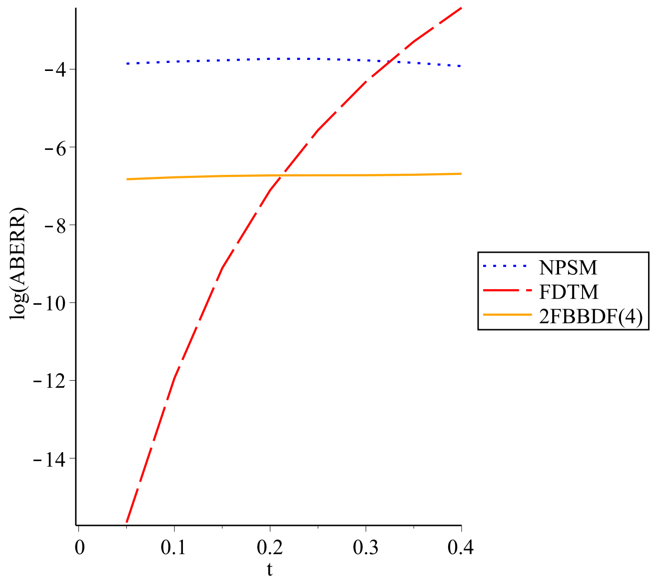


Figure 6: Efficiency curves of Example 5.3 when $\alpha = 1.0$.

Fractional Riccati Differential Equation (FRDE) was defined by Zabidi et al. [34] as an application problem considered in this article. The problem is resolved by employing the 2FBBDF(4) method and comparing it to the FEAM3 method. The results are presented in Table 5, and the efficiency curves are depicted in Figure 8. As shown in the figure, 2FBBDF(4) produces a reduced ABERR in comparison to FEAM3. In addition, the approximation solutions to Example 5.4 for different values of α are illustrated in Figure 7.

Table 5: Method of comparison for Example 5.4 as $\alpha = 1.0$ in terms of ABERR.

t	Method	ABERR
0.2	FEAM3	5.0289E-07
	2FBBDF(4)	9.3634E-09
0.4	FEAM3	2.0725E-06
	2FBBDF(4)	1.1229E-07
0.6	FEAM3	2.8479E-06
	2FBBDF(4)	3.0807E-07
0.8	FEAM3	3.3549E-06
	2FBBDF(4)	5.3024E-07
1.0	FEAM3	3.6079E-06
	2FBBDF(4)	7.0278E-07

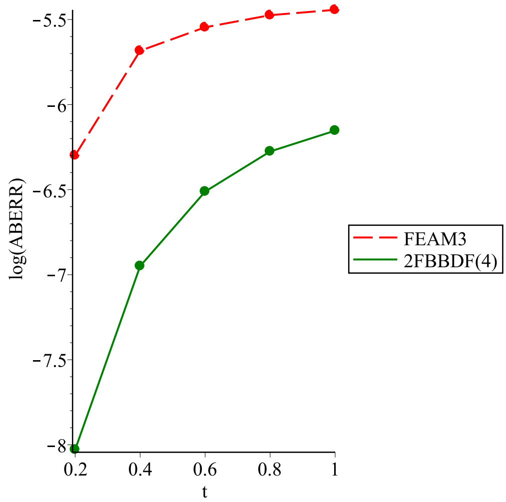


Figure 7: Efficiency curves of Example 5.4 when $\alpha = 1.0$.

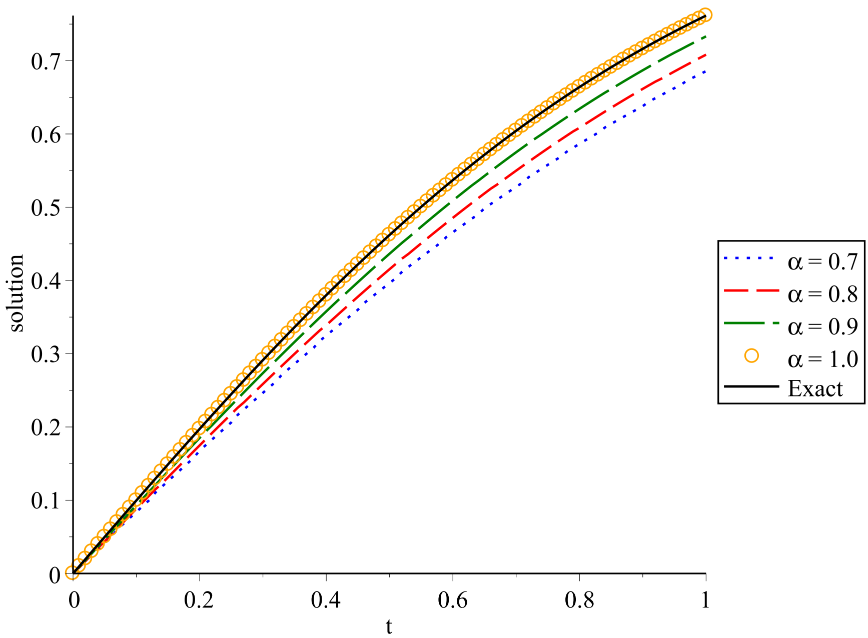
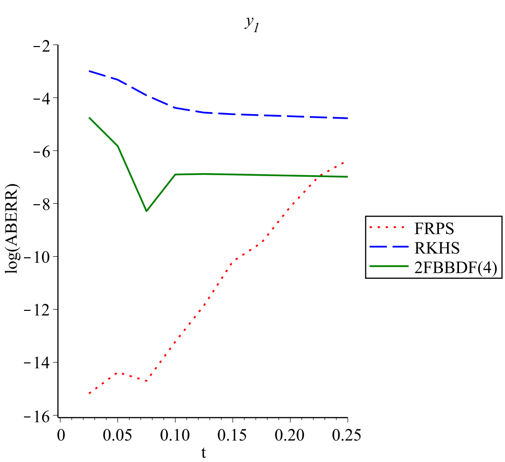


Figure 8: Graph of the approximate solution of Example 5.4 when $\alpha = 0.7, 0.8, 0.9$, and 1.0.

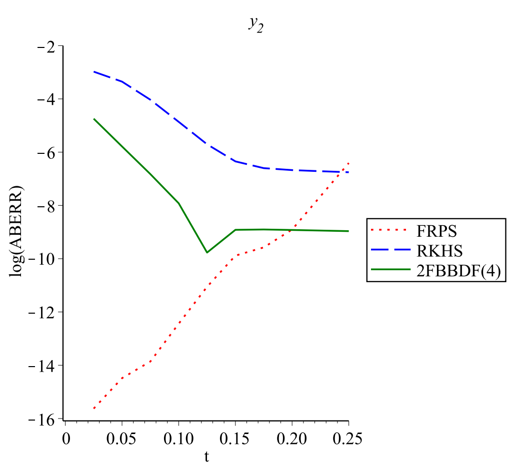
Next, we compare the absolute error, ABERR, obtained from the existing method with the 2FBBDF(4) for Example 5.5. The results are presented in Table 6, and the simulations of the data are illustrated in Figure 9. The figure shows that all the methods give good performance since the ABERR produced is within the tolerance, but the 2FBBDF(4) gives better prediction as compared to the FRPS and RKHS methods because the ABERR for the 2FBBDF(4) is getting smaller as the t increases. It shows that the method is more stable when solving stiff system problems. In addition, we plotted the approximation solution of 2FBBDF(4) for the problem with different fractional orders of α in Figure 10, as the exact solution is provided along with it.

Table 6: Method of comparison for Example 5.5 as $\alpha_1 = \alpha_2 = 1.0$ in terms of ABERR.

t	Method	$y_1(t)$	$y_2(t)$
0.0	FRPS	0.00000e+00	0.00000e+00
	RKHS	0.00000e+00	0.00000e+00
	2FBBDF(4)	0.00000e+00	0.00000e+00
0.4	FRPS	5.55112e-17	0.00000e+00
	RKHS	1.20214e-06	1.23629e-06
	2FBBDF(4)	1.09827e-08	1.41156e-08
0.8	FRPS	5.55112e-16	5.55112e-15
	RKHS	1.28401e-06	2.47593e-02
	2FBBDF(4)	3.97618e-10	9.46857e-09
1.2	FRPS	1.70219e-12	0.00000e+00
	RKHS	9.10870e-07	1.19237e-01
	2FBBDF(4)	3.75421e-09	6.35140e-09
1.6	FRPS	6.92656e-10	5.55112e-16
	RKHS	5.59235e-07	2.62552e-01
	2FBBDF(4)	4.08416e-09	4.08416e-09
2.0	FRPS	7.27625e-08	3.75810e-14
	RKHS	3.18632e-07	4.38604e-01
	2FBBDF(4)	3.44232e-09	2.85784e-09

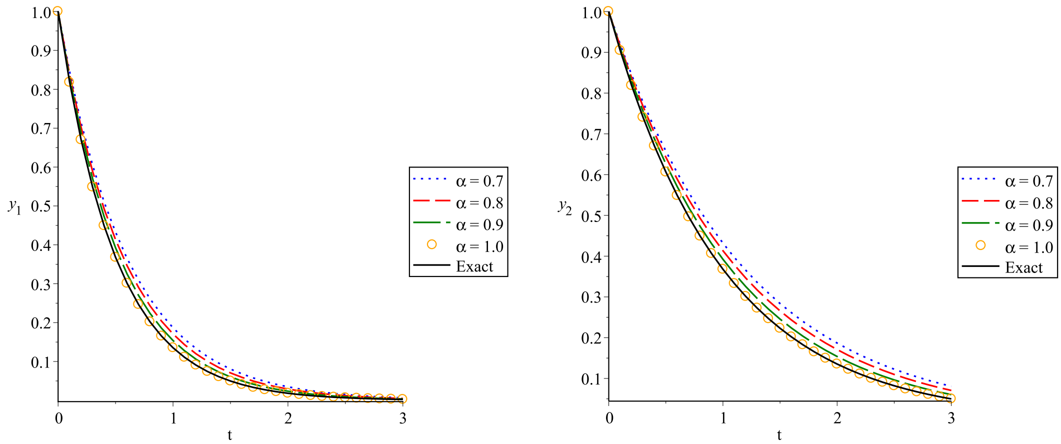


(a) $y_1(t)$



(b) $y_2(t)$

Figure 9: Efficiency curves of Example 5.5 for $\alpha_1 = \alpha_2 = 1$.



(a) $y_1(t)$ (b) $y_2(t)$

Figure 10: Graph of the approximate solution of Example 5.5 when $\alpha = 0.7, 0.8, 0.9$, and 1.0.

6 Conclusion

In conclusion, a new numerical method known as the fourth-order 2-point Fractional Block Backward Differentiation Formula (2FBBDF(4)) has been proposed in this paper. The analysis of stability showed that the derived method (15) is A-stable for the fractional order, $\alpha < 1$ and almost A-stable for $\alpha = 1$. The numerical examples were solved using the proposed method (15), and it was discovered that the 2FBBDF(4) method can achieve comparable results to the existing methods for solving FDEs. Therefore, the fractional order differential equations of linear, non-linear, and systems can be solved using the 2FBBDF(4) method as an alternative solver. In addition, the 2FBBDF(4) method is suitable as the FDE solver, especially when dealing with models that require memory effects, such as dynamical systems.

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Conflicts of Interest The authors declare no conflict of interest.

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